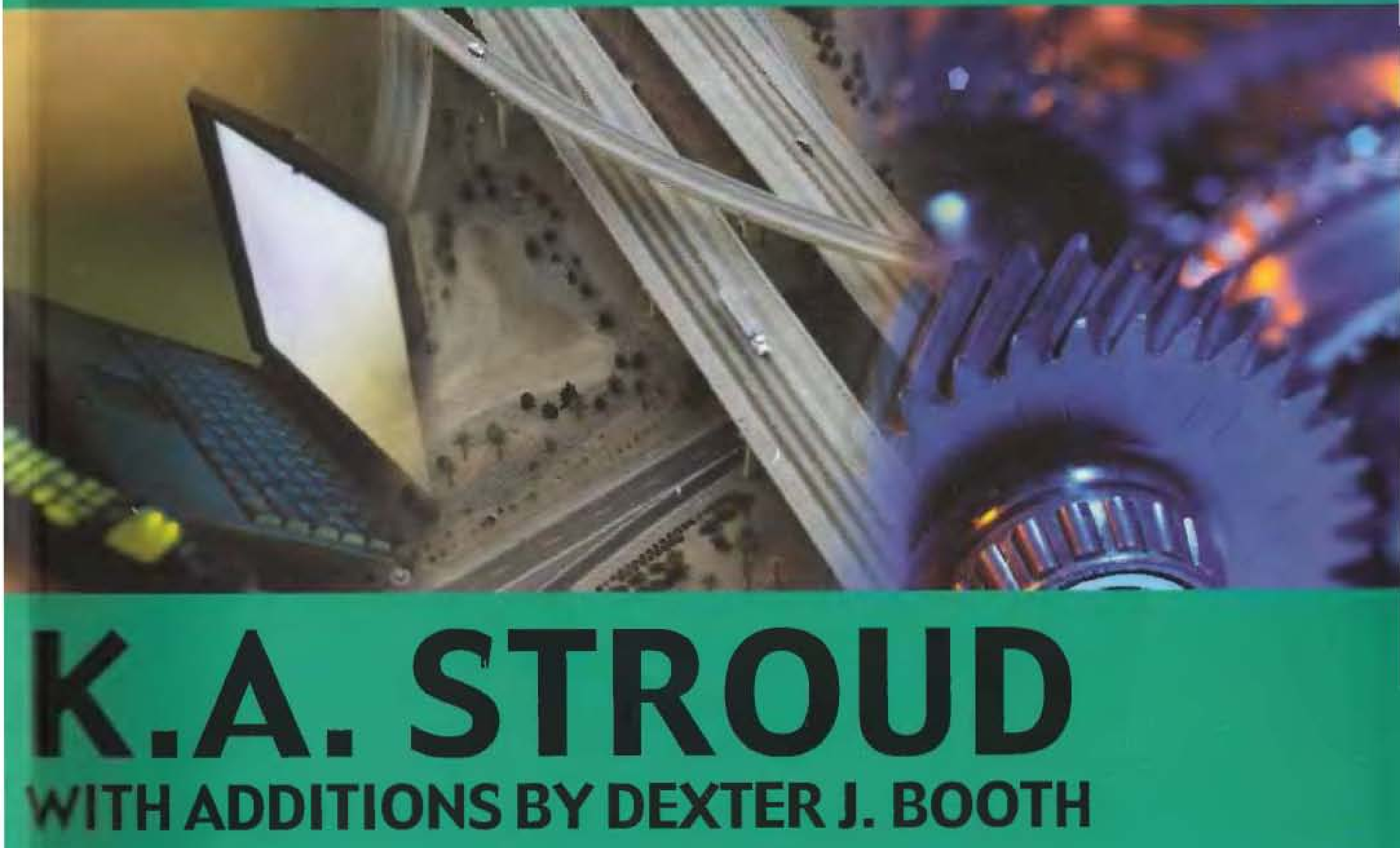
ENGINEERING MATHEMATICS



K.A. STROUD

WITH ADDITIONS BY DEXTER J. BOOTH

FIFTH EDITION

Engineering Mathematics

K. A. Stroud
Formerly Principal Lecturer
Department of Mathematics
Coventry University
United Kingdom

with additions by

Dexter J. Booth
Principal Lecturer
School of Computing and Mathematics
University of Huddersfield
United Kingdom

FIFTH EDITION

Review Board for the fifth edition:

Dr Charles Fall, University of Northumbria at Newcastle
Dr Pat Lewis, Staffordshire University
Dr Mark Kermode, University of Liverpool
Dr Hazel Shute, University of Plymouth
Dr Mike Gover, University of Bradford

INDUSTRIAL PRESS, INC.
NEW YORK

Library of Congress Cataloging-in-Publication Data

Stroud, K. A.

Engineering mathematics / K.A. Stroud ; with additions by Dexter J. Booth.—5th ed.

p. cm.

Includes index.

ISBN 0-8311-3152-7

1. Engineering mathematics—Programmed instruction. I. Booth, Dexter J. II. Title.

TA330 .S78 2001

510í.246—dc21 2001039100

Published in North America under license from Palgrave Publishers Ltd, Houndmills, Basingstoke, Hants RG21 6XS, United Kingdom.

Industrial Press, Inc.

200 Madison Avenue

New York, NY 10016-4078

10 9 8 7 6 5 4

Contents

Programme 26 Introduction to Laplace transforms 1097

Programme 27 Statistics 1119

Hints on using the book and Personal Tutor CD-ROM

This book contains forty lessons called *Programmes*. Each Programme has been written in such a way as to make learning more effective and more interesting. It is like having a personal tutor because you proceed at your own rate of learning and any difficulties you may have are cleared before you have the chance to practise incorrect ideas or techniques.

You will find that each Programme is divided into numbered sections called *frames*. When you start a Programme, begin at Frame 1. Read each frame carefully and carry out any instructions or exercise that you are asked to do. In almost every frame, you are required to make a response of some kind, testing your understanding of the information in the frame, and you can immediately compare your answer with the correct answer given in the next frame. To obtain the greatest benefit, you are strongly advised to cover up the following frame until you have made your response. When a series of dots occurs, you are expected to supply the missing word, phrase, number or mathematical expression. At every stage you will be guided along the right path. There is no need to hurry: read the frames carefully and follow the directions exactly. In this way, you must learn.

Each Programme opens with a list of **Learning outcomes** which specify exactly what you will learn by studying the contents of the Programme. The Programme ends with a matching checklist of **Can You?** questions that enables you to rate your success in having achieved the **Learning outcomes**. If you feel sufficiently confident then tackle the short **Test exercise** which follows. This is set directly on what you have learned in the Programme: the questions are straightforward and contain no tricks. To provide you with the necessary practice, a set of **Further problems** is also included: do as many of these problems as you can. Remember, that in mathematics, as in many other situations, practice makes perfect – or more nearly so.

Of the forty Programmes, the first twelve are at Foundation level. Some of these will undoubtedly contain material with which you are already familiar. However, read the Programme's **Learning outcomes** and if you feel confident about them try the **Quiz** that immediately follows – you will soon find out if you need a refresher course. Indeed, even if you feel you have done some of the topics before, it would still be worthwhile to work steadily through the Programme: it will serve as useful revision and fill any gaps in your knowledge that you may have.

When you have come to the end of a Foundation level Programme and have rated your success in achieving the **Learning outcomes** using the **Can**

You? checklist, go back to the beginning of the Programme and try the **Quiz** before you complete the Programme with the **Test exercise** and the **Further problems**. This way you will get even more practice.

The Personal Tutor CD-ROM

Alongside this text is a CD-ROM containing a bank of questions for you to answer using your computer. There are no scores for the questions and it is not possible to enter an incorrect answer – you are guided every inch of the way without having to worry about getting any answers wrong. As with the exercises in the book, take your time, make mistakes, correct them using all the assistance available to you, and you will surely learn.

The bank consists of odd-numbered questions from the **Quizzes** and **Test exercises**. Using the CD-ROM will give you more practice and increase your confidence in your learning of the mathematics. The questions require their solution to be entered in several steps, and each step is accompanied by a hint and a step solution. In addition, the complete solution to the entire question is also available. A PERSONAL TUTOR symbol next to a **Quiz** or **Test exercise** question indicates that it is also on the CD-ROM.

The Personal Tutor On-line

The odd-numbered questions from the **Further problems** are available from March 2001 on the Stroud website (see below), offering all the functionality of the Personal Tutor CD-ROM. Again, these are marked by the PERSONAL TUTOR symbol in the book.

The book's website at www.palgrave.com/stroud

You are recommended to visit the book's website that is maintained by its publisher in the United Kingdom. There you will find a growing resource to accompany the text including, from April 2001, mathematical questions set within engineering and scientific contexts. There is also an email address for you to communicate your comments on the book, critical or otherwise. This book is for you, and so the more that we know about your wishes and desires the more likely they are to be accommodated in future editions.

Useful background information

Symbols used in the text

$=$	is equal to	$\rightarrow$	tends to
$\approx$	is approximately equal to	$\neq$	is not equal to
$>$	is greater than	$\equiv$	is identical to
$\geq$	is greater than or equal to	$<$	is less than
$n!$	factorial $n = 1 \times 2 \times 3 \times \ldots \times n$	$\leq$	is less than or equal to
$\lvert k \rvert$	modulus of k, i.e. size of k irrespective of sign	∞	infinity
$\sum$	summation	$\underset{n\to\infty}{Lim}$	limiting value as $n \rightarrow \infty$

Useful mathematical information

1 Algebraic identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$a^2 - b^2 = (a-b)(a+b) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

2 Trigonometrical identities

(a) $\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta = 1 + \tan^2\theta$; $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

(c) Let $A = B = \theta$ $\therefore$ $\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

(d) Let $\theta = \frac{\phi}{2}$ $\therefore$ $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$

$$\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = 1 - 2\sin^2\frac{\phi}{2} = 2\cos^2\frac{\phi}{2} - 1$$

$$\tan\phi = \frac{2\tan\frac{\phi}{2}}{1 - 2\tan^2\frac{\phi}{2}}$$

(e) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(f) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(g) Negative angles: $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

(h) Angles having the same trigonometrical ratios:

(i) Same sine: θ and $(180^\circ - \theta)$

(ii) Same cosine: θ and $(360^\circ - \theta)$, i.e. $(-\theta)$

(iii) Same tangent: θ and $(180^\circ + \theta)$

(i) $a\sin\theta + b\cos\theta = A\sin(\theta + \alpha)$

$a\sin\theta - b\cos\theta = A\sin(\theta - \alpha)$

$a\cos\theta + b\sin\theta = A\cos(\theta - \alpha)$

$a\cos\theta - b\sin\theta = A\cos(\theta + \alpha)$

$$\text{where} \quad \begin{cases} A = \sqrt{a^2 + b^2} \\ \alpha = \tan^{-1}\frac{b}{a} \quad (0^\circ < \alpha < 90^\circ) \end{cases}$$

3 Standard curves

(a) *Straight line*

Slope, $m = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$

Angle between two lines, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

For parallel lines, $m_2 = m_1$

For perpendicular lines, $m_1 m_2 = -1$

Equation of a straight line (slope $= m$)

(i) Intercept c on real y-axis: $y = mx + c$

(ii) Passing through (x_1, y_1): $y - y_1 = m(x - x_1)$

(iii) Joining (x_1, y_1) and (x_2, y_2): $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

(b) *Circle*

Centre at origin, radius r: $x^2 + y^2 = r^2$

Centre (h, k), radius r: $(x - h)^2 + (y - k)^2 = r^2$

General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

with centre $(-g, -f)$: radius $= \sqrt{g^2 + f^2 - c}$

Parametric equations: $x = r\cos\theta,\ y = r\sin\theta$

(c) *Parabola*

Vertex at origin, focus $(a, 0)$: $y^2 = 4ax$

Parametric equations : $x = at^2,\ y = 2at$

(d) *Ellipse*

Centre at origin, foci $\left(\pm\sqrt{a^2 + b^2}, 0\right)$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where $a =$ semi-major axis, $b =$ semi-minor axis

Parametric equations: $x = a\cos\theta,\ y = b\sin\theta$

(e) *Hyperbola*

Centre at origin, foci $\left(\pm\sqrt{a^2 + b^2}, 0\right)$: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Parametric equations: $x = a\sec\theta,\ y = b\tan\theta$

Rectangular hyperbola:

Centre at origin, vertex $\pm\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$: $xy = \dfrac{a^2}{2} = c^2$

i.e. $xy = c^2$ where $c = \dfrac{a}{\sqrt{2}}$

Parametric equations: $x = ct,\ y = c/t$

4 Laws of mathematics

(a) *Associative laws* – for addition and multiplication

$a + (b + c) = (a + b) + c$

$a(bc) = (ab)c$

(b) *Commutative laws* – for addition and multiplication

$a + b = b + a$

$ab = ba$

(c) *Distributive laws* – for multiplication and division

$a(b + c) = ab + ac$

$\dfrac{b + c}{a} = \dfrac{b}{a} + \dfrac{c}{a}$ (provided $a \neq 0$)

Preface to the Fifth Edition

Engineering Mathematics by Ken Stroud has been a favoured textbook of science and engineering students for over 30 years and to have been asked to contribute to a new edition of this remarkable book has given me great pleasure and no little trepidation. A clear requirement of any additions in this new edition was the retention of the very essence of the book that has contributed to so many students' mathematical abilities over the years. In line with this, the majority of the Programmes in Part II are retained largely unaltered and I have of course taken great care to preserve the time-tested Stroud format and close attention to technique development throughout the book which have made *Engineering Mathematics* the tremendous success that it is. I trust therefore that I have succeeded in meeting this requirement. The largest part of my work has been to re-structure, re-organize and expand the Foundation section, which has been done in response to the many changes in the university sector, meaning that students from ever more diverse backgrounds are starting courses in engineering mathematics. Towards the end of the book I have also added a Programme on Laplace transforms in place of an original Programme that dealt with the D-Operator method of solving ordinary differential equations. In a book of this size, space is crucial and I felt that an early introduction to transform techniques in the solution of differential equations was more valuable than to retain what is becoming a less widely used method.

To give as much assistance as possible in organizing the student's study I have introduced specific **Learning outcomes** at the beginning and **Can You?** checklists at the end of each Programme. In this way, learning experience is made more explicit and the student is given greater confidence in what has been learnt.

In addition to the text, a CD-ROM is available that provides a large number of questions for students to work through. These are a selection of questions from the text for which, in the text, no worked solutions are given. Using the CD-ROM, students will be closely guided through the solutions to these questions, so confirming and adding to their skills in mathematical techniques and their knowledge of mathematical ideas. A growing collection of interactive questions plus additional problems set within engineering and scientific contexts is available at the book's website **www.palgrave.com/stroud**

The work involved in creating a new edition of an established textbook is always a cooperative, team effort and this book is no exception. I was fortunate enough to be able to meet Ken Stroud in the few months before he died and to be able to discuss with him ideas for the new edition. He was very enthusiastic about taking the book forward with new technology and both his eagerness and his concerns have been taken into account in the development of the CD-ROM. The enormous task which Ken undertook in writing the original book and three subsequent editions cannot be underestimated. Ken's achievement was an extraordinary one and it has been a great privilege to be able to work on such a book. I should like to thank the Stroud family for their support in my work for this new edition and the editorial team for their close attention to detail, their appropriate comments on the text and their assiduous checking of everything that I have written. I should also like to thank Richard Law of *Lexis Interactive* in Mulhouse for his care and dedicated professionalism in the construction of the software. As with any team, the role of the leader is paramount and I should particularly like to thank my Editor Helen Bugler whose good humour and care has made this extensive exercise one of enjoyment and pleasure.

Huddersfield
January 2001

Dexter J. Booth

Preface to the First Edition

The purpose of this book is to provide a complete year's course in mathematics for those studying in the engineering, technical and scientific fields. The material has been specially written for courses leading to

(i) Part I of B.Sc. Engineering Degrees,
(ii) Higher National Diploma and Higher National Certificate in technological subjects, and for other courses of a comparable level. While formal proofs are included where necessary to promote understanding, the emphasis throughout is on providing the student with sound mathematical skills and with a working knowledge and appreciation of the basic concepts involved. The programmed structure ensures that the book is highly suited for general class use and for individual self-study, and also provides a ready means for remedial work or subsequent revision.

The book is the outcome of some eight years' work undertaken in the development of programmed learning techniques in the Department of Mathematics at the Lanchester College of Technology, Coventry. For the past four years, the whole of the mathematics of the first year of various Engineering Degree courses has been presented in programmed form, in conjunction with seminar and tutorial periods. The results obtained have proved to be highly satisfactory, and further extension and development of these learning techniques are being pursued.

Each programme has been extensively validated before being produced in its final form and has consistently reached a success level above 80/80, i.e. at least 80% of the students have obtained at least 80% of the possible marks in carefully structured criteria tests. In a research programme, carried out against control groups receiving the normal lectures, students working from programmes have attained significantly higher mean scores than those in the control groups and the spread of marks has been considerably reduced. The general pattern has also been reflected in the results of the sessional examinations.

The advantages of working at one's own rate, the intensity of the student involvement and the immediate assessment of responses, are well known to those already acquainted with programmed learning activities. Programmed learning in the first year of a student's course at a college or university provides the additional advantage of bridging the gap between the rather highly organised aspect of school life and the freer environment and greater personal responsibility for his own progress which faces every student on entry to the realms of higher education.

Acknowledgement and thanks are due to all those who have assisted in any way in the development of the work, including whose who have been actively engaged in validation processes. I especially wish to record my sincere thanks for the continued encouragement and support which I received from my present head of Department at the College, Mr. J.E. Sellars, M.Sc., A.F.R.Ae.S., F.I.M.A., and also from Mr. R. Wooldridge, M.C., B.Sc., F.I.M.A., formerly Head of Department, now Principal of Derby College of Technology. Acknowledgement is also made of the many sources, too numerous to list, from which the selected examples quoted in the programmes have been gleaned over the years. Their inclusion contributes in no small way to the success of the work.

K.A. Stroud

Preface to the Second Edition

The continued success of *Engineering Mathematics* since its first publication has been reflected in the number of courses for which it has been adopted as the official class text and also in the correspondence from numerous individuals who have welcomed the self-instructional aspects of the work.

Over the years, however, syllabuses of existing courses have undergone some modification and new courses have been established. As a result, suggestions have been received from time to time requesting the inclusion of further programme topics in addition to those already provided as core material for relevant undergraduate and comparable courses. Unlimited expansion of the book to accommodate all the topics requested is hardly feasible on account of the physical size of the book and the commercial aspects of production. However, in the light of these representations and as a result of further research undertaken by the author and the publishers, it has now been found possible to provide a new edition of *Engineering Mathematics* incorporating three of the topics for which there is clearly a wide demand.

The additional programmes cover the following topics:

(a) Matrices: definitions; types of matrices; operations; transpose; inverse; solution of linear equations; eigenvalues and eigenvectors.
(b) Curves and curve fitting: standard curves; asymptotes; systematic curve sketching; curve recognition; curve fitting; method of least squares.
(c) Statistics: discrete and continuous data; grouped data; frequency and relative frequency; histograms; central tendency – mean, mode and median; coding; frequency polygons and frequency curves; dispersion – range, variance and standard deviation; normal distribution and standardised normal curve.

The three new programmes follow the structure of the previous material and each is provided with numerous worked examples and exercises. As before, each programme concludes with a short Test Exercise for self-assessment and set of Further Problems provides valuable extra practice. A complete set of answers is available at the end of the book.

Advantage has also been taken during the revision of the book to amend a small number of minor points in other parts of the text and it is anticipated that, in its new updated form, the book will have an even greater appeal and continue to provide a worthwhile service.

K.A. Stroud

Preface to the Third Edition

Following the publication of the enlarged second edition of *Engineering Mathematics*, which included a programme on the introduction to Statistics, requests were again received for an associated programme on Probability. This has now been incorporated as Programme XXVIII of the current third edition of the book.

The additional programme follows the established pattern and structure of the previous sections of the text, including the customary worked examples through which the student is guided with progressive responsibility and concluding with the Text Exercise and set of Further Problems for essential practice. Answers to all problems are provided. The opportunity has also been taken to make one or two minor modifications to the remainder of the text.

Engineering Mathematics, originally devised as a first year mathematics course for engineering and science degree undergraduates and students of comparable courses, is widely sought both for general class use and for individual study. A companion volume and sequel, *Further Engineering Mathematics*, dealing with core material of a typical second/third year course, is also now available through the normal channels. The two texts together provide a comprehensive and integrated course of study and have been well received as such.

My thanks are due, once again, to the publishers for their ready cooperation and helpful advice in the preparation of the material for publication.

K.A.S.

Preface to the Fourth Edition

Since the publication of the third edition of *Engineering Mathematics*, considerable changes in the syllabus and options for A-level qualifications in Mathematics have been introduced nationally, as a result of which numbers of students with various levels of mathematical background have been enrolling for undergraduate courses in engineering and science. In view of the widespread nature of the situation, requests have been received from several universities for the inclusion in the new edition of *Engineering Mathematics* of what amounts to a bridging course of material in relevant topics to ensure a solid foundation on which the main undergraduate course can be established.

Accordingly, the fourth edition now includes ten new programmes ranging from Number Systems and algebraic processes to an introduction to the Calculus. These Foundation Topics constitute Part I of the current book and precede the well-established section of the text now labelled as Part II.

For students already well versed in the contents of the Part I programmes the Test Exercises and Further Problems afford valuable revision and should not be ignored.

With the issue of the new edition, the publishers have undertaken the task of changing the format of the pages and of resetting the whole of the text to provide a more open presentation and improved learning potential for the student.

Once again, I am indebted to the publishers for their continued support and close cooperation in the preparation of the text in its new form and particularly to all those directly involved in the editorial, production and marketing processes both at home and overseas.

K.A.S.

PART I
Foundation topics

Programme F.1

Arithmetic

Frames 1 to 156

Learning outcomes

When you have completed this Programme you will be able to:

- Carry out the basic rules of arithmetic with integers
- Write a natural number as a product of prime numbers
- Find the highest common factor and lowest common multiple of two natural numbers
- Check the result of a calculation making use of rounding
- Manipulate fractions, ratios and percentages
- Manipulate decimal numbers
- Manipulate powers
- Use standard or preferred standard form and complete a calculation to the required level of accuracy
- Understand the construction of various number systems and convert from one number system to another.

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.1

Frames

1 Place the appropriate symbol < or > between each of the following pairs of numbers:
(a) $-3 \quad -2$ (b) $8 \quad -13$ (c) $-25 \quad 0$ — 1 to 4

2 Find the value of each of the following:
(a) $13 + 9 \div 3 - 2 \times 5$ (b) $(13 + 9) \div (3 - 2) \times 5$ — 5 to 12

3 Write each of the following as a product of prime factors:
(a) 170 (b) 455 (c) 9075 (d) 1140 — 16 to 19

4 Find the HCF and the LCM of each pair of numbers:
(a) 84, 88 (b) 105, 66 — 20 to 21

5 Round each number to the nearest 10, 100 and 1000:
(a) 1354 (b) 2501 (c) -2452 (d) $-23\,625$ — 22 to 24

6 Reduce each of the following fractions to their lowest terms:
(a) $\frac{12}{18}$ (b) $\frac{144}{21}$ (c) $-\frac{49}{14}$ (d) $\frac{64}{4}$ — 28 to 36

7 Evaluate the following:
(a) $\frac{3}{7} \times \frac{2}{3}$ (b) $\frac{11}{30} \div \frac{5}{6}$ (c) $\frac{3}{7} + \frac{4}{13}$ (d) $\frac{5}{16} - \frac{4}{3}$ — 37 to 46

8 Write the following proportions as ratios:
(a) $\frac{1}{2}$ of A, $\frac{1}{5}$ of B and $\frac{3}{10}$ of C
(b) $\frac{1}{4}$ of P, $\frac{1}{3}$ of Q, $\frac{1}{5}$ of R and the remainder S — 47 to 48

9 Complete the following:
(a) $\frac{4}{5} = \quad \%$ (b) 48% of $50 =$
(c) $\frac{9}{14} = \quad \%$ (d) 15% of $25 =$ — 49 to 52

10 Round each of the following decimal numbers, first to 3 significant figures and then to 2 decimal places:
(a) 21·355 (b) 0·02456
(c) 0·3105 (d) 5134·555 — 56 to 65

		Frames
11	Convert each of the following to decimal form to 3 decimal places: (a) $\frac{4}{15}$ (b) $-\frac{7}{13}$ (c) $\frac{9}{5}$ (d) $-\frac{28}{13}$	66 to 67
12	Convert each of the following to fractional form in lowest terms: (a) $0{\cdot}8$ (b) $2{\cdot}8$ (c) $3{\cdot}\dot{3}\dot{2}$ (d) $-5{\cdot}5$	68 to 73
13	Write each of the following in abbreviated form: (a) $1{\cdot}010101\ldots$ (b) $9{\cdot}2456456456\ldots$	70 to 71
14	Write each of the following as a number raised to a power: (a) $3^6 \times 3^3$ (b) $4^3 \div 2^5$ (c) $(9^2)^3$ (d) $(7^0)^{-8}$	78 to 89
15	Find the value of each of the following to 3 dp: (a) $15^{\frac{1}{3}}$ (b) $\sqrt[5]{5}$ (c) $(-27)^{\frac{1}{3}}$ (d) $(-9)^{\frac{1}{2}}$	90 to 94
16	Write each of the following as a single decimal number: (a) $3{\cdot}2044 \times 10^3$ (b) $16{\cdot}1105 \div 10^{-2}$	95 to 98
17	Write each of the following in standard form: (a) $134{\cdot}65$ (b) $0{\cdot}002401$	99 to 101
18	Write each of the following in preferred standard form: (a) $16{\cdot}1105 \div 10^{-2}$ (b) $9{\cdot}3304$	102 to 104
19	In each of the following the numbers have been obtained by measurement. Evaluate each calculation to the appropriate level of accuracy: (a) $11{\cdot}4 \times 0{\cdot}0013 \div 5{\cdot}44 \times 8{\cdot}810$ (b) $\dfrac{1{\cdot}01 \div 0{\cdot}00335}{9{\cdot}12 \times 6{\cdot}342}$	105 to 108
20	Express the following numbers in denary form: (a) $1011{\cdot}01_2$ (b) $456{\cdot}721_8$ (c) $123{\cdot}A29_{12}$ (d) $CA1{\cdot}B22_{16}$	112 to 129
21	Convert $15{\cdot}605_{10}$ to the equivalent octal, binary, duodecimal and hexadecimal forms.	130 to 152

Types of numbers

1

The natural numbers

The first numbers we ever meet are the *whole numbers*, also called the *natural numbers*, and these are written down using *numerals*.

Numerals and place value

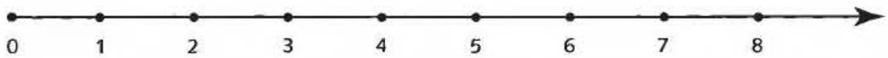
The *whole numbers* or *natural numbers* are written using the ten numerals 0, 1, ..., 9 where the position of a numeral dictates the value that it represents. For example:

246 stands for 2 hundreds and 4 tens and 6 units. That is $200 + 40 + 6$

Here the numerals 2, 4 and 6 are called the hundreds, tens and unit *coefficients* respectively. This is the place value principle.

Points on a line and order

The natural numbers can be represented by equally spaced points on a straight line where the first natural number is zero 0.



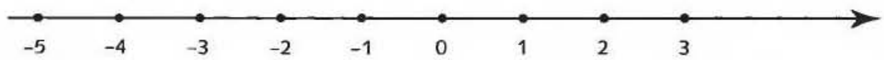
The natural numbers are ordered – they progress from small to large. As we move along the line from left to right the numbers increase as indicated by the arrow at the end of the line. On the line, numbers to the left of a given number are *less than* ($<$) the given number and numbers to the right are *greater than* ($>$) the given number. For example, $8 > 5$ because 8 is represented by a point on the line to the right of 5. Similarly, $3 < 6$ because 3 is to the left of 6.

Now move on to the next frame

2

The integers

If the straight line displaying the natural numbers is extended to the left we can plot equally spaced points to the left of zero.



These points represent *negative* numbers which are written as the natural number preceded by a minus sign, for example -4. These positive and negative whole numbers and zero are collectively called the *integers*. The notion of order still applies. For example, $-5 < 3$ and $-2 > -4$ because the point on the line representing -5 is to the *left* of the point representing 3. Similarly, -2 is to the *right* of -4.

The numbers -10, 4, 0, -13 are of a type called ....

You can check your answer in the next frame

3

$\boxed{\text{Integers}}$

They are integers. The natural numbers are all positive. Now try this:

Place the appropriate symbol $<$ or $>$ between each of the following pairs of numbers:

(a) $-3 \quad -6$
(b) $\ \ 2 \quad -4$
(c) $-7 \quad 12$

Complete these and check your results in the next frame

4

(a) $-3 > -6$
(b) $\ \ 2 > -4$
(c) $-7 < 12$

The reasons being:

(a) $-3 > -6$ because -3 is represented on the line to the *right* of -6
(b) $2 > -4$ because 2 is represented on the line to the *right* of -4
(c) $-7 < 12$ because -7 is represented on the line to the *left* of 12

Now move on to the next frame

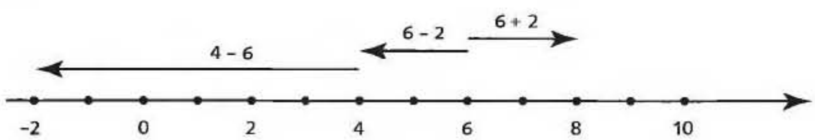
5

Brackets

Brackets should be used around negative numbers to separate the minus sign attached to the number from the arithmetic operation symbol. For example, $5 - -3$ should be written $5 - (-3)$ and 7×-2 should be written $7 \times (-2)$. *Never write two arithmetic operation symbols together without using brackets.*

Addition and subtraction

Adding two numbers gives their *sum* and subtracting two numbers gives their *difference*. For example, $6 + 2 = 8$. Adding moves to the right of the first number and subtracting moves to the left of the first number, so that $6 - 2 = 4$ and $4 - 6 = -2$:



▶

Adding a negative number is the same as subtracting its positive counterpart. For example $7 + (-2) = 7 - 2$. The result is 5. Subtracting a negative number is the same as adding its positive counterpart. For example $7 - (-2) = 7 + 2 = 9$.

So what is the value of:

(a) $8 + (-3)$
(b) $9 - (-6)$
(c) $(-4) + (-8)$
(d) $(-14) - (-7)$?

When you have finished these check your results with the next frame

6

(a)	5
(b)	15
(c)	−12
(d)	−7

Move now to Frame 7

7

Multiplication and division

Multiplying two numbers gives their *product* and dividing two numbers gives their *quotient*. Multiplying and dividing two positive or two negative numbers gives a positive number. For example:

$12 \times 2 = 24$ and $(-12) \times (-2) = 24$
$12 \div 2 = 6$ and $(-12) \div (-2) = 6$

Multiplying or dividing a positive number by a negative number gives a negative number. For example:

$12 \times (-2) = -24$, $(-12) \div 2 = -6$ and $8 \div (-4) = -2$

So what is the value of:

(a) $(-5) \times 3$
(b) $12 \div (-6)$
(c) $(-2) \times (-8)$
(d) $(-14) \div (-7)$?

When you have finished these check your results with the next frame

8

(a)	−15
(b)	−2
(c)	16
(d)	2

Move on to Frame 9

Brackets and precedence rules

9

Brackets and the precedence rules are used to remove ambiguity in a calculation. For example, $14 - 3 \times 4$ could be either:

$11 \times 4 = 44$ or $14 - 12 = 2$

depending on which operation is performed first.

To remove the ambiguity we rely on the precedence rules:

In any calculation involving all four arithmetic operations we proceed as follows:

(a) Working from the left evaluate divisions and multiplications as they are encountered;

this leaves a calculation involving just addition and subtraction.

(b) Working from the left evaluate additions and subtractions as they are encountered.

For example, to evaluate:

$$4 + 5 \times 6 \div 2 - 12 \div 4 \times 2 - 1$$

a first sweep from left to right produces:

$$4 + 30 \div 2 - 3 \times 2 - 1$$

a second sweep from left to right produces:

$$4 + 15 - 6 - 1$$

and a final sweep produces:

$$19 - 7 = 12$$

If the calculation contains brackets then these are evaluated first, so that:

$$\begin{aligned}(4 + 5 \times 6) \div 2 - 12 \div 4 \times 2 - 1 &= 34 \div 2 - 6 - 1\\ &= 17 - 7\\ &= 10\end{aligned}$$

This means that:

$$\begin{aligned}14 - 3 \times 4 &= 14 - 12\\ &= 2\end{aligned}$$

because, reading from the left we multiply before we subtract. Brackets must be used to produce the alternative result:

$$\begin{aligned}(14 - 3) \times 4 &= 11 \times 4\\ &= 44\end{aligned}$$

because the precedence rules state that brackets are evaluated first.

So that $34 + 10 \div (2 - 3) \times 5 =$ . . . .

Result in the next frame

10

$\boxed{-16}$

Because

$$\begin{aligned} 34 + 10 \div (2 - 3) \times 5 &= 34 + 10 \div (-1) \times 5 && \text{we evaluate the bracket first} \\ &= 34 + (-10) \times 5 && \text{by dividing} \\ &= 34 + (-50) && \text{by multiplying} \\ &= 34 - 50 && \text{finally we subtract} \\ &= -16 \end{aligned}$$

Notice that when brackets are used we can omit the multiplication signs and replace the division sign by a line, so that:

$5 \times (6 - 4)$ becomes $5(6 - 4)$

and

$(25 - 10) \div 5$ becomes $(25 - 10)/5$ or $\dfrac{25 - 10}{5}$

When evaluating expressions containing *nested* brackets the innermost brackets are evaluated first. For example:

$$\begin{aligned} 3(4 - 2[5 - 1]) &= 3(4 - 2 \times 4) && \text{evaluating the innermost bracket } [\ldots] \text{ first} \\ &= 3(4 - 8) && \text{multiplication before subtraction inside the } (\ldots) \text{ bracket} \\ &= 3(-4) && \text{subtraction completes the evaluation of the } (\ldots) \text{ bracket} \\ &= -12 && \text{multiplication completes the calculation} \end{aligned}$$

so that $5 - \{8 + 7[4 - 1] - 9/3\} = \ldots\ldots\ldots\ldots$

Work this out, the result is in the following frame

11

$\boxed{-21}$

Because

$$\begin{aligned} 5 - \{8 + 7[4 - 1] - 9/3\} &= 5 - \{8 + 7 \times 3 - 9 \div 3\} \\ &= 5 - \{8 + 21 - 3\} \\ &= 5 - \{29 - 3\} \\ &= 5 - 26 \\ &= -21 \end{aligned}$$

Now move to Frame 12

Basic laws of arithmetic

12

All the work that you have done so far has been done under the assumption that you know the rules that govern the use of arithmetic operations as, indeed, you no doubt do. However, there is a difference between knowing the rules innately and being consciously aware of them, so here they are. The four basic arithmetic operations are:

addition and subtraction

multiplication and division

where each pair may be regarded as consisting of 'opposites' – in each pair one operation is the reverse operation of the other.

1 Commutativity

Two integers can be added or multiplied in either order without affecting the result. For example:

$5 + 8 = 8 + 5 = 13$ and $5 \times 8 = 8 \times 5 = 40$

We say that addition and multiplication are commutative operations

The order in which two integers are subtracted or divided *does* affect the result. For example:

$4 - 2 \neq 2 - 4$ because $4 - 2 = 2$ and $2 - 4 = -2$

Notice that $\neq$ means *is not equal to*. Also

$4 \div 2 \neq 2 \div 4$

We say that subtraction and division are not commutative operations

2 Associativity

The way in which three or more integers are associated under addition or multiplication does not affect the result. For example:

$3 + (4 + 5) = (3 + 4) + 5 = 3 + 4 + 5 = 12$

and

$3 \times (4 \times 5) = (3 \times 4) \times 5 = 3 \times 4 \times 5 = 60$

We say that addition and multiplication are associative operations

The way in which three or more integers are associated under subtraction or division does affect the result. For example:

$3 - (4 - 5) \neq (3 - 4) - 5$ because
$3 - (4 - 5) = 3 - (-1) = 3 + 1 = 4$ and $(3 - 4) - 5 = (-1) - 5 = -6$

Also

$24 \div (4 \div 2) \neq (24 \div 4) \div 2$ because
$24 \div (4 \div 2) = 24 \div 2 = 12$ and $(24 \div 4) \div 2 = 6 \div 2 = 3$

We say that subtraction and division are not associative operations ▶

3 Distributivity

Multiplication is distributed over addition and subtraction from both the left and the right. For example:

$$3 \times (4+5) = (3 \times 4) + (3 \times 5) = 27 \text{ and } (3+4) \times 5 = (3 \times 5) + (4 \times 5) = 35$$

$$3 \times (4-5) = (3 \times 4) - (3 \times 5) = -3 \text{ and } (3-4) \times 5 = (3 \times 5) - (4 \times 5) = -5$$

Division is distributed over addition and subtraction from the right but not from the left. For example:

$$(60+15) \div 5 = (60 \div 5) + (15 \div 5) \text{ because}$$

$$(60+15) \div 5 = 75 \div 5 = 15 \text{ and } (60 \div 5) + (15 \div 5) = 12+3 = 15$$

However, $60 \div (15+5) \neq (60 \div 15) + (60 \div 5)$ because

$$60 \div (15+5) = 60 \div 20 = 3 \text{ and } (60 \div 15) + (60 \div 5) = 4+12 = 16$$

Also:

$$(20-10) \div 5 = (20 \div 5) - (10 \div 5) \text{ because}$$

$$(20-10) \div 5 = 10 \div 5 = 2 \text{ and } (20 \div 5) - (10 \div 5) = 4-2 = 2$$

but $20 \div (10-5) \neq (20 \div 10) - (20 \div 5)$ because

$$20 \div (10-5) = 20 \div 5 = 4 \text{ and } (20 \div 10) - (20 \div 5) = 2-4 = -2$$

At this point let us pause and summarize the main facts so far

13

Revision summary

1 The integers consist of the positive and negative whole numbers and zero.

2 The integers are ordered so that they range from large negative to small negative through zero to small positive and then large positive. They are written using the ten numerals 0 to 9 according to the principle of place value where the place of a numeral in a number dictates the value it represents.

3 The integers can be represented by equally spaced points on a line.

4 The four arithmetic operations of addition, subtraction, multiplication and division obey specific precedence rules that govern the order in which they are to be executed:

In any calculation involving all four arithmetic operations we proceed as follows:

(a) *working from the left evaluate divisions and multiplications as they are encountered.*

This leaves an expression involving just addition and subtraction:

(b) *working from the left evaluate additions and subtractions as they are encountered.*

5 Brackets are used to group numbers and operations together. In any arithmetic expression, the contents of brackets are evaluated first.

Revision exercise

14

1 Place the appropriate symbol < or > between each of the following pairs of numbers:

(a) $-1 \quad -6$ (b) $5 \quad -29$ (c) $-14 \quad 7$

2 Find the value of each of the following:

(a) $16 - 12 \times 4 + 8 \div 2$

(b) $(16 - 12) \times (4 + 8) \div 2$

(c) $9 - 3(17 + 5[5 - 7])$

(d) $8(3[2 + 4] - 2[5 + 7])$

3 Show that:

(a) $6 - (3 - 2) \neq (6 - 3) - 2$

(b) $100 \div (10 \div 5) \neq (100 \div 10) \div 5$

(c) $24 \div (2 + 6) \neq (24 \div 2) + (24 \div 6)$

(d) $24 \div (2 - 6) \neq (24 \div 2) - (24 \div 6)$

15

1 (a) $-1 > -6$ because -1 is represented on the line to the right of -6

(b) $5 > -29$ because 5 is represented on the line to the right of -29

(c) $-14 < 7$ because -14 is represented on the line to the left of 7

2 (a) $16 - 12 \times 4 + 8 \div 2 = 16 - 48 + 4 = 16 - 44 = -28$
divide and multiply before adding and subtracting

(b) $(16 - 12) \times (4 + 8) \div 2 = (4) \times (12) \div 2 = 4 \times 12 \div 2 = 4 \times 6 = 24$
brackets are evaluated first

(c)
$$\begin{aligned} 9 - 3(17 + 5[5 - 7]) &= 9 - 3(17 + 5[-2]) \\ &= 9 - 3(17 - 10) \\ &= 9 - 3(7) \\ &= 9 - 21 = -12 \end{aligned}$$

(d)
$$\begin{aligned} 8(3[2 + 4] - 2[5 + 7]) &= 8(3 \times 6 - 2 \times 12) \\ &= 8(18 - 24) \\ &= 8(-6) = -48 \end{aligned}$$

3 (a) Left-hand side (LHS) $= 6 - (3 - 2) = 6 - (1) = 5$
Right-hand side (RHS) $= (6 - 3) - 2 = (3) - 2 = 1 \neq$ LHS

(b) Left-hand side (LHS) $= 100 \div (10 \div 5) = 100 \div 2 = 50$
Right-hand side (RHS) $= (100 \div 10) \div 5 = 10 \div 5 = 2 \neq$ LHS

(c) Left-hand side (LHS) $= 24 \div (2 + 6) = 24 \div 8 = 3$
Right-hand side (RHS) $= (24 \div 2) + (24 \div 6) = 12 + 4 = 16 \neq$ LHS

(d) Left-hand side (LHS) $= 24 \div (2 - 6) = 24 \div (-4) = -6$
Right-hand side (RHS) $= (24 \div 2) - (24 \div 6) = 12 - 4 = 8 \neq$ LHS

So now on to Frame 16

Factors and prime numbers

16

Factors

Any pair of natural numbers are called *factors* of their product. For example, the numbers 3 and 6 are factors of 18 because $3 \times 6 = 18$. These are not the only factors of 18. The complete collection of factors of 18 is 1, 2, 3, 6, 9, 18 because

$$\begin{aligned} 18 &= 1 \times 18 \\ &= 2 \times 9 \\ &= 3 \times 6 \end{aligned}$$

So the factors of:

(a) 12
(b) 25
(c) 17 are ....

The results are in the next frame

17

(a) 1, 2, 3, 4, 6, 12
(b) 1, 5, 25
(c) 1, 17

Because

(a) $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$
(b) $25 = 1 \times 25 = 5 \times 5$
(c) $17 = 1 \times 17$

Now move to the next frame

18

Prime numbers

If a natural number has only two factors which are itself and the number 1, the number is called a *prime number*. The first six prime numbers are 2, 3, 5, 7, 11 and 13. The number 1 is *not* a prime number because it only has one factor, namely, itself.

▶

Prime factorization

Every natural number can be written as a product involving only prime factors. For example, the number 126 has the factors 1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63 and 126, of which 2, 3 and 7 are prime numbers and 126 can be written as:

$126 = 2 \times 3 \times 3 \times 7$

To obtain this *prime factorization* the number is divided by successively increasing prime numbers thus:

$$\begin{array}{r|r}
2 & 126 \\ \hline
3 & 63 \\ \hline
3 & 21 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array} \quad \text{so that } 126 = 2 \times 3 \times 3 \times 7$$

Notice that a prime factor may occur more than once in a prime factorization.

Now find the prime factorization of:

(a) 84
(b) 512

Work these two out and check the working in Frame 19

19

> (a) $84 = 2 \times 2 \times 3 \times 7$
> (b) $512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Because

(a)
$$\begin{array}{r|r}
2 & 84 \\ \hline
2 & 42 \\ \hline
3 & 21 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array} \quad \text{so that } 84 = 2 \times 2 \times 3 \times 7$$

(b) The only prime factor of 512 is 2 which occurs 9 times. The prime factorization is:

$512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Move to Frame 20

▶

20

Highest common factor (HCF)

The *highest common factor* (HCF) of two natural numbers is the largest factor that they have in common. For example, the prime factorizations of 144 and 66 are:

$$\begin{aligned} 144 &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \\ 66 &= 2 \qquad\qquad\qquad \times 3 \quad \times 11 \end{aligned}$$

Only the 2 and the 3 are common to both factorizations and so the highest factor that these two numbers have in common (HCF) is $2 \times 3 = 6$.

Lowest common multiple (LCM)

The smallest natural number that each one of a pair of natural numbers divides into a whole number of times is called their *lowest common multiple* (LCM). This is also found from the prime factorization of each of the two numbers. For example:

$$\begin{aligned} 144 &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \\ 66 &= 2 \qquad\qquad\qquad \times 3 \quad \times 11 \\ \text{LCM} &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 11 = 1584 \end{aligned}$$

The HCF and LCM of 84 and 512 are ....

21

$$\boxed{\begin{array}{l} \text{HCF: } 4 \\ \text{LCM: } 10\,752 \end{array}}$$

Because

84 and 512 have the prime factorizations:

$$\begin{aligned} 84 &= 2 \times 2 \qquad\qquad\qquad\qquad\qquad\qquad \times 3 \times 7 \\ 512 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \qquad\qquad \text{HCF} = 2 \times 2 = 4 \\ \text{LCM} &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 7 = 10\,752 \end{aligned}$$

On now to Frame 22

22

Estimating

Arithmetic calculations are easily performed using a calculator. However, by pressing a wrong key, wrong answers can just as easily be produced. Every calculation made using a calculator should at least be checked for the reasonableness of the final result and this can be done by *estimating* the result using *rounding*. For example, using a calculator the sum $39 + 53$ is incorrectly found to be 62 if $39 + 23$ is entered by mistake. If, now, 39 is rounded up to 40, and 53 is rounded down to 50 the reasonableness of the calculator result can be simply checked by adding 40 to 50 to give 90. This indicates that the answer 62 is wrong and that the calculation should be done again. The correct answer 92 is then seen to be close to the approximation of 90. ▶

Rounding

An integer can be rounded to the nearest 10 as follows:

If the number is less than halfway to the next multiple of 10 then the number is rounded *down* to the previous multiple of 10. For example, 53 is rounded down to 50.

If the number is more than halfway to the next multiple of 10 then the number is rounded *up* to the next multiple of 10. For example, 39 is rounded up to 40.

If the number is exactly halfway to the next multiple of 10 then the number is rounded *up or down* to the next even multiple of 10. For example, 35 is rounded up to 40 but 65 is rounded down to 60.

This principle also applies when rounding to the nearest 100, 1000, 10 000 or more. For example, 349 rounds up to 350 to the nearest 10 but rounds down to 300 to the nearest 100, and 2501 rounds up to 3000 to the nearest 1000. Notice that 2500 to the nearest 1000 rounds down to 2000 – it being the nearest even thousand.

Try rounding each of the following to the nearest 10, 100 and 1000 respectively:

(a) 1846
(b) -638
(c) 445

Finish all three and check your results with the next frame

23

(a) 1850, 1800, 2000
(b) -640, -600, -1000
(c) 440, 400, 0

Because

(a) 1846 is nearer to 1850 than to 1840, nearer to 1800 than to 1900 and nearer to 2000 than to 1000.

(b) -638 is nearer to -640 than to -630, nearer to -600 than to -700 and nearer to -1000 than to 0. The negative sign does not introduce any complications.

(c) 445 rounds to 440 because it is the nearest even multiple of 10, 445 is nearer to 400 than to 500 and nearer to 0 than 1000.

How about estimating each of the following using rounding to the nearest 10:

(a) $18 \times 21 - 19 \div 11$
(b) $99 \div 101 - 49 \times 8$

Check your results in Frame 24

24

(a) 398
(b) −499

Because

(a) $18 \times 21 - 19 \div 11$ rounds to $20 \times 20 - 20 \div 10 = 398$

(b) $99 \div 101 - 49 \times 11$ rounds to $100 \div 100 - 50 \times 10 = -499$

At this point let us pause and summarize the main facts on factors, prime numbers and estimating

25

Revision summary

1. A pair of natural numbers are called factors of their product.
2. If a natural number only has one and itself as factors it is called a prime number.
3. Every natural number can be written as a product of its prime factors, some of which may be repeated.
4. The highest common factor (HCF) is the highest factor that two natural numbers have in common.
5. The lowest common multiple (LCM) is the lowest natural number that two natural numbers will divide into a whole number of times.
6. Integers can be rounded to the nearest 10, 100 etc. and the rounded values used as estimates for the result of a calculation.

26

Revision exercise

1. Write each of the following as a product of prime factors:

   (a) 429 (b) 1820 (c) 2992 (d) 3185
2. Find the HCF and the LCM of each pair of numbers:

   (a) 63, 42 (b) 92, 34
3. Round each number to the nearest 10, 100 and 1000:

   (a) 2562 (b) 1500 (c) −3451 (d) −14 525

27

1 (a)

$$\begin{array}{r|r} 3 & 429 \\ \hline 11 & 143 \\ \hline 13 & 13 \\ \hline & 1 \end{array}$$

$429 = 3 \times 11 \times 13$

(b)
$$\begin{array}{r|r} 2 & 1820 \\ \hline 2 & 910 \\ \hline 5 & 455 \\ \hline 7 & 91 \\ \hline 13 & 13 \\ \hline & 1 \\ \hline \end{array} \qquad 1820 = 2 \times 2 \times 5 \times 7 \times 13$$

(c)
$$\begin{array}{r|r} 2 & 2992 \\ \hline 2 & 1496 \\ \hline 2 & 748 \\ \hline 2 & 374 \\ \hline 11 & 187 \\ \hline 17 & 17 \\ \hline & 1 \\ \hline \end{array} \qquad 2992 = 2 \times 2 \times 2 \times 2 \times 11 \times 17$$

(d)
$$\begin{array}{r|r} 5 & 3185 \\ \hline 7 & 637 \\ \hline 7 & 91 \\ \hline 13 & 13 \\ \hline & 1 \\ \hline \end{array} \qquad 3185 = 5 \times 7 \times 7 \times 13$$

2 (a) The prime factorizations of 63 and 42 are:

$$\begin{aligned} 63 &= \phantom{2 \times {}} 3 \times 3 \times 7 \\ 42 &= 2 \times 3 \phantom{{} \times 3} \times 7 \qquad \text{HCF } 3 \times 7 = 21 \\ \text{LCM} &= 2 \times 3 \times 3 \times 7 = 126 \end{aligned}$$

(b) The prime factorizations of 34 and 92 are:

$$\begin{aligned} 34 &= 2 \phantom{{} \times 2} \times 17 \\ 92 &= 2 \times 2 \phantom{{} \times 17} \times 23 \qquad \text{HCF } 2 \\ \text{LCM} &= 2 \times 2 \times 17 \times 23 = 1564 \end{aligned}$$

3 (a) 2560, 2600, 3000

(b) 1500, 1500, 2000

(c) −3450, −3500, −3000

(d) −14 520, −14 500, −15 000

Now on to the next topic

Fractions

28

Division of integers

A fraction is a number which is represented by one integer – the *numerator* – divided by another integer – the *denominator* (or the *divisor*). For example, $\frac{3}{5}$ is a fraction with numerator 3 and denominator 5. Because fractions are written as one integer divided by another – a *ratio* – they are called *rational* numbers. Fractions are either *proper, improper* or *mixed*:

- in a proper fraction the numerator is less than the denominator, for example $\dfrac{4}{7}$
- in an improper fraction the numerator is greater than the denominator, for example $\dfrac{12}{5}$
- a mixed fraction is in the form of an integer and a fraction, for example $6\frac{2}{3}$

So that $-\dfrac{8}{11}$ is a .... fraction?

The answer is in the next frame

29

$$\boxed{\text{Proper}}$$

Fractions can be either positive or negative.

Now to the next frame

30

Multiplying fractions

Two fractions are multiplied by multiplying their respective numerators and denominators independently. For example:

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Try this one for yourself. $\dfrac{5}{9} \times \dfrac{2}{7} =$ ....

31

$$\boxed{\frac{10}{63}}$$

Because

$$\frac{5}{9} \times \frac{2}{7} = \frac{5 \times 2}{9 \times 7} = \frac{10}{63}$$

Correct? Then on to the next frame

Of

32

The word 'of' when interposed between two fractions means multiply. For example:

Half of half a cake is one-quarter of a cake. That is $\frac{1}{2}$ *of* $\frac{1}{2}=\frac{1}{2}\times\frac{1}{2}=\frac{1\times 1}{2\times 2}=\frac{1}{4}$

So that, for example:

$\frac{1}{3}$ *of* $\frac{2}{5}=\frac{1}{3}\times\frac{2}{5}=\frac{1\times 2}{3\times 5}=\frac{2}{15}$

So that $\frac{3}{8}$ *of* $\frac{5}{7}=$ ....

33

$$\boxed{\frac{15}{56}}$$

Because

$\frac{3}{8}$ *of* $\frac{5}{7}=\frac{3}{8}\times\frac{5}{7}=\frac{3\times 5}{8\times 7}=\frac{15}{56}$

On now to the next frame

Equivalent fractions

34

Multiplying the numerator and denominator by the same number is equivalent to multiplying the fraction by unity, that is by 1:

$\frac{4\times 3}{5\times 3}=\frac{4}{5}\times\frac{3}{3}=\frac{4}{5}\times 1=\frac{4}{5}$

Now, $\frac{4\times 3}{5\times 3}=\frac{12}{15}$ so that the fraction $\frac{4}{5}$ and the fraction $\frac{12}{15}$ *both represent the same number* and for this reason we call $\frac{4}{5}$ and $\frac{12}{15}$ *equivalent fractions*.

A second fraction, equivalent to a first fraction, can be found by multiplying the numerator and the denominator of the first fraction by the same number.

So that if we multiply the numerator and denominator of the fraction $\frac{7}{5}$ by 4 we obtain the equivalent fraction ....

Check your result in Frame 35

35

$$\boxed{\frac{28}{20}}$$

Because

$$\frac{7 \times 4}{5 \times 4} = \frac{28}{20}$$

We can reverse this process and find the equivalent fraction that has the smallest numerator by *cancelling out* common factors. This is known as reducing the fraction to its *lowest terms*. For example:

$\frac{16}{96}$ can be reduced to its lower terms as follows:

$$\frac{16}{96} = \frac{4 \times 4}{24 \times 4} = \frac{4 \times \cancel{4}}{24 \times \cancel{4}} = \frac{4}{24}$$

by cancelling out the 4 in the numerator and the denominator

The fraction $\frac{4}{24}$ can also be reduced:

$$\frac{4}{24} = \frac{4}{6 \times 4} = \frac{\cancel{4}}{6 \times \cancel{4}} = \frac{1}{6}$$

Because $\frac{1}{6}$ cannot be reduced further we see that $\frac{16}{96}$ reduced to its lowest terms is $\frac{1}{6}$.

How about this one? The fraction $\frac{84}{108}$ reduced to its lowest terms is ....

Check with the next frame

36

$$\boxed{\frac{7}{9}}$$

Because

$$\frac{84}{108} = \frac{7 \times 3 \times 4}{9 \times 3 \times 4} = \frac{7 \times \cancel{3} \times \cancel{4}}{9 \times \cancel{3} \times \cancel{4}} = \frac{7}{9}$$

Now move on to the next frame

37

Dividing fractions

The expression $6 \div 3$ means the number of 3's in 6, which is 2. Similarly, the expression $1 \div \frac{1}{4}$ means the number of $\frac{1}{4}$'s in 1, which is, of course, 4. That is:

$1 \div \frac{1}{4} = 4 = 1 \times \frac{4}{1}$. Notice how the numerator and the denominator of the divisor are switched and the division replaced by multiplication.

Two fractions are divided by switching the numerator and the denominator of the divisor and multiplying the result. For example:

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

So that $\frac{7}{13} \div \frac{3}{4} =$ ....

38

$$\boxed{\frac{28}{39}}$$

Because

$$\frac{7}{13} \div \frac{3}{4} = \frac{7}{13} \times \frac{4}{3} = \frac{28}{39}$$

In particular:

$$1 \div \frac{3}{5} = 1 \times \frac{5}{3} = \frac{5}{3}$$

The fraction $\frac{5}{3}$ is called the *reciprocal* of $\frac{3}{5}$

So that the reciprocal of $\frac{17}{4}$ is ....

39

$$\boxed{\frac{4}{17}}$$

Because

$$1 \div \frac{17}{4} = 1 \times \frac{4}{17} = \frac{4}{17}$$

And the reciprocal of -5 is ....

40

$$\boxed{-\frac{1}{5}}$$

Because

$$1 \div (-5) = 1 \div \left(-\frac{5}{1}\right) = 1 \times \left(-\frac{1}{5}\right) = -\frac{1}{5}$$

Move on to the next frame

41

Adding and subtracting fractions

Two fractions can only be added or subtracted immediately if they both possess the same denominator, in which case we add or subtract the numerators and divide by the common denominator. For example:

$$\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$$

If they do not have the same denominator they must be rewritten in equivalent form so that they do have the same denominator – called the *common denominator*. For example:

$$\frac{2}{3}+\frac{1}{5}=\frac{10}{15}+\frac{3}{15}=\frac{10+3}{15}=\frac{13}{15}$$

The common denominator of the equivalent fractions is the LCM of the two original denominators. That is:

$$\frac{2}{3}+\frac{1}{5}=\frac{2\times 5}{3\times 5}+\frac{1\times 3}{5\times 3}=\frac{10}{15}+\frac{3}{15} \text{ where 15 is the LCM of 3 and 5}$$

So that $\frac{5}{9}+\frac{1}{6}=$ ....

The result is in Frame 42

42

$$\boxed{\frac{13}{18}}$$

Because

$$\text{The LCM of 9 and 6 is 18 so that } \frac{5}{9}+\frac{1}{6}=\frac{5\times 2}{9\times 2}+\frac{1\times 3}{6\times 3}=\frac{10}{18}+\frac{3}{18}$$
$$=\frac{10+3}{18}=\frac{13}{18}$$

There's another one to try in the next frame

43

Now try $\frac{11}{15}-\frac{2}{3}=$ ....

44

$$\boxed{\frac{1}{15}}$$

Because

$$\frac{11}{15}-\frac{2}{3}=\frac{11}{15}-\frac{2\times 5}{3\times 5}=\frac{11}{15}-\frac{10}{15}$$
$$=\frac{11-10}{15}=\frac{1}{15} \qquad \text{(15 is the LCM of 3 and 15)}$$

Correct? Then on to Frame 45

45

Fractions on a calculator

The $a^b\!/_c$ button on a calculator enables fractions to be entered and manipulated with the results given in fractional form. For example, to evaluate $\frac{2}{3}\times 1\frac{3}{4}$ using your calculator [*note*: your calculator may not produce the identical display in what follows]:

Enter the number 2
Press the $a^b\!/_c$ key
Enter the number 3
The display now reads 2 ⌟ 3 to represent $\frac{2}{3}$
Press the × key
Enter the number 1
Press the $a^b\!/_c$ key
Enter the number 3
Press the $a^b\!/_c$ key
Enter the number 4
The display now reads 1 ⌟ 3 ⌟ 4 to represent $1\frac{3}{4}$
Press the = key to display the result 1 ⌟ 1 ⌟ 6 $= 1\frac{1}{6}$, that is:

$$\tfrac{2}{3}\times 1\tfrac{3}{4}=\frac{2}{3}\times\frac{7}{4}=\frac{14}{12}=1\tfrac{1}{6}$$

Now use your calculator to evaluate each of the following:

(a) $\frac{5}{7}+3\frac{2}{3}$

(b) $\dfrac{8}{3}-\dfrac{5}{11}$

(c) $\dfrac{13}{5}\times\dfrac{4}{7}-\dfrac{2}{9}$

(d) $4\frac{1}{11}\div\left(-\dfrac{3}{5}\right)+\dfrac{1}{8}$

Check your answers in Frame 46

46

(a) $4 \lrcorner 8 \lrcorner 21 = 4\frac{8}{21}$
(b) $2 \lrcorner 7 \lrcorner 33 = 2\frac{7}{33}$
(c) $1 \lrcorner 83 \lrcorner 315 = 1\frac{83}{315}$
(d) $-6 \lrcorner 61 \lrcorner 88 = -6\frac{61}{88}$

In (d) enter the $\frac{3}{5}$ and then press the +/− key.

On now to the next frame

47 **Ratios**

If a whole number is separated into a number of fractional parts where each fraction has the same denominator, the numerators of the fractions form a *ratio*. For example, if a quantity of brine in a tank contains $\frac{1}{3}$ salt and $\frac{2}{3}$ water, the salt and water are said to be in the ratio 'one-to-two' – written $1:2$.

What ratio do the components A, B and C form if a compound contains $\frac{3}{4}$ of A, $\frac{1}{6}$ of B and $\frac{1}{12}$ of C?

Take care here and check your results with Frame 48

48

$$\boxed{9:2:1}$$

Because the LCM of the denominators 4, 6 and 12 is 12, then:

$\frac{3}{4}$ of A is $\frac{9}{12}$ of A, $\frac{1}{6}$ of B is $\frac{2}{12}$ of B and the remaining $\frac{1}{12}$ is of C. This ensures that the components are in the ratio of their numerators. That is:

$9:2:1$

Notice that the sum of the numbers in the ratio is the common denominator.

On now to the next frame

49 **Percentages**

A percentage is a fraction whose denominator is equal to 100. For example, if 5 out of 100 people are left-handed then the fraction of left-handers is $\frac{5}{100}$ which is written as 5%, that is 5 *per cent* (%).

So if 13 out of 100 cars on an assembly line are red, the percentage of red cars on the line is . . . .

50

$\boxed{13\%}$

Because

The fraction of cars that are red is $\frac{13}{100}$ which is written as 13%.

Try this. What is the percentage of defective resistors in a batch of 25 if 12 of them are defective?

51

$\boxed{48\%}$

Because

The fraction of defective resistors is $\frac{12}{25}=\frac{12\times 4}{25\times 4}=\frac{48}{100}$ which is written as 48%. Notice that this is the same as:

$$\left(\frac{12}{25}\times 100\right)\%=\left(\frac{12}{25}\times 25\times 4\right)\%=(12\times 4)\%=48\%$$

A fraction can be converted to a percentage by multiplying the fraction by 100.

To find the percentage part of a quantity we multiply the quantity by the percentage written as a fraction. For example, 24% of 75 is:

$$\begin{aligned}24\%\ \textit{of}\ 75&=\frac{24}{100}\ \textit{of}\ 75=\frac{24}{100}\times 75=\frac{6\times 4}{25\times 4}\times 25\times 3=\frac{6\times\cancel{4}}{\cancel{25}\times\cancel{4}}\times\cancel{25}\times 3\\&=6\times 3=18\end{aligned}$$

So that 8% of 25 is ....

Work it through and check your results with the next frame

52

$\boxed{2}$

Because

$$\frac{8}{100}\times 25=\frac{2\times 4}{25\times 4}\times 25=\frac{2\times\cancel{4}}{\cancel{25}\times\cancel{4}}\times\cancel{25}=2$$

At this point let us pause and summarize the main facts on fractions, ratios and percentages

53

Revision summary

1 A fraction is a number represented as one integer (the numerator) divided by another integer (the denominator or divisor).

2 The same number can be represented by different but equivalent fractions.

3 A fraction with no common factors other than unity in its numerator and denominator is said to be in its lowest terms.

4 Two fractions are multiplied by multiplying the numerators and denominators independently.

5 Two fractions can only be added or subtracted immediately when their denominators are equal.

6 A ratio consists of the numerators of fractions with identical denominators.

7 The numerator of a fraction whose denominator is 100 is called a percentage.

54

Revision exercise

1 Reduce each of the following fractions to their lowest terms:

(a) $\dfrac{24}{30}$ (b) $\dfrac{72}{15}$ (c) $-\dfrac{52}{65}$ (d) $\dfrac{32}{8}$

2 Evaluate the following:

(a) $\dfrac{5}{9}\times\dfrac{2}{5}$ (b) $\dfrac{13}{25}\div\dfrac{2}{15}$ (c) $\dfrac{5}{9}+\dfrac{3}{14}$ (d) $\dfrac{3}{8}-\dfrac{2}{5}$

(e) $\dfrac{12}{7}\times\left(-\dfrac{3}{5}\right)$ (f) $\left(-\dfrac{3}{4}\right)\div\left(-\dfrac{12}{7}\right)$ (g) $\dfrac{19}{2}+\dfrac{7}{4}$ (h) $\dfrac{1}{4}-\dfrac{3}{8}$

3 Write the following proportions as ratios:

(a) $\dfrac{1}{2}$ of A, $\dfrac{2}{5}$ of B and $\dfrac{1}{10}$ of C

(b) $\dfrac{1}{3}$ of P, $\dfrac{1}{5}$ of Q, $\dfrac{1}{4}$ of R and the remainder S

4 Complete the following:

(a) $\dfrac{2}{5}=$ % (b) 58% of 25 =

(c) $\dfrac{7}{12}=$ % (d) 17% of 50 =

55

1 (a) $\frac{24}{30}=\frac{2\times2\times2\times3}{2\times3\times5}=\frac{2\times2}{5}=\frac{4}{5}$

(b) $\frac{72}{15}=\frac{2\times2\times2\times3\times3}{3\times5}=\frac{2\times2\times2\times3}{5}=\frac{24}{5}$

(c) $-\frac{52}{65}=-\frac{2\times2\times13}{5\times13}=-\frac{2\times2}{5}=-\frac{4}{5}$

(d) $\frac{32}{8}=\frac{2\times2\times2\times2\times2}{2\times2\times2}=4$

2 (a) $\frac{5}{9}\times\frac{2}{5}=\frac{5\times2}{9\times5}=\frac{2}{9}$

(b) $\frac{13}{25}\div\frac{2}{15}=\frac{13}{25}\times\frac{15}{2}=\frac{13\times15}{25\times2}=\frac{13\times3\times5}{5\times5\times2}=\frac{39}{10}$

(c) $\frac{5}{9}+\frac{3}{14}=\frac{5\times14}{9\times14}+\frac{3\times9}{14\times9}=\frac{70}{126}+\frac{27}{126}=\frac{70+27}{126}=\frac{97}{126}$

(d) $\frac{3}{8}-\frac{2}{5}=\frac{3\times5}{8\times5}-\frac{2\times8}{5\times8}=\frac{15}{40}-\frac{16}{40}=\frac{15-16}{40}=-\frac{1}{40}$

(e) $\frac{12}{7}\times\left(-\frac{3}{5}\right)=-\frac{12\times3}{7\times5}=-\frac{36}{35}$

(f) $\left(-\frac{3}{4}\right)\div\left(-\frac{12}{7}\right)=\frac{3}{4}\times\frac{7}{12}=\frac{3\times7}{4\times3\times4}=\frac{7}{16}$

(g) $\frac{19}{2}+\frac{7}{4}=\frac{38}{4}+\frac{7}{4}=\frac{45}{4}$

(h) $\frac{1}{4}-\frac{3}{8}=\frac{2}{8}-\frac{3}{8}=-\frac{1}{8}$

3 (a) $\frac{1}{2}, \frac{2}{5}, \frac{1}{10}=\frac{5}{10}, \frac{4}{10}, \frac{1}{10}$ so ratio is 5 : 4 : 1

(b) $\frac{1}{3}, \frac{1}{5}, \frac{1}{4}=\frac{20}{60}, \frac{12}{60}, \frac{15}{60}$ and $\frac{20}{60}+\frac{12}{60}+\frac{15}{60}=\frac{47}{60}$

so the fraction of S is $\frac{13}{60}$

so P, Q , R and S are in the ratio 20 : 12 : 15 : 13

4 (a) $\frac{2}{5}=\frac{2\times20}{5\times20}=\frac{40}{100}$ that is 40%

(b) $\frac{58}{100}\times25=\frac{58}{4}=\frac{29}{2}=14\tfrac{1}{2}$

(c) $\frac{7}{12}=\frac{7}{12}\times\frac{100}{1}\%=\frac{700}{12}\%=\frac{58\times12+4}{12}\%=58\tfrac{4}{12}\%=58\tfrac{1}{3}\%$

(d) $\frac{17}{100}\times50=\frac{17}{2}=8\tfrac{1}{2}$

Now let's look at decimal numbers

Decimal numbers

56 Division of integers

If one integer is divided by a second integer that is not one of the first integer's factors the result will not be another integer. Instead, the result will lie between two integers. For example, using a calculator it is seen that:

$25 \div 8 = 3{\cdot}125$

which is a number greater than 3 but less than 4. As with integers, the position of a numeral within the number indicates its value. Here the number 3·125 represents

3 units + 1 tenth + 2 hundredths + 5 thousandths.

That is $3 + \frac{1}{10} + \frac{2}{100} + \frac{5}{1000}$

where the decimal point shows the separation of the units from the tenths. Numbers written in this format are called *decimal numbers*.

On to the next frame

57 Rounding

All the operations of arithmetic that we have used with the integers apply to decimal numbers. However, when performing calculations involving decimal numbers it is common for the end result to be a number with a large quantity of numerals after the decimal point. For example:

$$15{\cdot}11 \div 8{\cdot}92 = 1{\cdot}6939461883\ldots$$

To make such numbers more manageable or more reasonable as the result of a calculation, they can be rounded either to a specified number of *significant figures* or to a specified number of *decimal places*.

Now to the next frame

58 Significant figures

Significant figures are counted from the first non-zero numeral encountered starting from the left of the number. When the required number of significant figures has been counted off, the remaining numerals are deleted with the following proviso:

If the first of a group of numerals to be deleted is a 5 or more, the last significant numeral is increased by 1. For example:

9·4534 to two significant figures is 9·5, to three significant figures is 9·45, and 0·001354 to two significant figures is 0·0014

Try this one for yourself. To four significant figures the number 18·7249 is ....

Check your result with the next frame

59

$\boxed{18{\cdot}72}$

Because

The first numeral deleted is a 4 which is less than 5.

There is one further proviso. If the only numeral to be dropped is a 5 then the last numeral retained is rounded to the next even numeral. So that 12·235 to four significant figures (abbreviated to *sig fig*) is 12·24 and 3·465 to three sig fig is 3·46.

So 8·1265 to four sig fig is ....

Check with the next frame

60

$\boxed{8{\cdot}126}$

Because

The only numeral deleted is a 5 and the last numeral retained is already even.

Now on to the next frame

61

Decimal places

Decimal places are counted to the right of the decimal point and the same rules as for significant figures apply for rounding to a specified number of decimal places (abbreviated to *dp*). For example:

123·4467 to one decimal place is 123·4 and to two dp is 123·45

So, 47·0235 to three dp is ....

62

$\boxed{47{\cdot}024}$

Because

The only numeral dropped is a 5 and the last numeral retained is odd so is increased to the next even numeral.

Now move on to the next frame

63

Trailing zeros

Sometimes zeros must be inserted within a number to satisfy a condition for a specified number of either significant figures or decimal places. For example:

12 645 to two significant figures is 13 000, and 13·1 to three decimal places is 13·100.

These zeros are referred to as *trailing zeros*.

So that 1515 to two sig fig is ....

64

$\boxed{1500}$

And:

25·13 to four dp is ....

65

$\boxed{25{\cdot}1300}$

On to the next frame

66

Fractions as decimals

Because a fraction is one integer divided by another it can be represented in decimal form simply by executing the division. For example:

$$\frac{7}{4} = 7 \div 4 = 1{\cdot}75$$

So that the decimal form of $\frac{3}{8}$ is ....

67

$\boxed{0{\cdot}375}$

Because

$$\frac{3}{8} = 3 \div 8 = 0{\cdot}375$$

Now move on to the next frame

68

Decimals as fractions

A decimal can be represented as a fraction. For example:

$$1{\cdot}224 = \frac{1224}{1000} \text{ which in lowest terms is } \frac{153}{125}$$

So that 0·52 as a fraction in lowest terms is ....

69

$\boxed{\dfrac{13}{25}}$

Because

$$0{\cdot}52 = \frac{52}{100} = \frac{13}{25}$$

Now move on to the next frame

70

Unending decimals

Converting a fraction into its decimal form by performing the division always results in an infinite string of numerals after the decimal point. This string of numerals may contain an infinite sequence of zeros or it may contain an infinitely repeated pattern of numerals. A repeated pattern of numerals can be written in an abbreviated format. For example:

$$\frac{1}{3} = 1 \div 3 = 0{\cdot}3333\ldots$$

Here the pattern after the decimal point is of an infinite number of 3's. We abbreviate this by placing a dot over the first 3 to indicate the repetition, thus:

$0{\cdot}3333\ldots = 0{\cdot}\dot{3}$ (described as zero point 3 recurring)

For other fractions the repetition may consist of a sequence of numerals, in which case a dot is placed over the first and last numeral in the sequence. For example:

$$\frac{1}{7} = 0{\cdot}142857142857142857\ldots = 0{\cdot}\dot{1}4285\dot{7}$$

So that we write $\dfrac{2}{11} = 0{\cdot}181818\ldots$ as ....

71

$$\boxed{0{\cdot}\dot{1}\dot{8}}$$

Sometimes the repeating pattern is formed by an infinite sequence of zeros, in which case we simply omit them. For example:

$\dfrac{1}{5} = 0{\cdot}20000\ldots$ is written as $0{\cdot}2$

Next frame

72

Unending decimals as fractions

Any decimal that displays an unending repeating pattern can be converted to its fractional form. For example:

To convert $0{\cdot}181818\ldots = 0{\cdot}\dot{1}\dot{8}$ to its fractional form we note that because there are two repeating numerals we multiply by 100 to give:

$100 \times 0{\cdot}\dot{1}\dot{8} = 18{\cdot}\dot{1}\dot{8}$

Subtracting $0{\cdot}\dot{1}\dot{8}$ from both sides of this equation gives:

$100 \times 0{\cdot}\dot{1}\dot{8} - 0{\cdot}\dot{1}\dot{8} = 18{\cdot}\dot{1}\dot{8} - 0{\cdot}\dot{1}\dot{8} = 18$

That is:

$99 \times 0{\cdot}\dot{1}\dot{8} = 18{\cdot}0$

This means that:

$0{\cdot}\dot{1}\dot{8} = \dfrac{18}{99} = \dfrac{2}{11}$

Similarly, the fractional form of $2{\cdot}0\dot{3}1\dot{5}$ is found as follows:

$2{\cdot}0\dot{3}1\dot{5} = 2{\cdot}0 + 0{\cdot}0\dot{3}1\dot{5}$ and, because there are three repeating numerals:

$1000 \times 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5\dot{3}1\dot{5}$

Subtracting $0{\cdot}0\dot{3}1\dot{5}$ from both sides of this equation gives:

$1000 \times 0{\cdot}0\dot{3}1\dot{5} - 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5\dot{3}1\dot{5} - 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5$

That is:

$999 \times 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5$ so that $0{\cdot}0\dot{3}1\dot{5} = \dfrac{31{\cdot}5}{999} = \dfrac{315}{9990}$

This means that:

$2{\cdot}0\dot{3}1\dot{5} = 2{\cdot}0 + 0{\cdot}0\dot{3}1\dot{5} = 2 + \dfrac{315}{9990} = 2\frac{35}{1110} = 2\frac{7}{222}$

What are the fractional forms of $0{\cdot}\dot{2}\dot{1}$ and $3{\cdot}2\dot{1}$?

The answers are in the next frame

73

$\boxed{\frac{7}{33} \text{ and } 3\frac{19}{90}}$

Because

$100 \times 0{\cdot}\dot{2}\dot{1} = 21{\cdot}\dot{2}\dot{1}$ so that $99 \times 0{\cdot}\dot{2}\dot{1} = 21$

giving $0{\cdot}\dot{2}\dot{1} = \dfrac{21}{99} = \dfrac{7}{33}$ and

$3{\cdot}2\dot{1} = 3{\cdot}2 + 0{\cdot}0\dot{1}$ and $10^1 \times 0{\cdot}0\dot{1} = 0{\cdot}1\dot{1}$ so that $9 \times 0{\cdot}0\dot{1} = 0{\cdot}1$ giving

$0{\cdot}0\dot{1} = \dfrac{0{\cdot}1}{9} = \dfrac{1}{90}$, hence $3\frac{1}{5} + \frac{1}{90} = 3\frac{19}{90}$

Rational, irrational and real numbers 74

A number that can be expressed as a fraction is called a *rational* number. An *irrational* number is one that *cannot* be expressed as a fraction and has a decimal form consisting of an infinite string of numerals that does not display a repeating pattern. As a consequence it is not possible either to write down the complete decimal form or to devise an abbreviated decimal format. Instead, we can only round them to a specified number of significant figures or decimal places. Alternatively, we may have a numeral representation for them such as, for example $\sqrt{2}$, e or π. The complete collection of rational and irrational numbers is called the collection of *real* numbers.

At this point let us pause and summarize the main facts so far on decimal numbers.

Revision summary 75

1. Every fraction can be written as a decimal number by performing the division.
2. The decimal number obtained will consist of an infinitely repeating pattern of numerals to the right of one of its digits.
3. Other decimals, with an infinite, non-repeating sequence of numerals after the decimal point are the irrational numbers.
4. A decimal number can be rounded to a specified number of significant figures (sig fig) by counting from the first non-zero numeral on the left.
5. A decimal number can be rounded to a specified number of decimal places (dp) by counting from the decimal point.

Revision exercise 76

1. Round each of the following decimal numbers, first to 3 significant figures and then to 2 decimal places:

   (a) 12·455 (b) 0·01356 (c) 0·1005 (d) 1344·555

2. Write each of the following in abbreviated form:

   (a) 12·110110110... (b) 0·123123123...
   (c) $-3{\cdot}11111$ (d) $-9360{\cdot}936093609360\ldots$

3. Convert each of the following to decimal form to 3 decimal places:

   (a) $\dfrac{3}{16}$ (b) $-\dfrac{5}{9}$
   (c) $\dfrac{7}{6}$ (d) $-\dfrac{24}{11}$

4. Convert each of the following to fractional form in lowest terms:

   (a) $0{\cdot}6$ (b) $1{\cdot}\dot{4}$
   (c) $1{\cdot}\dot{2}\dot{4}$ (d) $-7{\cdot}3$

77

1 (a) $12{\cdot}5$, $12{\cdot}46$

(b) $0{\cdot}0136$, $0{\cdot}01$

(c) $0{\cdot}100$, $0{\cdot}10$

(d) 1340, $1344{\cdot}56$

2 (a) $12{\cdot}\dot{1}1\dot{0}$

(b) $0{\cdot}\dot{1}2\dot{3}$

(c) $-3{\cdot}\dot{1}$

(d) $-9360{\cdot}\dot{9}36\dot{0}$

3 (a) $\dfrac{3}{16} = 0{\cdot}1875 = 0{\cdot}188$ to 3 dp

(b) $-\dfrac{5}{9} = -0{\cdot}555\ldots = -0{\cdot}556$ to 3 dp

(c) $\dfrac{7}{6} = 1{\cdot}1666\ldots = 1{\cdot}167$ to 3 dp

(d) $-\dfrac{24}{11} = -2{\cdot}1818\ldots = -2{\cdot}182$ to 3 dp

4 (a) $0{\cdot}6 = \dfrac{6}{10} = \dfrac{3}{5}$

(b) $1{\cdot}\dot{4} = 1 + \dfrac{4}{9} = \dfrac{13}{9}$

(c) $1{\cdot}\dot{2}\dot{4} = 1 + \dfrac{24}{99} = \dfrac{123}{99} = \dfrac{41}{33}$

(d) $-7{\cdot}3 = -\dfrac{73}{10}$

Now move on to the next topic

Powers

78

Raising a number to a power

The arithmetic operation of raising a number to a *power* is devised from repetitive multiplication. For example:

$10 \times 10 \times 10 \times 10 = 10^4$ – the number 10 multiplied by itself 4 times

The power is also called an *index* and the number to be raised to the power is called the *base*. Here the number 4 is the power (index) and 10 is the base.

So $5 \times 5 \times 5 \times 5 \times 5 \times 5 = \ldots\ldots\ldots\ldots$ (in the form of 5 raised to a power)

Compare your answer with the next frame

79

$\boxed{5^6}$

Because the number 5 (the base) is multiplied by itself 6 times (the power or index).

Now to the next frame

The laws of powers

80

The laws of powers are contained within the following set of rules:

- *Power unity*
  Any number raised to the power 1 equals itself.

  $3^1 = 3$

So $99^1 = \ldots\ldots\ldots\ldots$

On to the next frame

81

$\boxed{99}$

Because any number raised to the power 1 equals itself.

- *Multiplication of numbers and the addition of powers*
  If two numbers are each written as a given base raised to some power then the *product of the two numbers* is equal to the same base raised to the *sum of the powers*. For example, $16 = 2^4$ and $8 = 2^3$ so:

$$\begin{aligned}16 \times 8 &= 2^4 \times 2^3\\ &= (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2)\\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2\\ &= 2^7\\ &= 2^{4+3}\\ &= 128\end{aligned}$$

Multiplication requires powers to be added.

So $8^3 \times 8^5 = \ldots\ldots\ldots\ldots$ (in the form of 8 raised to a power)

Next frame

82

$\boxed{8^8}$

Because multiplication requires powers to be added.

Notice that we cannot combine different powers with different bases. For example:

$2^2 \times 4^3$ cannot be written as 8^5

but we can combine different bases to the same power. For example:

$3^4 \times 5^4$ can be written as 15^4 because

$$\begin{aligned} 3^4 \times 5^4 &= (3 \times 3 \times 3 \times 3) \times (5 \times 5 \times 5 \times 5) \\ &= 15 \times 15 \times 15 \times 15 \\ &= 15^4 \\ &= (3 \times 5)^4 \end{aligned}$$

So that $2^3 \times 4^3$ can be written as .... (in the form of a number raised to a power)

83

$\boxed{8^3}$

Next frame

84

- *Division of numbers and the subtraction of powers*
  If two numbers are each written as a given base raised to some power then the *quotient of the two numbers* is equal to the same base raised to the *difference of the powers*. For example:

$$\begin{aligned} 15\,625 \div 25 &= 5^6 \div 5^2 \\ &= (5 \times 5 \times 5 \times 5 \times 5 \times 5) \div (5 \times 5) \\ &= \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5} \\ &= 5 \times 5 \times 5 \times 5 \\ &= 5^4 \\ &= 5^{6-2} \\ &= 625 \end{aligned}$$

Division requires powers to be subtracted.

So $12^7 \div 12^3 =$ .... (in the form of 12 raised to a power)

Check your result in the next frame

85

$\boxed{12^4}$

Because division requires the powers to be subtracted.

- *Power zero*

  Any number raised to the power 0 equals unity. For example:

$$\begin{aligned} 1 &= 3^1 \div 3^1 \\ &= 3^{1-1} \\ &= 3^0 \end{aligned}$$

So $193^0 = \ldots\ldots\ldots\ldots$

86

$\boxed{1}$

Because any number raised to the power 0 equals unity.

- *Negative powers*

  A number raised to a negative power denotes the reciprocal. For example:

$$\begin{aligned} 6^{-2} &= 6^{0-2} \\ &= 6^0 \div 6^2 && \text{subtraction of powers means division} \\ &= 1 \div 6^2 && \text{because } 6^0 = 1 \\ &= \frac{1}{6^2} \end{aligned}$$

Also $6^{-1} = \dfrac{1}{6}$

A negative power denotes the reciprocal.

So $3^{-5} = \ldots\ldots\ldots\ldots$

87

$\boxed{\dfrac{1}{3^5}}$

Because

$$3^{-5} = 3^{0-5} = 3^0 \div 3^5 = \frac{1}{3^5}$$

A negative power denotes the reciprocal.

Now to the next frame

88

- *Multiplication of powers*

If a number is written as a given base raised to some power then that number *raised to a further power* is equal to the base raised to the *product of the powers*. For example:

$$\begin{aligned}(25)^3 &= (5^2)^3\\ &= 5^2 \times 5^2 \times 5^2\\ &= 5 \times 5 \times 5 \times 5 \times 5 \times 5\\ &= 5^6\\ &= 5^{2\times 3}\\ &= 15\,625\end{aligned}$$

Notice that $(5^2)^3 \neq 5^{2^3}$ because $5^{2^3} = 5^8 = 390\,625$.

Raising to a power requires powers to be multiplied.

So $(4^2)^4 =$ .... (in the form of 4 raised to a power)

89

$\boxed{4^8}$

Because raising to a power requires powers to be multiplied.

Now to the next frame.

90

Powers on a calculator

Powers on a calculator can be evaluated by using the x^y key. For example, enter the number 4, press the x^y key, enter the number 3 and press $=$. The result is 64 which is 4^3.

Try this one for yourself. To two decimal places, the value of $1{\cdot}3^{3\cdot 4}$ is ....

The result is in the following frame

91

$\boxed{2{\cdot}44}$

Because

Enter the number 1·3
Press the x^y key
Enter the number 3·4
Press the $=$ key

The number displayed is 2·44 to 2 dp.

Now try this one using the calculator:

$8^{\frac{1}{3}} =$ .... The 1/3 is a problem, use the $a^b\!/\!_c$ key.

Check your answer in the next frame

92

$\boxed{2}$

Because

Enter the number 8
Press the x^y key
Enter the number 1
Press the $a^b\!/_c$ key
Enter the number 3
Press = the number 2 is displayed.

Now move on to the next frame

93

Fractional powers and roots

We have just seen that $8^{\frac{1}{3}} = 2$. We call $8^{\frac{1}{3}}$ the *third root* or, alternatively, the *cube root* of 8 because:

$\left(8^{\frac{1}{3}}\right)^3 = 8$ *the number 8 is the result of raising the 3rd root of 8 to the power 3*

Roots are denoted by such fractional powers. For example, the 5th root of 6 is given as $6^{\frac{1}{5}}$ because:

$$\left(6^{\frac{1}{5}}\right)^5 = 6$$

and by using a calculator $6^{\frac{1}{5}}$ can be seen to be equal to 1·431 to 3 dp. Odd roots are unique in the real number system but even roots are not. For example, there are two 2nd roots – *square roots* – of 4, namely:

$$4^{\frac{1}{2}} = 2 \text{ and } 4^{\frac{1}{2}} = -2 \text{ because } 2 \times 2 = 4 \text{ and } (-2) \times (-2) = 4$$

Similarly:

$$81^{\frac{1}{4}} = \pm 3$$

Odd roots of negative numbers are themselves negative. For example:

$$(-32)^{\frac{1}{5}} = -2 \text{ because } \left[(-32)^{\frac{1}{5}}\right]^5 = (-2)^5 = -32$$

Even roots of negative numbers, however, pose a problem. For example, because

$$\left[(-1)^{\frac{1}{2}}\right]^2 = (-1)^1 = -1$$

we conclude that the square root of -1 is $(-1)^{\frac{1}{2}}$. Unfortunately, we cannot write this as a decimal number – we cannot find its decimal value because there is no decimal number which when multiplied by itself gives -1. We decide to accept the fact that, for now, we cannot find the even roots of a negative number. We shall return to this problem in a later programme when we introduce complex numbers.

▶

Surds

An alternative notation for the square root of 4 is the surd notation $\sqrt{4}$ and, by convention, this is always taken to mean the positive square root. This notation can also be extended to other roots, for example, $\sqrt[7]{9}$ is an alternative notation for $9^{\frac{1}{7}}$.

Use your calculator to find the value of each of the following roots to 3 dp:

(a) $16^{\frac{1}{7}}$ (b) $\sqrt{8}$ (c) $19^{\frac{1}{4}}$ (d) $\sqrt{-4}$

Answers in the next frame

94

(a) 1·486	use the $a^{b}/_{c}$ key
(b) 2·828	the positive value only
(c) $\pm 2{\cdot}088$	there are two values for even roots
(d) We cannot find the square root of a negative number	

On now to Frame 95

95

Multiplication and division by integer powers of 10

If a decimal number is multiplied by 10 raised to an integer power, the decimal point moves the integer number of places to the right if the integer is positive and to the left if the integer is negative. For example:

$1{\cdot}2345 \times 10^3 = 1234{\cdot}5$ (3 places to the right) and

$1{\cdot}2345 \times 10^{-2} = 0{\cdot}012345$ (2 places to the left).

Notice that, for example:

$1{\cdot}2345 \div 10^3 = 1{\cdot}2345 \times 10^{-3}$ and

$1{\cdot}2345 \div 10^{-2} = 1{\cdot}2345 \times 10^2$

So now try these:

(a) $0{\cdot}012045 \times 10^4$

(b) $13{\cdot}5074 \times 10^{-3}$

(c) $144{\cdot}032 \div 10^5$

(d) $0{\cdot}012045 \div 10^{-2}$

Work all four out and then check your results with the next frame

96

(a) 120·45
(b) 0·0135074
(c) 0·00144032
(d) 1·2045

Because

(a) multiplying by 10^4 moves the decimal point 4 places to the right
(b) multiplying by 10^{-3} moves the decimal point 3 places to the left
(c) $144{\cdot}032 \div 10^5 = 144{\cdot}032 \times 10^{-5}$ move the decimal point 5 places to the left
(d) $0{\cdot}012045 \div 10^{-2} = 0{\cdot}012045 \times 10^2$ move the decimal point 2 places to the right

Now move on to the next frame

Precedence rules

97

With the introduction of the arithmetic operation of raising to a power we need to amend our earlier precedence rules – *evaluating powers is performed before dividing and multiplying.* For example:

$$\begin{aligned} 5(3 \times 4^2 \div 6 - 7) &= 5(3 \times 16 \div 6 - 7) \\ &= 5(48 \div 6 - 7) \\ &= 5(8 - 7) \\ &= 5 \end{aligned}$$

So that:

$14 \div (125 \div 5^3 \times 4 + 3) = \dots\dots\dots\dots$

Check your result in the next frame

98

$\boxed{2}$

Because

$$\begin{aligned} 14 \div (125 \div 5^3 \times 4 + 3) &= 14 \div (125 \div 125 \times 4 + 3) \\ &= 14 \div (4 + 3) \\ &= 2 \end{aligned}$$

99

Standard form

Any decimal number can be written as a decimal number greater than or equal to 1 and less than 10 (called the *mantissa*) multiplied by the number 10 raised to an appropriate power (the power being called the *exponent*). For example:

$$57{\cdot}3 = 5{\cdot}73 \times 10^1$$
$$423{\cdot}8 = 4{\cdot}238 \times 10^2$$
$$6042{\cdot}3 = 6{\cdot}0423 \times 10^3$$

and

$$0{\cdot}267 = 2{\cdot}67 \div 10 = 2{\cdot}67 \times 10^{-1}$$
$$0{\cdot}000485 = 4{\cdot}85 \div 10^4 = 4{\cdot}85 \times 10^{-4} \text{ etc.}$$

So, written in standard form:

(a) $52\,674 = \ldots\ldots\ldots\ldots$ (c) $0{\cdot}0582 = \ldots\ldots\ldots\ldots$
(b) $0{\cdot}00723 = \ldots\ldots\ldots\ldots$ (d) $1\,523\,800 = \ldots\ldots\ldots\ldots$

100

(a) $5{\cdot}2674 \times 10^4$	(c) $5{\cdot}82 \times 10^{-2}$
(b) $7{\cdot}23 \times 10^{-3}$	(d) $1{\cdot}5238 \times 10^6$

Working in standard form

Numbers written in standard form can be multiplied or divided by multiplying or dividing the respective mantissas and adding or subtracting the respective exponents. For example:

$$\begin{aligned} 0{\cdot}84 \times 23\,000 &= (8{\cdot}4 \times 10^{-1}) \times (2{\cdot}3 \times 10^4) \\ &= (8{\cdot}4 \times 2{\cdot}3) \times 10^{-1} \times 10^4 \\ &= 19{\cdot}32 \times 10^3 \\ &= 1{\cdot}932 \times 10^4 \end{aligned}$$

Another example:

$$\begin{aligned} 175{\cdot}4 \div 6340 &= (1{\cdot}754 \times 10^2) \div (6{\cdot}34 \times 10^3) \\ &= (1{\cdot}754 \div 6{\cdot}34) \times 10^2 \div 10^3 \\ &= 0{\cdot}2767 \times 10^{-1} \\ &= 2{\cdot}767 \times 10^{-2} \text{ to 4 sig fig} \end{aligned}$$

Where the result obtained is not in standard form, the mantissa is written in standard number form and the necessary adjustment made to the exponent.

In the same way, then, giving the results in standard form to 4 dp:

(a) $472{\cdot}3 \times 0{\cdot}000564 = \ldots\ldots\ldots\ldots$
(b) $752\,000 \div 0{\cdot}862 = \ldots\ldots\ldots\ldots$

101

(a) $2{\cdot}6638 \times 10^{-1}$
(b) $8{\cdot}7239 \times 10^{5}$

Because

(a) $472{\cdot}3 \times 0{\cdot}000564 = (4{\cdot}723 \times 10^2) \times (5{\cdot}64 \times 10^{-4})$
$= (4{\cdot}723 \times 5{\cdot}64) \times 10^2 \times 10^{-4}$
$= 26{\cdot}638 \times 10^{-2} = 2{\cdot}6638 \times 10^{-1}$

(b) $752\,000 \div 0{\cdot}862 = (7{\cdot}52 \times 10^5) \div (8{\cdot}62 \times 10^{-1})$
$= (7{\cdot}52 \div 8{\cdot}62) \times 10^5 \times 10^1$
$= 0{\cdot}87239 \times 10^6 = 8{\cdot}7239 \times 10^5$

For *addition and subtraction in standard form* the approach is slightly different.

Example 1

$4{\cdot}72 \times 10^3 + 3{\cdot}648 \times 10^4$

Before these can be added, the powers of 10 must be made the same:

$4{\cdot}72 \times 10^3 + 3{\cdot}648 \times 10^4 = 4{\cdot}72 \times 10^3 + 36{\cdot}48 \times 10^3$
$= (4{\cdot}72 + 36{\cdot}48) \times 10^3$
$= 41{\cdot}2 \times 10^3 = 4{\cdot}12 \times 10^4$ in standard form

Similarly in the next example.

Example 2

$13{\cdot}26 \times 10^{-3} - 1{\cdot}13 \times 10^{-2}$

Here again, the powers of 10 must be equalized:

$13{\cdot}26 \times 10^{-3} - 1{\cdot}13 \times 10^{-2} = 1{\cdot}326 \times 10^{-2} - 1{\cdot}13 \times 10^{-2}$
$= (1{\cdot}326 - 1{\cdot}13) \times 10^{-2}$
$= 0{\cdot}196 \times 10^{-2} = 1{\cdot}96 \times 10^{-3}$ in standard form

▶

Using a calculator

Numbers given in standard form can be manipulated on a calculator by making use of the EXP key. For example, to enter the number $1{\cdot}234 \times 10^3$, enter 1·234 and then press the EXP key. The display then changes to:

$$1{\cdot}234 \quad 00$$

Now enter the power 3 and the display becomes:

$$1{\cdot}234 \quad 03$$

Manipulating numbers in this way produces a result that is in ordinary decimal format. If the answer is required in standard form then it will have to be converted by hand. For example, using the EXP key on a calculator to evaluate $(1{\cdot}234 \times 10^3) + (2{\cdot}6 \times 10^2)$ results in the display 1494 which is then converted by hand to $1{\cdot}494 \times 10^3$.

Therefore, working in standard form:

(a) $43{\cdot}6 \times 10^2 + 8{\cdot}12 \times 10^3 = \ldots\ldots\ldots\ldots$

(b) $7{\cdot}84 \times 10^5 - 12{\cdot}36 \times 10^3 = \ldots\ldots\ldots\ldots$

(c) $4{\cdot}25 \times 10^{-3} + 1{\cdot}74 \times 10^{-2} = \ldots\ldots\ldots\ldots$

102

(a) $1{\cdot}248 \times 10^4$
(b) $7{\cdot}7164 \times 10^5$
(c) $2{\cdot}165 \times 10^{-2}$

Preferred standard form

In the SI system of units, it is recommended that when a number is written in standard form, the power of 10 should be restricted to powers of 10^3, i.e. 10^3, 10^6, 10^{-3}, 10^{-6}, etc. Therefore in this *preferred standard form* up to three figures may appear in front of the decimal point.

In practice it is best to write the number first in standard form and to adjust the power of 10 to express this in preferred standard form.

Example 1

$5{\cdot}2746 \times 10^4$ in standard form

$= 5{\cdot}2746 \times 10 \times 10^3$

$= 52{\cdot}746 \times 10^3$ in preferred standard form

Example 2

$3{\cdot}472 \times 10^8$ in standard form

$= 3{\cdot}472 \times 10^2 \times 10^6$

$= 347{\cdot}2 \times 10^6$ in preferred standard form

▶

Example 3

$3{\cdot}684 \times 10^{-2}$ in standard form

$= 3{\cdot}684 \times 10 \times 10^{-3}$

$= 36{\cdot}84 \times 10^{-3}$ in preferred standard form

So, rewriting the following in preferred standard form, we have

(a) $8{\cdot}236 \times 10^{7} = \ldots\ldots\ldots\ldots$
(b) $1{\cdot}624 \times 10^{-4} = \ldots\ldots\ldots\ldots$
(c) $4{\cdot}827 \times 10^{4} = \ldots\ldots\ldots\ldots$
(d) $6{\cdot}243 \times 10^{5} = \ldots\ldots\ldots\ldots$
(e) $3{\cdot}274 \times 10^{-2} = \ldots\ldots\ldots\ldots$
(f) $5{\cdot}362 \times 10^{-7} = \ldots\ldots\ldots\ldots$

103

(a) $82{\cdot}36 \times 10^{6}$	(d) $624{\cdot}3 \times 10^{3}$
(b) $162{\cdot}4 \times 10^{-6}$	(e) $32{\cdot}74 \times 10^{-3}$
(c) $48{\cdot}27 \times 10^{3}$	(f) $536{\cdot}2 \times 10^{-9}$

One final exercise on this piece of work:

Example 4

The product of $(4{\cdot}72 \times 10^{2})$ and $(8{\cdot}36 \times 10^{5})$

(a) in standard form $= \ldots\ldots\ldots\ldots$
(b) in preferred standard form $= \ldots\ldots\ldots\ldots$

104

(a) $3{\cdot}9459 \times 10^{8}$
(b) $394{\cdot}59 \times 10^{6}$

Because

(a) $(4{\cdot}72 \times 10^{2}) \times (8{\cdot}36 \times 10^{5}) = (4{\cdot}72 \times 8{\cdot}36) \times 10^{2} \times 10^{5}$
$= 39{\cdot}459 \times 10^{7}$
$= 3{\cdot}9459 \times 10^{8}$ in standard form

(b) $(4{\cdot}72 \times 10^{2}) \times (8{\cdot}36 \times 10^{5}) = 3{\cdot}9459 \times 10^{2} \times 10^{6}$
$= 394{\cdot}59 \times 10^{6}$ in preferred standard form

Now move on to the next frame

105

Checking calculations

When performing a calculation involving decimal numbers it is always a good idea to check that your result is reasonable and that an arithmetic blunder or an error in using the calculator has not been made. This can be done using standard form. For example:

$$\begin{aligned} 59{\cdot}2347 \times 289{\cdot}053 &= 5{\cdot}92347 \times 10^1 \times 2{\cdot}89053 \times 10^2 \\ &= 5{\cdot}92347 \times 2{\cdot}89053 \times 10^3 \end{aligned}$$

This product can then be estimated for reasonableness as:

$6 \times 3 \times 1000 = 18\,000$ (see Frames 22–24)

The answer using the calculator is 17 121·968 to three decimal places, which is 17 000 when rounded to the nearest 1000. This compares favourably with the estimated 18 000, indicating that the result obtained could be reasonably expected.

So, the estimated value of $800{\cdot}120 \times 0{\cdot}007953$ is ....

Check with the next frame

106

$\boxed{6{\cdot}4}$

Because

$$\begin{aligned} 800{\cdot}120 \times 0{\cdot}007953 &= 8{\cdot}00120 \times 10^2 \times 7{\cdot}953 \times 10^{-3} \\ &= 8{\cdot}00120 \times 7{\cdot}9533 \times 10^{-1} \end{aligned}$$

This product can then be estimated for reasonableness as:

$8 \times 8 \div 10 = 6{\cdot}4$

The exact answer is 6·36 to two decimal places.

Now move on to the next frame

Accuracy

107

Many calculations are made using numbers that have been obtained from measurements. Such numbers are only accurate to a given number of significant figures but using a calculator can produce a result that contains as many figures as its display will permit. Because any calculation involving measured values will not be accurate to *more significant figures than the least number of significant figures in any measurement*, we can justifiably round the result down to a more manageable number of significant figures. For example:

> The base length and height of a rectangle are measured as 114·8 mm and 18 mm respectively. The area of the rectangle is given as the product of these lengths. Using a calculator this product is 2066·4 mm^2. Because one of the lengths is only measured to 2 significant figures, the result cannot be accurate to more than 2 significant figures. It should therefore be read as 2100 mm^2.

Assuming the following contains numbers obtained by measurement, use a calculator to find the value to the correct level of accuracy:

$19{\cdot}1 \times 0{\cdot}0053 \div 13{\cdot}345$

108

$\boxed{0{\cdot}0076}$

Because

> The calculator gives the result as 0·00758561 but because 0·0053 is only accurate to 2 significant figures the result cannot be accurate to more than 2 significant figures, namely 0·0076.

At this point let us pause and summarize the main facts so far on powers

Revision summary

109

1. Powers are devised from repetitive multiplication of a given number.
2. Negative powers denote reciprocals and any number raised to the power 0 is unity.
3. Multiplication of a decimal number by 10 raised to an integer power moves the decimal point to the right if the power is positive and to the left if the power is negative.
4. A decimal number written in standard form is in the form of a mantissa (a number between 1 and 10 but excluding 10) multiplied by 10 raised to an integer power, the power being called the exponent.
5. Writing decimal numbers in standard form permits an estimation of the reasonableness of a calculation.
6. In preferred standard form the powers of 10 in the exponent are restricted to multiples of 3.
7. If numbers used in a calculation are obtained from measurement, the result of the calculation is a number accurate to no more than the least number of significant figures in any measurement.

110

Revision exercise

1 Write each of the following as a number raised to a power:

(a) $5^8 \times 5^2$ (b) $6^4 \div 6^6$ (c) $(7^4)^3$ (d) $(19^{-8})^0$

2 Find the value of each of the following to 3 dp:

(a) $16^{\frac{1}{4}}$ (b) $\sqrt[3]{3}$ (c) $(-8)^{\frac{1}{5}}$ (d) $(-7)^{\frac{1}{4}}$

3 Write each of the following as a single decimal number:

(a) $1{\cdot}0521 \times 10^3$ (b) $123{\cdot}456 \times 10^{-2}$

(c) $0{\cdot}0135 \div 10^{-3}$ (d) $165{\cdot}21 \div 10^4$

4 Write each of the following in standard form:

(a) 125·87 (b) 0·0101 (c) 1·345 (d) 10·13

5 Write each of the following in preferred standard form:

(a) $1{\cdot}3204 \times 10^5$ (b) 0·0101 (c) 1·345 (d) $9{\cdot}5032 \times 10^{-8}$

6 In each of the following the numbers have been obtained by measurement. Evaluate each calculation to the appropriate level of accuracy:

(a) $13{\cdot}6 \div 0{\cdot}012 \times 7{\cdot}63 - 9015$ (b) $\dfrac{0{\cdot}003 \times 194}{13{\cdot}6}$

(c) $19{\cdot}3 \times 1{\cdot}04^{2{\cdot}00}$ (d) $\dfrac{18 \times 2{\cdot}1 - 3{\cdot}6 \times 0{\cdot}54}{8{\cdot}6 \times 2{\cdot}9 + 5{\cdot}7 \times 9{\cdot}2}$

111

1 (a) $5^8 \times 5^2 = 5^{8+2} = 5^{10}$ (b) $6^4 \div 6^6 = 6^{4-6} = 6^{-2}$

(c) $(7^4)^3 = 7^{4\times 3} = 7^{12}$ (d) $(19^{-8})^0 = 1$ as any number raised to the power 0 equals unity

2 (a) $16^{\frac{1}{4}} = \pm 2{\cdot}000$ (b) $\sqrt[3]{3} = 1{\cdot}442$

(c) $(-8)^{\frac{1}{5}} = -1{\cdot}516$ You may not be able to use a calculator as the problem stands, so think about this one

(d) $(-7)^{\frac{1}{4}}$ You cannot find the even root of a negative number

3 (a) $1{\cdot}0521 \times 10^3 = 1052{\cdot}1$ (b) $123{\cdot}456 \times 10^{-2} = 1{\cdot}23456$

(c) $0{\cdot}0135 \div 10^{-3} = 0{\cdot}0135 \times 10^3 = 13{\cdot}5$

(d) $165{\cdot}21 \div 10^4 = 165{\cdot}21 \times 10^{-4} = 0{\cdot}016521$

4 (a) $125{\cdot}87 = 1{\cdot}2587 \times 10^2$ (b) $0{\cdot}0101 = 1{\cdot}01 \times 10^{-2}$

(c) $1{\cdot}345 = 1{\cdot}345 \times 10^0$ (d) $10{\cdot}13 = 1{\cdot}013 \times 10^1 = 1{\cdot}013 \times 10$

5 (a) $1{\cdot}3204 \times 10^5 = 132{\cdot}04 \times 10^3$ (b) $0{\cdot}0101 = 10{\cdot}1 \times 10^{-3}$

(c) $1{\cdot}345 = 1{\cdot}345 \times 10^0$ (d) $9{\cdot}5032 \times 10^{-8} = 95{\cdot}032 \times 10^{-9}$

6 (a) $13{\cdot}6 \div 0{\cdot}012 \times 7{\cdot}63 - 9015 = -367{\cdot}\dot{6} = -370$ to 2 sig fig

(b) $\dfrac{0{\cdot}003 \times 194}{13{\cdot}6} = 0{\cdot}042794\ldots = 0{\cdot}04$ to 1 sig fig

(c) $19{\cdot}3 \times 1{\cdot}04^{2{\cdot}00} = 19{\cdot}3 \times 1{\cdot}0816 = 20{\cdot}87488 = 20{\cdot}9$ to 2 sig fig

(d) $\dfrac{18 \times 2{\cdot}1 - 3{\cdot}6 \times 0{\cdot}54}{8{\cdot}6 \times 2{\cdot}9 + 5{\cdot}7 \times 9{\cdot}2} = \dfrac{35{\cdot}856}{77{\cdot}38} = 0{\cdot}46337554\ldots = 0{\cdot}463$ to 3 sig fig

On now to the next topic

Number systems

112

1 Denary (or decimal) system

This is our basic system in which quantities large or small can be represented by use of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 together with appropriate place values according to their positions.

For example	2	7	6	5	·	3	2	1_{10}
has place values	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
	1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

In this case, the place values are powers of 10, which gives the name *denary* (or *decimal*) to the system. The denary system is said to have a *base* of 10. You are, of course, perfectly familiar with this system of numbers, but it is included here as it leads on to other systems which have the same type of structure but which use different place values.

So let us move on to the next system

113

2 Binary system

This is widely used in all forms of switching applications. The only symbols used are 0 and 1 and the place values are powers of 2, i.e. the system has a base of 2.

For example	1	0	1	1	·	1	0	1_2
has place values	2^3	2^2	2^1	2^0		2^{-1}	2^{-2}	2^{-3}
i.e.	8	4	2	1		$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

So: 1 0 1 1 · 1 0 1 in the binary system

$= \quad 1\times 8 \quad 0\times 4 \quad 1\times 2 \quad 1\times 1 \quad 1\times\frac{1}{2} \quad 0\times\frac{1}{4} \quad 1\times\frac{1}{8}$

$= \quad 8 + 0 + 2 + 1 + \frac{1}{2} + 0 + \frac{1}{8}$ in denary

$= \quad 11\frac{5}{8} = 11{\cdot}625$ in the denary system. Therefore $1011{\cdot}101_2 = 11{\cdot}625_{10}$

The small subscripts 2 and 10 indicate the bases of the two systems. In the same way, the denary equivalent of $1\ 1\ 0\ 1 \cdot 0\ 1\ 1_2$ is ....

114

$\boxed{13{\cdot}375_{10}}$

Because

$$\begin{array}{lcccccccccccccc} & 1 & & 1 & & 0 & & 1 & \cdot & 0 & & 1 & & 1_2 \\ = & 8 & + & 4 & + & 0 & + & 1 & + & 0 & + & \frac{1}{4} & + & \frac{1}{8} \end{array}$$

$= 13\frac{3}{8} = 13{\cdot}375_{10}$

3 Octal system (base 8)

This system uses the symbols

0, 1, 2, 3, 4, 5, 6, 7

with place values that are powers of 8.

For example	3	5	7	·	3	2	1	in the octal system
has place values	8^2	8^1	8^0		8^{-1}	8^{-2}	8^{-3}	
i.e.	64	8	1		$\frac{1}{8}$	$\frac{1}{64}$	$\frac{1}{512}$	

$$\begin{array}{llllllllllll} \text{So} & 3 & & 5 & & 7 & \cdot & 3 & & 2 & & 1_8 \\ = & 3 \times 64 & & 5 \times 8 & & 7 \times 1 & & 3 \times \frac{1}{8} & & 2 \times \frac{1}{64} & & 1 \times \frac{1}{512} \\ = & 192 & + & 40 & + & 7 & + & \frac{3}{8} & + & \frac{1}{32} & + & \frac{1}{512} \\ = & 239\frac{209}{512} & & = 239{\cdot}4082_{10} \end{array}$$

That is

$357{\cdot}321_8 = 239{\cdot}408_{10}$

As you see, the method is very much as before: the only change is in the base of the place values.

In the same way then, $263{\cdot}452_8$ expressed in denary form is ....

115

$\boxed{179{\cdot}582_{10}}$

Because

$2\,6\,3 \cdot 4\,5\,2_8$

$= 2 \times 8^2 + 6 \times 8^1 + 3 \times 8^0 + 4 \times 8^{-1} + 5 \times 8^{-2} + 2 \times 8^{-3}$

$= 2 \times 64 + 6 \times 8 + 3 \times 1 + 4 \times \frac{1}{8} + 5 \times \frac{1}{64} + 2 \times \frac{1}{512}$

$= 128 + 48 + 3 + \frac{1}{2} + \frac{5}{64} + \frac{1}{256}$

$= 179\frac{149}{256} = 179{\cdot}582_{10}$

Now we come to the duodecimal system, which has a base of 12.

So move on to the next frame

4 Duodecimal system (base 12) 116

With a base of 12, the units column needs to accommodate symbols up to 11 before any carryover to the second column occurs. Unfortunately, our denary symbols go up to only 9, so we have to invent two extra symbols to represent the values 10 and 11. Several suggestions for these have been voiced in the past, but we will adopt the symbols X and Λ for 10 and 11 respectively. The first of these calls to mind the Roman numeral for 10 and the Λ symbol may be regarded as the two strokes of 11 tilted together 1 1 to join at the top.

The duodecimal system, therefore, uses the symbols

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X, Λ

and has place values that are powers of 12.

For example	2	X	5	·	1	3	6_{12}
has place values	12^2	12^1	12^0		12^{-1}	12^{-2}	12^{-3}
i.e.	144	12	1		$\frac{1}{12}$	$\frac{1}{144}$	$\frac{1}{1728}$

So 2 X 5 · 1 3 6_{12}

$$= 2\times 144 + 10\times 12 + 5\times 1 + 1\times\tfrac{1}{12} + 3\times\tfrac{1}{144} + 6\times\tfrac{1}{1728}$$

$= \ldots\ldots\ldots\ldots_{10}$

Finish it off

117

$$\boxed{413{\cdot}108_{10}}$$

Because

$$2\,X\,5\cdot 1\,3\,6_{12}$$
$$= 288 + 120 + 5 + \tfrac{1}{12} + \tfrac{1}{48} + \tfrac{1}{288} = 413\tfrac{31}{288}$$

Therefore $2X5{\cdot}136_{12} = 413{\cdot}108_{10}$

5 Hexadecimal system (base 16)

This system has computer applications. The symbols here need to go up to an equivalent denary value of 15, so, after 9, letters of the alphabet are used as follows:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

The place values in this system are powers of 16.

For example	2	A	7	·	3	E	2_{16}
has place values	16^2	16^1	16^0		16^{-1}	16^{-2}	16^{-3}
i.e.	256	16	1		$\frac{1}{16}$	$\frac{1}{256}$	$\frac{1}{4096}$

Therefore 2 A 7 · 3 E $2_{16} = \ldots\ldots\ldots\ldots$

118

$\boxed{679{\cdot}243_{10}}$

Here it is: 2 A 7 · 3 E 2_{16}

$$= 2 \times 256 + 10 \times 16 + 7 \times 1 + 3 \times \frac{1}{16} + 14 \times \frac{1}{256} + 2 \times \frac{1}{4096}$$
$$= 679\tfrac{497}{2048} = 679{\cdot}243_{10}$$

And now, two more by way of practice.

Express the following in denary form:

(a) 3 Λ 4 · 2 6 5_{12}
(b) 3 C 4 · 2 1 F_{16}

Finish both of them and check the working with the next frame

119

Here they are:

(a)

	3	Λ	4	·	2	6	5_{12}
Place values	144	12	1		$\frac{1}{12}$	$\frac{1}{144}$	$\frac{1}{1728}$

So 3 Λ 4 · 2 6 5_{12}

$$= 3 \times 144 + 11 \times 12 + 4 \times 1 + 2 \times \frac{1}{12} + 6 \times \frac{1}{144} + 5 \times \frac{1}{1728}$$
$$= 432 + 132 + 4 + \frac{1}{6} + \frac{1}{24} + \frac{5}{1728}$$
$$= 568\tfrac{365}{1728} \qquad = 568{\cdot}211_{10}$$

Therefore 3 Λ 4 · 265_{12} $= 568{\cdot}211_{10}$

(b)

	3	C	4	·	2	1	F_{16}
Place values	256	16	1		$\frac{1}{16}$	$\frac{1}{256}$	$\frac{1}{4096}$

So 3 C 4 · 2 1 F_{16}

$$= 3 \times 256 + 12 \times 16 + 4 \times 1 + 2 \times \frac{1}{16} + 1 \times \frac{1}{256} + 15 \times \frac{1}{4096}$$
$$= 768 + 192 + 4 + \frac{1}{8} + \frac{1}{256} + \frac{15}{4096}$$
$$= 964\tfrac{543}{4096} \qquad = 964{\cdot}133_{10}$$

Therefore 3 C 4 · 2 1 F_{16} $= 964{\cdot}133_{10}$

At this point let us pause and summarize the main facts so far

120

Revision summary

1 *Denary (or decimal) system*: Base 10. Place values powers of 10. Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

2 *Binary system*: Base 2. Place values powers of 2. Symbols 0, 1.

3 *Octal system*: Base 8. Place values powers of 8. Symbols 0, 1, 2, 3, 4, 5, 6, 7.

4 *Duodecimal system*: Base 12. Place values powers of 12. Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X, Λ

5 *Hexadecimal system*: Base 16. Place values powers of 16. Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.

Now move on to the next frame

121

So far, we have changed numbers in various bases into the equivalent denary numbers from first principles. Another way of arriving at the same results is by using the fact that two adjacent columns differ in place values by a factor which is the base of the particular system. An example will show the method.

Express the octal $357{\cdot}121_8$ in denary form.

First of all, we will attend to the whole-number part 357_8.

Starting at the left-hand end, multiply the first column by the base 8 and add the result to the entry in the next column (making 29).

$$\begin{array}{rcrcr}
3 & & 5 & & 7 \\
\underline{\times\ 8} & \to & \underline{24} & \to & \underline{232} \\
\underline{24} & & 29 & & \underline{239} \\
 & & \underline{\times\ 8} & & \\
 & & \underline{232} & &
\end{array}$$

Now repeat the process. Multiply the second column total by 8 and add the result to the next column. This gives 239 in the units column.

$$\text{So } 357_8 = 239_{10}$$

Now we do much the same with the decimal part of the octal number

122

$\boxed{\text{The decimal part is } 0 \cdot 1\,2\,1_8}$

$$
\begin{array}{ccccc}
0 & \cdot & 1 & 2 & 1 \\
 & & \times\ 8 & 8 & 80 \\
\hline
 & & 8 & 10 & 81 \\
 & & & \times\ 8 & \\
\hline
 & & & 80 &
\end{array}
$$

Starting from the left-hand column immediately following the decimal point, multiply by 8 and add the result to the next column. Repeat the process, finally getting a total of 81 in the end column.

But the place value of this column is ....

123

$\boxed{8^{-3}}$

The denary value of $0{\cdot}121_8$ is 81×8^{-3} i.e. $81 \times \frac{1}{8^3} = \frac{81}{512} = 0{\cdot}1582_{10}$

Collecting the two partial results together, $357{\cdot}121_8 = 239{\cdot}1582_{10}$

In fact, we can set this out across the page to save space, thus:

$$
\begin{array}{ccc|ccc}
3 & 5 & 7\ \cdot & 1 & 2 & 1 \\
\times\ 8 & 24 & 232 & \times\ 8 & 8 & 80 \\
\hline
24 & 29 & 239 & 8 & 10 & 81 \\
 & \times\ 8 & & & \times\ 8 & \\
\hline
 & 232 & & & 80 &
\end{array}
$$

$81 \times \frac{1}{8^3} = \frac{81}{512} = 0{\cdot}1582_{10}$ Therefore $357{\cdot}121_8 = 239{\cdot}158_{10}$

Now you can set this one out in similar manner.

Express the duodecimal $245{\cdot}136_{12}$ in denary form.

Space out the duodecimal digits to give yourself room for the working:

$2 \quad 4 \quad 5 \quad \cdot \quad 1 \quad 3 \quad 6_{12}$

Then off you go. $245{\cdot}136_{12} =$ ....

124

$\boxed{341{\cdot}1076_{10}}$

Here is the working as a check:

$$
\begin{array}{ccc|ccc}
2 & 4 & 5 \;\cdot & 1 & 3 & 6_{12} \\
\times\,12 & 24 & 336 & \times\,12 & 12 & 180 \\
\hline
24 & 28 & 341 & 12 & 15 & 186 \\
 & \times\,12 & & & \times\,12 & \\
\hline
 & 336 & & & 180 &
\end{array}
$$

Place value of last column is 12^{-3}, therefore

$$0{\cdot}136_{12} = 186 \times 12^{-3} = \frac{186}{1728} = 0{\cdot}1076_{10}$$

$$\text{So } 245{\cdot}136_{12} = 341{\cdot}1076_{10}$$

On to the next

125

Now for an easy one. Find the denary equivalent of the binary number $1\,1\,0\,1\,1 \cdot 1\,0\,1\,1_2$.

Setting it out in the same way, the result is ....

126

$\boxed{27{\cdot}6875_{10}}$

$$
\begin{array}{ccccc|cccc}
1 & 1 & 0 & 1 & 1 \;\cdot & 1 & 0 & 1 & 1 \\
\times 2 & 2 & 6 & 12 & 26 & \times 2 & 2 & 4 & 10 \\
\hline
2 & 3 & 6 & 13 & 27 & 2 & 2 & 5 & 11 \\
 & \times 2 & \times 2 & \times 2 & & & \times 2 & \times 2 & \\
\hline
 & 6 & 12 & 26 & & & 4 & 10 &
\end{array}
$$

$11 \times 2^{-4} = \dfrac{11}{16} = 0{\cdot}6875_{10}$ Therefore $1\,1\,0\,1\,1 \cdot 1\,0\,1\,1_2 = 27{\cdot}6875_{10}$

And now a hexadecimal. Express $4\;C\;5 \cdot 2\;B\;8_{16}$ in denary form. Remember that $C = 12$ and $B = 11$. There are no snags.

$4\;C\;5 \cdot 2\;B\;8_{16} =$ ....

127

$$\boxed{1221{\cdot}1699_{10}}$$

Here it is:

$$\begin{array}{cccccc}
4 & C & 5\,\cdot & 2 & B & 8 \\
\times\, 16 & 64 & 1216 & \times\, 16 & 32 & 688 \\
\hline
64 & 76 & 1221 & 32 & 43 & 696 \\
 & \times\, 16 & & & \times\, 16 & \\
 & 456 & & & 258 & \\
 & 76 & & & 43 & \\
 & 1216 & & & 688 &
\end{array}$$

Place value 16^{-3}

So $\dfrac{696}{4096} = 0{\cdot}16992_{10}$

Therefore $4\,C\,5 \cdot 2\,B\,8_{16} = 1221{\cdot}1699_{10}$

They are all done in the same way.

By now then, we can change any binary, octal, duodecimal or hexadecimal number – including a decimal part – into the equivalent denary number. So here is a short revision exercise for valuable practice. Complete the set and then check your results with the next frame.

128

Revision exercise

Express each of the following in denary form

(a) $1\,1\,0\,0\,1 \cdot 1\,1_2$ (b) $7\,7\,6 \cdot 1\,4\,3_8$

(c) $4\,X\,9 \cdot 2\,\Lambda\,5_{12}$ (d) $6\,F\,8 \cdot 3\,D\,5_{16}$

129

(a)	$25{\cdot}75_{10}$
(b)	$510{\cdot}193_{10}$
(c)	$705{\cdot}246_{10}$
(d)	$1784{\cdot}240_{10}$

Just in case you have made a slip anywhere, here is the working.

(a)
$$\begin{array}{ccccc|cc}
1 & 1 & 0 & 0 & 1\ \cdot & 1 & 1 \\
\times 2 & 2 & 6 & 12 & 24 & \times 2 & 2 \\
\hline
2 & 3 & 6 & 12 & 25_{10} & 2 & 3\times 2^{-2} = \frac{3}{4} = 0{\cdot}75_{10} \\
 & \times 2 & \times 2 & \times 2 & & & \\
 & 6 & 12 & 24 & & &
\end{array}$$

$$11001{\cdot}11_2 = 25{\cdot}75_{10}$$

(b)
$$\begin{array}{ccc|ccc}
7 & 7 & 6\ \cdot & 1 & 4 & 3 \\
\times 8 & 56 & 504 & \times 8 & 8 & 96 \\
\hline
56 & 63 & 510_{10} & 8 & 12 & 99\times 8^{-3} = \frac{99}{512} \\
 & \times 8 & & & \times 8 & = 0{\cdot}19336_{10} \\
 & 504 & & & 96 &
\end{array}$$

$$776{\cdot}143_8 = 510{\cdot}193_{10}$$

(c)
$$\begin{array}{ccc|ccc}
4 & \mathrm{X} & 9\ \cdot & 2 & \Lambda & 5 \\
\times 12 & 48 & 696 & \times 12 & 24 & 420 \\
\hline
48 & 58 & 705_{10} & 24 & 35 & 425\times 12^{-3} = \frac{425}{1728} \\
 & \times 12 & & & \times 12 & = 0{\cdot}2459_{10} \\
 & 696 & & & 420 &
\end{array}$$

$$4\mathrm{X}9{\cdot}2\Lambda5_{12} = 705{\cdot}246_{10}$$

(d)
$$\begin{array}{ccc|ccc}
6 & \mathrm{F} & 8\ \cdot & 3 & \mathrm{D} & 5 \\
\times 16 & 96 & 1776 & \times 16 & 48 & 976 \\
\hline
96 & 111 & 1784_{10} & 48 & 61 & 981\times 16^{-3} = \frac{981}{4096} \\
 & \times 16 & & & \times 16 & = 0{\cdot}2395_{10} \\
 & 666 & & & 366 & \\
 & 111 & & & 61 & \\
 & 1776 & & & 976 &
\end{array}$$

$$6\mathrm{F}8{\cdot}3\mathrm{D}5_{16} = 1784{\cdot}240_{10}$$

In all the previous examples, we have changed binary, octal, duodecimal and hexadecimal numbers into their equivalent denary forms. The reverse process is also often required, so we will now see what is involved.

So on then to the next frame

Change of base from denary to a new base

130

1 To express a denary number in binary form

The simplest way to do this is by repeated division by 2 (the new base), noting the remainder at each stage. Continue dividing until a final zero quotient is obtained.

For example, to change 245_{10} to binary:

$$\begin{array}{r|l l}
2 & 245_{10} & \\
2 & 122 & -1 \\
2 & 61 & -0 \\
2 & 30 & -1 \\
2 & 15 & -0 \\
2 & 7 & -1 \\
2 & 3 & -1 \\
2 & 1 & -1 \\
 & 0 & -1
\end{array}$$

Now write all the remainders in the reverse order, i.e. from bottom to top.

Then $245_{10} = 11110101_2$

2 To express a denary number in octal form

The method here is exactly the same except that we divide repeatedly by 8 (the new base). So, without more ado, changing 524_{10} to octal gives ....

131

$$\boxed{1014_8}$$

$$\begin{array}{r|l l}
\text{For: } 8 & 524_{10} & \\
8 & 65 & -4 \\
8 & 8 & -1 \\
8 & 1 & -0 \\
 & 0 & -1
\end{array}$$

As before, write the remainders in order, i.e. from bottom to top.

$\therefore\ 524_{10} = 1014_8$

3 To express a denary number in duodecimal form

Method as before, but this time we divide repeatedly by 12.

So $897_{10} =$ ....

132

$\boxed{629_{12}}$

Because

$$\begin{array}{r|c l}
12 & 897_{10} & \\
12 & 74 & -9 \\
12 & 6 & -2 \\
 & 0 & -6
\end{array} \quad \uparrow \quad \therefore\ 897_{10} = 629_{12}$$

Now move to the next frame

133

The method we have been using is quick and easy enough when the denary number to be changed is a whole number. When it contains a decimal part, we must look further.

4 To change a denary decimal to octal form

To change $0{\cdot}526_{10}$ to octal form, we multiply the decimal repeatedly by the new base, in this case 8, but on the second and subsequent multiplication, we do not multiply the whole-number part of the previous product.

$$\begin{array}{rcl l}
0 & \cdot & 526_{10} & \\
 & & 8 & \\
\hline
4 & \cdot & 208 & \text{Now multiply again by 8, but treat only the decimal part} \\
 & & 8 & \\
\hline
1 & \cdot & 664 & \\
 & & 8 & \\
\hline
5 & \cdot & 312 & \\
 & & 8 & \\
\hline
2 & \cdot & 496 & \text{and so on}
\end{array}$$

Finally, we write the whole-number numerals downwards to form the required octal decimal.

Be careful *not* to include the zero unit digit in the original denary decimal. In fact, it may be safer simply to write the decimal as $\cdot 526_{10}$ in the working.

So $$0{\cdot}526_{10} = 0{\cdot}4152_8$$

Converting a denary decimal into any new base is done in the same way. If we express $0{\cdot}306_{10}$ as a duodecimal, we get ....

Set it out in the same way: there are no snags

134

$$\boxed{0{\cdot}3809_{12}}$$

$$\begin{array}{r l l}
 & \cdot & 306_{10} \\
 & & \underline{12} \\
3 & \cdot & 672 \\
 & & \underline{12} \\
8 & \cdot & 064 \\
 & & \underline{12} \\
0 & \cdot & 768 \\
 & & \underline{12} \\
9 & \cdot & 216 \\
 & & \underline{12} \\
2 & \cdot & \underline{592}
\end{array}$$

There is no carryover into the units column, so enter a zero.

etc.

$$\therefore \; 0{\cdot}306_{10} = 0{\cdot}3809_{12}$$

Now let us go one stage further – so on to the next frame

135

If the denary number consists of both a whole-number and a decimal part, the two parts are converted separately and united in the final result. The example will show the method.

Express $492{\cdot}731_{10}$ in octal form:

$$\begin{array}{r|l l}
8 & 492_{10} & \\
\hline
8 & 61 & -4 \\
\hline
8 & 7 & -5 \\
\hline
 & 0 & -7
\end{array}
\qquad\qquad
\begin{array}{r l l}
 & \cdot & 731_{10} \\
 & & \underline{8} \\
5 & \cdot & 848 \\
 & & \underline{8} \\
6 & \cdot & 784 \\
 & & \underline{8} \\
6 & \cdot & 272 \\
 & & \underline{8} \\
2 & \cdot & \underline{176}
\end{array}$$

Then $492{\cdot}731_{10} = 754{\cdot}5662_8$

In similar manner, $384{\cdot}426_{10}$ expressed in duodecimals becomes ....

Set the working out in the same way

136

$$\boxed{280{\cdot}5142_{12}}$$

Because we have:

$$\begin{array}{r|ll}
12 & 384_{10} & \\
\hline
12 & 32 & -0 \\
\hline
12 & 2 & -8 \\
\hline
 & 0 & -2 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{rcl}
 & \cdot & 426_{10} \\
 & & 12 \\
\hline
5 & \cdot & 112 \\
 & & 12 \\
\hline
1 & \cdot & 344 \\
 & & 12 \\
\hline
4 & \cdot & 128 \\
 & & 12 \\
\hline
1 & \cdot & 536 \\
 & & 12 \\
\hline
6 & \cdot & 432 \\
\hline
\end{array}$$

$$\therefore\ 384{\cdot}426_{10} = 280{\cdot}5142_{12}$$

That is straightforward enough, so let us now move on to see a very helpful use of octals in the next frame.

Use of octals as an intermediate step

137

This gives us an easy way of converting denary numbers into binary and hexadecimal forms. As an example, note the following.

Express the denary number $348{\cdot}654_{10}$ in octal, binary and hexadecimal forms.

(a) First we change $348{\cdot}654_{10}$ into octal form by the usual method.

This gives $348{\cdot}654_{10} = \ldots\ldots\ldots\ldots$

138

$$\boxed{534{\cdot}517_8}$$

(b) Now we take this octal form and write the binary equivalent of each digit in groups of three binary digits, thus:

$$101\quad 011\quad 100\ \cdot\ 101\quad 001\quad 111$$

Closing the groups up we have the binary equivalent of $534{\cdot}517_8$

i.e. $\quad 348{\cdot}654_{10} = 534{\cdot}517_8$

$\quad\quad = 101011100{\cdot}101001111_2$

(c) Then, starting from the decimal point and working in each direction, regroup the same binary digits in groups of four. This gives $\ldots\ldots\ldots\ldots$

139

$$\boxed{0001 \quad 0101 \quad 1100 \;\cdot\; 1010 \quad 0111 \quad 1000}$$

completing the group at either end by addition of extra zeros, as necessary.

(d) Now write the hexadecimal equivalent of each group of four binary digits, so that we have ....

140

$$\boxed{1 \quad 5 \quad (12) \;\cdot\; (10) \quad 7 \quad 8}$$

Replacing (12) and (10) with the corresponding hexadecimal symbols, C and A, this gives $1\ 5\ C \cdot A\ 7\ 8_{16}$

So, collecting the partial results together:

$$\begin{aligned} 348{\cdot}654_{10} &= 534{\cdot}517_8 \\ &= 101011100{\cdot}101001111_2 \\ &= 15C{\cdot}A78_{16} \end{aligned}$$

Next frame

141

We have worked through the previous example in some detail. In practice, the method is more concise. Here is another example.

Change the denary number $428{\cdot}371_{10}$ into its octal, binary and hexadecimal forms.

(a) First of all, the octal equivalent of $428{\cdot}371_{10}$ is ....

142

$$\boxed{654{\cdot}276_8}$$

(b) The binary equivalent of each octal digit in groups of three is ....

143

$$\boxed{110 \quad 101 \quad 100 \;\cdot\; 010 \quad 111 \quad 110_2}$$

(c) Closing these up and rearranging in groups of four in each direction from the decimal point, we have ....

144

$$\boxed{0001 \quad 1010 \quad 1100 \;\cdot\; 0101 \quad 1111_2}$$

(d) The hexadecimal equivalent of each set of four binary digits then gives ....

145

$$\boxed{1AC{\cdot}5F_{16}}$$

$$\begin{aligned} \text{So } 428{\cdot}371_{10} &= 654{\cdot}276_8 \\ &= 110101100{\cdot}010111110_2 \\ &= 1AC{\cdot}5F_{16} \end{aligned}$$

This is important, so let us do one more.

Convert $163{\cdot}245_{10}$ into octal, binary and hexadecimal forms.

You can do this one entirely on your own. Work right through it and then check with the results in the next frame

146

$$\boxed{\begin{aligned} 163{\cdot}245_{10} &= 243{\cdot}175_8 \\ &= 010100011{\cdot}001111101_2 \\ &= 1010\ 0011\ \cdot\ 0011\ 1110\ 1000_2 \\ &= A3{\cdot}3E8_{16} \end{aligned}}$$

And that is it.

On to the next frame

Reverse method

147

Of course, the method we have been using can be used in reverse, i.e. starting with a hexadecimal number, we can change it into groups of four binary digits, regroup these into groups of three digits from the decimal point, and convert these into the equivalent octal digits. Finally, the octal number can be converted into denary form by the usual method.

Here is one you can do with no trouble.

Express the hexadecimal number $4B2{\cdot}1A6_{16}$ in equivalent binary, octal and denary forms.

(a) Rewrite $4B2{\cdot}1A6_{16}$ in groups of four binary digits.
(b) Regroup into groups of three binary digits from the decimal point.
(c) Express the octal equivalent of each group of three binary digits.
(d) Finally convert the octal number into its denary equivalent.

Work right through it and then check with the solution in the next frame

148

$4B2{\cdot}1A6_{16} =$
(a) $0100\ 1011\ 0010 \cdot 0001\ 1010\ 0110_2$
(b) $010\ 010\ 110\ 010 \cdot 000\ 110\ 100\ 110_2$
(c) $2\ \ 2\ \ 6\ \ 2 \cdot 0\ \ 6\ \ 4\ \ 6_8$
(d) $1202{\cdot}103_{10}$

Now one more for good measure.

Express $2E3{\cdot}4D_{16}$ in binary, octal and denary forms.

Check results with the next frame

149

$$
\begin{aligned}
2E3{\cdot}4D_{16} &= 0010\ 1110\ 0011 \cdot 0100\ 1101_2\\
&= 001\ 011\ 100\ 011 \cdot 010\ 011\ 010_2\\
&= 1\ \ 3\ \ 4\ \ 3 \cdot 2\ \ 3\ \ 2_8\\
&= 739{\cdot}301_{10}
\end{aligned}
$$

At this point let us pause and summarize the main facts so far on number systems

150

Revision summary

1	*Denary (or decimal) system*	Base 10 Numerals	Place values – powers of 10 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
2	*Binary system*	Base 2 Numerals	Place values – powers of 2 0, 1
3	*Octal system*	Base 8 Numerals	Place values – powers of 8 0, 1, 2, 3, 4, 5, 6, 7
4	*Duodecimal system*	Base 12 Numerals	Place values – powers of 12 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X, Λ (10, 11)
5	*Hexadecimal system*	Base 16 Numerals	Place values – powers of 16 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F (10,11,12,13,14,15)

151

Revision exercise

1 Express the following numbers in denary form:

(a) $1110{\cdot}11_2$ (b) $507{\cdot}632_8$ (c) $345{\cdot}2Λ7_{12}$ (d) $2B4{\cdot}CA3_{16}$

2 Express the denary number $427{\cdot}362_{10}$ as a duodecimal number.

3 Convert $139{\cdot}825_{10}$ to the equivalent octal, binary and hexadecimal forms.

152

1 (a) $14{\cdot}75_{10}$ (b) $327{\cdot}801_{10}$ (c) $485{\cdot}247_{10}$ (d) $692{\cdot}790_{10}$ to 3 dp

2 $2\text{A}7{\cdot}442_{12}$

3 $213{\cdot}646_8$, 10 001 011 · 110 100 110_2 and $8\text{B}{\cdot}\text{D}3_{16}$

153

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through these at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

Checklist F.1

154

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Carry out the basic rules of arithmetic with integers? 1 to 12
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Write a natural number as a product of prime numbers? 16 to 19
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Find the highest common factor and lowest common multiple of two natural numbers? 20 to 21
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Check the result of a calculation making use of rounding? 22 to 24
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Manipulate fractions, ratios and percentages? 28 to 52
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Manipulate decimal numbers? 56 to 74
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Manipulate powers? 78 to 98
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use standard or preferred standard form and complete a calculation to the required level of accuracy? 99 to 108
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Understand the construction of various number systems and convert from one number system to another? 112 to 149
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.1

155

Frames

1 Place the appropriate symbol < or > between each of the following pairs of numbers:
(a) $-12 \quad -15$ (b) $9 \quad -17$ (c) $-11 \quad 10$ — 1 to 4

2 Find the value of each of the following:
(a) $24 - 3 \times 4 + 28 \div 14$ (b) $(24 - 3) \times (4 + 28) \div 14$ — 5 to 12

3 Write each of the following as a product of prime factors:
(a) 156 (b) 546 (c) 1445 (d) 1485 — 16 to 19

4 Round each number to the nearest 10, 100 and 1000:
(a) 5045 (b) 1100 (c) -1552 (d) -4995 — 22 to 24

5 Find (i) the HCF and (ii) the LCM of:
(a) 1274 and 195 (b) 64 and 18 — 20 to 21

6 Reduce each of the following fractions to their lowest terms:
(a) $\dfrac{8}{14}$ (b) $\dfrac{162}{36}$ (c) $-\dfrac{279}{27}$ (d) $-\dfrac{81}{3}$ — 28 to 36

7 Evaluate each of the following, giving your answer as a fraction:
(a) $\dfrac{1}{3}+\dfrac{3}{5}$ (b) $\dfrac{2}{7}-\dfrac{1}{9}$ (c) $\dfrac{8}{3}\times\dfrac{6}{5}$ (d) $\dfrac{4}{5}$ *of* $\dfrac{2}{15}$
(e) $\dfrac{9}{2}\div\dfrac{3}{2}$ (f) $\dfrac{6}{7}-\dfrac{4}{5}\times\dfrac{3}{2}\div\dfrac{7}{5}+\dfrac{9}{4}$ — 37 to 46

8 In each of the following the proportions of a compound are given. Find the ratios of the components in each case:
(a) $\dfrac{3}{4}$ of A and $\dfrac{1}{4}$ of B
(b) $\dfrac{2}{3}$ of P, $\dfrac{1}{15}$ of Q and the remainder of R
(c) $\dfrac{1}{5}$ of R, $\dfrac{3}{5}$ of S, $\dfrac{1}{6}$ of T and the remainder of U — 47 to 48

9 What is:
(a) $\dfrac{3}{5}$ as a percentage?
(b) 16% as a fraction in its lowest terms?
(c) 17·5% of £12·50? — 49 to 52

Frames

10 Evaluate each of the following (i) to 4 sig fig and (ii) to 3 dp:
(a) $13{\cdot}6 \times 25{\cdot}8 \div 4{\cdot}2$
(b) $13{\cdot}6 \div 4{\cdot}2 \times 25{\cdot}8$
(c) $9{\cdot}1(17{\cdot}43 + 7{\cdot}2(8{\cdot}6 - 4{\cdot}1^2 \times 3{\cdot}1))$
(d) $-8{\cdot}4((6{\cdot}3 \times 9{\cdot}1 + 2{\cdot}2^{1{\cdot}3}) - (4{\cdot}1^{-3{\cdot}1} \div 3{\cdot}3^3 - 5{\cdot}4))$ — **56** to **65**

11 Convert each of the following to decimal form to 3 decimal places:
(a) $\dfrac{3}{17}$ (b) $-\dfrac{2}{15}$ (c) $\dfrac{17}{3}$ (d) $-\dfrac{24}{11}$ — **66** to **67**

12 Write each of the following in abbreviated form:
(a) $6{\cdot}7777\ldots$ (b) $0{\cdot}01001001001\ldots$ — **70** to **71**

13 Convert each of the following to fractional form in lowest terms:
(a) $0{\cdot}4$ (b) $3{\cdot}68$ (c) $1{\cdot}\dot{4}$ (d) $-6{\cdot}1$ — **68** to **73**

14 Write each of the following as a number raised to a power:
(a) $2^9 \times 2^2$ (b) $6^2 \div 5^2$ (c) $((-4)^4)^{-4}$ (d) $(3^{-5})^0$ — **78** to **89**

15 Find the value of each of the following to 3 dp:
(a) $11^{\frac{1}{4}}$ (b) $\sqrt[7]{3}$ (c) $(-81)^{\frac{1}{5}}$ (d) $(-81)^{\frac{1}{4}}$ — **90** to **94**

16 Express in standard form:
(a) $537{\cdot}6$ (b) $0{\cdot}364$ (c) 4902 (d) $0{\cdot}000125$ — **95** to **101**

17 Convert to preferred standard form:
(a) $6{\cdot}147 \times 10^7$ (b) $2{\cdot}439 \times 10^{-4}$ (c) $5{\cdot}286 \times 10^5$
(d) $4{\cdot}371 \times 10^{-7}$ — **102** to **104**

18 Determine the following product, giving the result in both standard form and preferred standard form:
$(6{\cdot}43 \times 10^3)(7{\cdot}35 \times 10^4)$ — **95** to **104**

19 Each of the following contains numbers obtained by measurement. Evaluate each to the appropriate level of accuracy:
(a) $18{\cdot}4^{1{\cdot}6} \times 0{\cdot}01$ (b) $\dfrac{7{\cdot}632 \times 2{\cdot}14 - 8{\cdot}32 \div 1{\cdot}1}{16{\cdot}04}$ — **105** to **108**

20 Express the following numbers in denary form:
(a) $1111{\cdot}11_2$ (b) $777{\cdot}701_8$ (c) $3\Lambda3{\cdot}9\Lambda1_{12}$
(d) $E02{\cdot}FAB_{16}$ — **112** to **129**

21 Convert $19{\cdot}872_{10}$ to the equivalent octal, binary, duodecimal and hexadecimal forms. — **130** to **152**

Further problems F.1

156

1 Place the appropriate symbol < or > between each of the following pairs of numbers:

(a) $-4 \quad -11$ (b) $7 \quad -13$ (c) $-15 \quad 13$

2 Find the value of each of the following:

(a) $6 + 14 \div 7 - 2 \times 3$ (b) $(6 + 14) \div (7 - 2) \times 3$

3 Round each number to the nearest 10, 100 and 1000:

(a) 3505 (b) 500 (c) -2465 (d) -9005

4 Calculate each of the following to:

(i) 5 sig fig;
(ii) 4 dp;
(iii) The required level of accuracy given that each number, other than those indicated in brackets, has been obtained by measurement.

(a) $\dfrac{3{\cdot}21^{2{\cdot}33} + (5{\cdot}77 - 3{\cdot}11)}{8{\cdot}32 - 2{\cdot}64 \times \sqrt{2{\cdot}56}}$

(b) $\dfrac{3{\cdot}142 \times 1{\cdot}95}{6}(3 \times 5{\cdot}44^2 + 1{\cdot}95^2)$ (power 2, divisor 6, multiplier 3)

(c) $\dfrac{3{\cdot}142 \times 1{\cdot}234}{12}(0{\cdot}424^2 + 0{\cdot}424 \times 0{\cdot}951 + 0{\cdot}951^2)$ (power 2, divisor 12)

(d) $\sqrt{\dfrac{2 \times 0{\cdot}577}{3{\cdot}142 \times 2{\cdot}64} + \dfrac{2{\cdot}64^2}{3}}$ (power 2, divisor 3, multiplier 2)

(e) $\dfrac{3{\cdot}26 + \sqrt{12{\cdot}13}}{14{\cdot}192 - 2{\cdot}4 \times 1{\cdot}63^2}$ (power 2)

(f) $\dfrac{4{\cdot}62^2 - (7{\cdot}16 - 2{\cdot}35)}{2{\cdot}63 + 1{\cdot}89 \times \sqrt{73{\cdot}24}}$ (power 2)

5 Find the prime factorization for each of the following:

(a) 924 (b) 825 (c) 2310 (d) 35 530

6 Find the HCF and LCM for each of the following pairs of numbers:

(a) 9, 21 (b) 15, 85 (c) 66, 42 (d) 64, 360

7 Reduce each of the following fractions to their lowest terms:

(a) $\dfrac{6}{24}$ (b) $\dfrac{104}{48}$ (c) $-\dfrac{120}{15}$ (d) $-\dfrac{51}{7}$

▶

8 Evaluate:

(a) $\frac{9}{2}-\frac{4}{5}\div\left(\frac{2}{3}\right)^2\times\frac{3}{11}$ (b) $\dfrac{\frac{3}{4}+\frac{7}{5}\div\frac{2}{9}\times\frac{1}{3}}{\frac{7}{3}-\frac{11}{2}\times\frac{2}{5}+\frac{4}{9}}$

(c) $\left(\frac{3}{4}+\frac{7}{5}\right)^2\div\left(\frac{7}{3}-\frac{11}{5}\right)^2$ (d) $\dfrac{\left(\frac{5}{2}\right)^3-\frac{2}{9}\div\left(\frac{2}{3}\right)^2\times\frac{3}{2}}{\frac{3}{11}+\left(\frac{11}{2}\times\frac{2}{5}\right)^2-\frac{7}{5}}$

9 Express each of the following as a fraction in its lowest terms:
(a) 36% (b) 17·5% (c) 8·7% (d) 72%

10 Express each of the following as a percentage accurate to 1 dp:

(a) $\frac{4}{5}$ (b) $\frac{3}{11}$ (c) $\frac{2}{9}$ (d) $\frac{1}{7}$

(e) $\frac{9}{19}$ (f) $\frac{13}{27}$ (g) $\frac{7}{101}$ (h) $\frac{199}{200}$

11 Find:
(a) 16% *of* 125 (b) 9·6% *of* 5·63
(c) 13·5% *of* (−13·5) (d) 0·13% *of* 92·66

12 In each of the following the properties of a compound are given. In each case find A : B : C.

(a) $\frac{1}{5}$ of A, $\frac{2}{3}$ of B and the remainder of C;

(b) $\frac{3}{8}$ of A with B and C in the ratio 1 : 2;

(c) A, B and C are mixed according to the ratios A : B = 2 : 5 and B : C = 10 : 11;

(d) A, B and C are mixed according to the ratios A : B = 1 : 7 and B : C = 13 : 9.

13 Write each of the following in abbreviated form:
(a) 8·767676... (b) 212·211211211...

14 Convert each of the following to fractional form in lowest terms:
(a) 0·12 (b) 5·25 (c) $5{\cdot}\dot{3}0\dot{6}$ (d) −9·3

15 Write each of the following as a number raised to a power:
(a) $8^4\times8^3$ (b) $2^9\div8^2$ (c) $(5^3)^5$ (d) $3^4\div9^2$

16 Find the value of each of the following to 3 dp:
(a) $17^{\frac{2}{5}}$ (b) $\sqrt[4]{13}$ (c) $(-5)^{\frac{2}{3}}$ (d) $\sqrt{(-5)^4}$

17 Convert each of the following to decimal form to 3 decimal places:

(a) $\frac{5}{21}$ (b) $-\frac{2}{17}$ (c) $\frac{8}{3}$ (d) $-\frac{32}{19}$

▶

18 Express in standard form:

(a) 52·876 (b) 15 243 (c) 0·08765
(d) 0·0000492 (e) 436·2 (f) 0·5728

19 Rewrite in preferred standard form:

(a) 4285 (b) 0·0169 (c) $8{\cdot}526 \times 10^{-4}$
(d) $3{\cdot}629 \times 10^{5}$ (e) $1{\cdot}0073 \times 10^{7}$ (f) $5{\cdot}694 \times 10^{8}$

20 Evaluate the following, giving the result in standard form and in preferred standard form:

$$\frac{(4{\cdot}26 \times 10^{4})(9{\cdot}38 \times 10^{5})}{3{\cdot}179 \times 10^{2}}$$

21 Convert each of the following decimal numbers to (i) binary, (ii) octal, (iii) duodecimal and (iv) hexadecimal format:

(a) 1·83 (b) $3{\cdot}425 \times 10^{2}$

22 Convert each of the following octal numbers to (i) binary, (ii) decimal, (iii) duodecimal and (iv) hexadecimal format:

(a) 0·577 (b) 563

23 Convert each of the following duodecimal numbers to (i) binary, (ii) octal, (iii) decimal and (iv) hexadecimal format:

(a) 0·ΛX (b) 9Λ1

24 Convert each of the following binary numbers to (i) decimal, (ii) octal, (iii) duodecimal and (iv) hexadecimal format:

(a) 0·10011 (b) 111001100

25 Convert each of the following hexadecimal numbers to (i) binary, (ii) octal, (iii) duodecimal and (iv) decimal format:

(a) 0·F4B (b) 3A5

Programme F.2

Introduction to algebra

Frames 1 to 80

Learning outcomes

When you have completed this Programme you will be able to:

- Use alphabetic symbols to supplement the numerals and to combine these symbols using all the operations of arithmetic
- Simplify algebraic expressions by collecting like terms and by abstracting common factors from similar terms
- Remove brackets and so obtain alternative algebraic expressions
- Manipulate expressions involving powers and multiply two expressions together
- Manipulate logarithms both numerically and symbolically
- Manipulate algebraic fractions and divide one expression by another
- Factorize algebraic expressions using standard factorizations
- Factorize quadratic algebraic expressions

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.2

Frames

1 Simplify each of the following:

(a) $3pq + 5pr - 2qr + qp - 6rp$

(b) $5l^2mn + 2nl^2m - 3mln^2 + l^2nm + 4n^2ml - nm^2$

(c) $w^4 \div w^{-a} \times w^{-b}$

(d) $\dfrac{(s^{\frac{1}{3}})^{\frac{3}{4}} \times (t^{\frac{1}{4}})^{-1} \div (s^{\frac{2}{7}})^{-\frac{7}{4}}}{(s^{-\frac{1}{4}})^{-1} \div (t^{\frac{1}{2}})^4}$ — 1 to 22

2 Remove the brackets in each of the following:

(a) $-4x(2x - y)(3x + 2y)$

(b) $(a - 2b)(2a - 3b)(3a - 4b)$

(c) $-\{-2[x - 3(y - 4)] - 5(z + 6)\}$ — 12 to 17

3 Evaluate by calculator or by change of base where necessary (to 3 dp):

(a) $\log 0{\cdot}0101$

(b) $\ln 3{\cdot}47$

(c) $\log_2 3{\cdot}16$ — 23 to 32

4 Express $\log F = \log G + \log m - \log\left(\dfrac{1}{M}\right) - 2\log r$ without logs.

5 Rewrite $T = 2\pi\sqrt{\dfrac{l}{g}}$ in log form. — 33 to 35

6 Perform the following multiplications and simplify your results:

(a) $(n^2 + 2n - 3)(4n + 5)$

(b) $(v^3 - v^2 - 2)(1 - 3v + 2v^2)$ — 39 to 43

7 Simplify each of the following:

(a) $\dfrac{p}{q^3} \div \dfrac{q}{p^3}$

(b) $\dfrac{a^2b}{2c} \times \dfrac{ac^2}{2b} \div \dfrac{b^2c}{2a}$ — 43 to 47

8 Perform the following divisions: **Frames**

(a) $(2y^2 - y - 10) \div (y + 2)$

(b) $\dfrac{q^3 - 8}{q - 2}$

(c) $\dfrac{2r^3 + 5r^2 - 4r - 3}{r^2 + 2r - 3}$ 48 to 55

9 Factorize the following:

(a) $18x^2y - 12xy^2$

(b) $x^3 + 4x^2y - 3xy^2 - 12y^3$

(c) $4(x - y)^2 - (x - 3y)^2$

(d) $12x^2 - 25x + 12$ 56 to 76

Algebraic expressions

1

Think of a number
Add 15 to it
Double the result
Add this to the number you first thought of
Divide the result by 3
Take away the number you first thought of

The answer is 10

Why?

Check your answer in the next frame

2

Symbols other than numerals

A letter of the alphabet can be used to represent a number when the specific number is unknown and because the number is unknown (except, of course, to the person who thought of it) we shall represent the number by the letter a:

Think of a number	a
Add 15 to it	$a+15$
Double the result	$2\times(a+15)=(2\times a)+(2\times 15)$ $=(2\times a)+30$
Add the result to the number you first thought of	$a+(2\times a)+30=(3\times a)+30$
Divide the result by 3	$[(3\times a)+30]\div 3=a+10$
Take away the number you first thought of	$a+10-a=10$
The result is 10	

Next frame

3

This little puzzle has demonstrated how:

an unknown number can be represented by a letter of the alphabet which can then be manipulated just like an ordinary numeral within an arithmetic expression.

So that, for example:

$a+a+a+a=4\times a$
$3\times a-a=2\times a$
$8\times a\div a=8$

and

$a\times a\times a\times a\times a=a^5$

▶

If a and b represent two unknown numbers then we speak of the:

sum of a and b	$a+b$
difference of a and b	$a-b$
product of a and b	$a \times b$, $a.b$ or simply ab (*we can and do suppress the multiplication sign*)
quotient of a and b	$a \div b$, a/b or $\frac{a}{b}$ provided $b \neq 0$

and

raising a to the power b	a^b

Using letters and numerals in this way is referred to as *algebra*.

Now move to the next frame

4

Constants

In the puzzle of Frame 1 we saw how to use the letter a to represent an unknown number – we call such a symbol a *constant*.

In many other problems we require a symbolism that can be used to represent not just one number but any one of a collection of numbers. Central to this symbolism is the idea of a variable.

Next frame

5

Variables

We have seen that the operation of addition is commutative. That is, for example:

$$2+3=3+2$$

To describe this rule as applying to any pair of numbers and not just 2 and 3 we resort to the use of alphabetic characters x and y and write:

$$x+y=y+x$$

where x and y represent any two numbers. Used in this way, the letters x and y are referred to as *variables* because they each represent, not just one number, but any one of a collection of numbers.

So how would you write down the fact that multiplication is an associative operation? (Refer to Frame 12 of Programme F.1.)

You can check your answer in the next frame

6

$$\boxed{x(yz) = (xy)z = xyz}$$

where x, y and z represent numbers. Notice the suppression of the multiplication sign.

While it is not a hard and fast rule, it is generally accepted that letters from the beginning of the alphabet, i.e. $a, b, c, d, \ldots$ are used to represent constants and letters from the end of the alphabet, i.e. $\ldots v, w, x, y, z$ are used to represent variables. In any event, when a letter of the alphabet is used it should be made clear whether the letter stands for a constant or a variable.

Now move on to the next frame

7

Rules of algebra

The rules of arithmetic that we met in the previous Programme for integers also apply to any type of number and we express this fact in the *rules of algebra* where we use variables rather than numerals as specific instances. The rules are:

1 Commutativity

Two numbers x and y can be added or multiplied in any order without affecting the result. That is:

$x + y = y + x$ and

$xy = yx$

Addition and multiplication are commutative operations

The order in which two numbers are subtracted or divided *does* affect the result. That is:

$x - y \neq y - x$ unless $x = y$ and

$x \div y \neq y \div x$, $\left(\dfrac{x}{y} \neq \dfrac{y}{x}\right)$ unless $x = y$ and neither equals 0

Subtraction and division are not commutative operations except in very special cases

2 Associativity

The way in which the numbers x, y and z are associated under addition or multiplication *does not* affect the result. That is:

$x + (y + z) = (x + y) + z = x + y + z$ and

$x(yz) = (xy)z = xyz$

Addition and multiplication are associative operations

The way in which the numbers are associated under subtraction or division *does* affect the result. That is:

$x - (y - z) \neq (x - y) - z$ unless $z = 0$ and

$x \div (y \div z) \neq (x \div y) \div z$ unless $z = 1$ and $y \neq 0$

Subtraction and division are not associative operations except in very special cases

3 Distributivity

Multiplication is distributed over addition and subtraction from both the left and the right. For example:

$$x(y+z) = xy + xz \text{ and } (x+y)z = xz + yz$$
$$x(y-z) = xy - xz \text{ and } (x-y)z = xz - yz$$

Division is distributed over addition and subtraction from the right but not from the left. For example:

$$(x+y) \div z = (x \div z) + (y \div z) \text{ but}$$
$$x \div (y+z) \neq (x \div y) + (x \div z)$$

that is:

$$\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z} \text{ but } \frac{x}{y+z} \neq \frac{x}{y} + \frac{x}{z}$$

Take care here because it is a common mistake to get this wrong

Rules of precedence

The familiar rules of precedence continue to apply when algebraic expressions involving mixed operations are to be manipulated.

Next frame

8

Terms and coefficients

An algebraic expression consists of alphabetic characters and numerals linked together with the arithmetic operators. For example:

$8x - 3xy$

is an algebraic expression in the two variables x and y. Each component of this expression is called a *term* of the expression. Here there are two terms, namely:

the x term and the xy term.

The numerals in each term are called the *coefficients* of the respective terms. So that:

8 is the coefficient of the x term and -3 is the coefficient of the xy term.

Collecting like terms

Terms which have the same variables are called *like* terms and like terms can be collected together by addition or subtraction. For example:

$4x + 3y - 2z + 5y - 3x + 4z$ can be rearranged as $4x - 3x + 3y + 5y - 2z + 4z$ and simplified to:

$x + 8y + 2z$

Similarly, $4uv - 7uz - 6wz + 2uv + 3wz$ can be simplified to ....

Check your answer with the next frame

9

$$\boxed{6uv - 7uz - 3wz}$$

Next frame

10 Similar terms

In the algebraic expression:

$ab + bc$

both terms contain the letter b and for this reason these terms, though not like terms, are called *similar* terms. Common symbols such as this letter b are referred to as *common factors* and by using brackets these common factors can be *factored out*. For example, the common factor b in this expression can be factored out to give:

$ab + bc = b(a + c)$ This process is known as *factorization*.

Numerical factors are factored out in the same way. For example, in the algebraic expression:

$3pq - 3qr$

the terms are similar, both containing the letter q. They also have a common coefficient 3 and this, as well as the common letter q, can be factored out to give:

$3pq - 3qr = 3q(p - r)$

So the algebraic expression $9st - 3sv - 6sw$ simplifies to ....

The answer is in the next frame

11

$$\boxed{3s(3t - v - 2w)}$$

Because

$$\begin{aligned} 9st - 3sv - 6sw &= 3s \times 3t - 3s \times v - 3s \times 2w \\ &= 3s(3t - v - 2w) \end{aligned}$$

Next frame

Expanding brackets

12

Sometimes it will be desired to reverse the process of factorizing an expression by *removing* the brackets. This is done by:

(a) multiplying or dividing each term inside the bracket by the term outside the bracket, but
(b) if the term outside the bracket is negative then each term inside the bracket changes sign.

For example, the brackets in the expression:

$3x(y-2z)$ are removed to give $3xy-6xz$ and the brackets in the expression $-2y(2x-4z)$ are removed to give $-4yx+8yz$.

As a further example, the expression:

$\dfrac{y+x}{8x}-\dfrac{y-x}{4x}$ is an alternative form of $(y+x)\div 8x-(y-x)\div 4x$ and the brackets can be removed as follows:

$$\begin{aligned}\frac{y+x}{8x}-\frac{y-x}{4x}&=\frac{y}{8x}+\frac{x}{8x}-\frac{y}{4x}+\frac{x}{4x}\\&=\frac{y}{8x}+\frac{1}{8}-\frac{y}{4x}+\frac{1}{4}\\&=\frac{3}{8}-\frac{y}{8x}\text{ which can be written as }\frac{1}{8}\left(3-\frac{y}{x}\right)\text{ or as }\frac{1}{8x}(3x-y)\end{aligned}$$

Nested brackets

13

Whenever an algebraic expression contains brackets nested within other brackets the innermost brackets are removed first. For example:

$$\begin{aligned}7(a-[4-5(b-3a)])&=7(a-[4-5b+15a])\\&=7(a-4+5b-15a)\\&=7a-28+35b-105a\\&=35b-98a-28\end{aligned}$$

So that the algebraic expression $4(2x+3[5-2(x-y)])$ becomes, after the removal of the brackets ....

Next frame

14

$$\boxed{24y-16x+60}$$

Because

$$\begin{aligned}4(2x+3[5-2(x-y)])&=4(2x+3[5-2x+2y])\\&=4(2x+15-6x+6y)\\&=8x+60-24x+24y\\&=24y-16x+60\end{aligned}$$

At this point let us pause and summarize the main facts so far

15 Revision summary

1. Alphabetic characters can be used to represent numbers and then be subjected to the arithmetic operations in much the same way as numerals.
2. An alphabetic character that represents a single number is called a *constant*.
3. An alphabetic character that represents any one of a collection of numbers is called a *variable*.
4. Some algebraic expressions contain terms multiplied by numerical coefficients.
5. Like terms contain identical alphabetic characters.
6. Similar terms have some but not all alphabetic characters in common.
7. Similar terms can be factorized by identifying their common factors and using brackets.

16 Revision exercise

1. Simplify each of the following by collecting like terms:
   (a) $4xy + 3xz - 6zy - 5zx + yx$
   (b) $-2a + 4ab + a - 4ba$
   (c) $3rst - 10str + 8ts - 5rt + 2st$
   (d) $2pq - 4pr + qr - 2rq + 3qp$
   (e) $5lmn - 6ml + 7lm + 8mnl - 4ln$
2. Simplify each of the following by collecting like terms and factorizing:
   (a) $4xy + 3xz - 6zy - 5zx + yx$
   (b) $3rst - 10str + 8ts - 5rt + 2st$
   (c) $2pq - 4pr + qr - 2rq + 3qp$
   (d) $5lmn - 6ml + 7lm + 8mnl - 4ln$
3. Expand the following and then refactorize where possible:
   (a) $8x(y - z) + 2y(7x + z)$ (b) $(3a - b)(b - 3a) + b^2$
   (c) $-3(w - 7[x - 8(3 - z)])$ (d) $\dfrac{2a - 3}{4b} + \dfrac{3a + 2}{6b}$

17

1 (a) $4xy + 3xz - 6zy - 5zx + yx = 4xy + xy + 3xz - 5xz - 6yz$
$= 5xy - 2xz - 6yz$

Notice that the characters are written in alphabetic order.

(b) $-2a + 4ab + a - 4ba = -2a + a + 4ab - 4ab$
$= -a$

(c) $3rst - 10str + 8ts - 5rt + 2st = 3rst - 10rst + 8st + 2st - 5rt$
$= -7rst + 10st - 5rt$

(d) $2pq - 4pr + qr - 2rq + 3qp = 2pq + 3pq - 4pr + qr - 2qr$
$= 5pq - 4pr - qr$

(e) $5lmn - 6ml + 7lm + 8mnl - 4ln = 5lmn + 8lmn - 6lm + 7lm - 4ln$
$= 13lmn + lm - 4ln$

2 (a) $4xy + 3xz - 6zy - 5zx + yx = 5xy - 2xz - 6yz$
$= x(5y - 2z) - 6yz$ or
$= 5xy - 2z(x + 3y)$ or
$= y(5x - 6z) - 2xz$

(b) $3rst - 10str + 8ts - 5rt + 2st = -7rst + 10st - 5rt$
$= st(10 - 7r) - 5rt$ or
$= 10st - rt(7s + 5)$ or
$= t(10s - 7rs - 5r) = t(s[10 - 7r] - 5r)$

(c) $2pq - 4pr + qr - 2rq + 3qp = 5pq - 4pr - qr$
$= p(5q - 4r) - qr$ or
$= q(5p - r) - 4pr$ or
$= 5pq - r(4p + q)$

(d) $5lmn - 6ml + 7lm + 8mnl - 4ln = 13lmn + lm - 4ln$
$= l(13mn + m - 4n)$
$= l(m[13n + 1] - 4n)$ or
$= l(n[13m - 4] + m)$

3 (a) $8x(y - z) + 2y(7x + z) = 8xy - 8xz + 14xy + 2yz$
$= 22xy - 8xz + 2yz$
$= 2(x[11y - 4z] + yz)$

(b) $(3a - b)(b - 3a) + b^2 = 3a(b - 3a) - b(b - 3a) + b^2$
$= 3ab - 9a^2 - b^2 + 3ab + b^2$
$= 6ab - 9a^2$
$= 3a(2b - 3a)$

(c) $-3(w - 7[x - 8(3 - z)]) = -3(w - 7[x - 24 + 8z])$
$= -3(w - 7x + 168 - 56z)$
$= -3w + 21x - 504 + 168z$

▶

(d) $$\frac{2a-3}{4b}+\frac{3a+2}{6b}=\frac{2a}{4b}-\frac{3}{4b}+\frac{3a}{6b}+\frac{2}{6b}$$
$$=\frac{a}{2b}-\frac{3}{4b}+\frac{a}{2b}+\frac{1}{3b}$$
$$=\frac{a}{b}-\frac{5}{12b}$$
$$=\frac{1}{12b}(12a-5)$$

So now on to the next topic

Powers

18 Powers

The use of *powers* (also called *indices* or *exponents*) provides a convenient form of algebraic shorthand. Repeated factors of the same base, for example $a \times a \times a \times a$ can be written as a^4, where the number 4 indicates the number of factors multiplied together. In general, the product of n such factors a, where a and n are positive integers, is written a^n, where a is called the *base* and n is called the *index* or *exponent* or *power*. Any number multiplying a^n is called the *coefficient* (as described in Frame 8)

$5a^3$ ← index or exponent or power

coefficient ↗ ↑

base

From the definitions above a number of rules of indices can immediately be established.

Rules of indices

1 $a^m \times a^n = a^{m+n}$ e.g. $a^5 \times a^2 = a^{5+2} = a^7$

2 $a^m \div a^n = a^{m-n}$ e.g. $a^5 \div a^2 = a^{5-2} = a^3$

3 $(a^m)^n = a^{mn}$ e.g. $(a^5)^2 = a^5 \times a^5 = a^{10}$

These three basic rules lead to a number of important results.

4 $a^0 = 1$ because $a^m \div a^n = a^{m-n}$ and also $a^m \div a^n = \dfrac{a^m}{a^n}$

Then if $n = m$, $a^{m-m} = a^0$ and $\dfrac{a^m}{a^m} = 1$. So $a^0 = 1$

5 $a^{-m} = \dfrac{1}{a^m}$ because $a^{-m} = \dfrac{a^{-m} \times a^m}{a^m} = \dfrac{a^0}{a^m} = \dfrac{1}{a^m}$. So $a^{-m} = \dfrac{1}{a^m}$

6 $a^{\frac{1}{m}} = \sqrt[m]{a}$ because $\left(a^{\frac{1}{m}}\right)^m = a^{\frac{m}{m}} = a^1 = a$. So $a^{\frac{1}{m}} = \sqrt[m]{a}$

From this it follows that $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ or $\left(\sqrt[m]{a}\right)^n$.

Make a note of any of these results that you may be unsure about.

Then move on to the next frame

19

So we have:

(a) $a^m \times a^n = a^{m+n}$ (e) $a^{-m} = \dfrac{1}{a^m}$

(b) $a^m \div a^n = a^{m-n}$ (f) $a^{\frac{1}{m}} = \sqrt[m]{a}$

(c) $(a^m)^n = a^{mn}$ (g) $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$

(d) $a^0 = 1$ or $\sqrt[m]{a^n}$

Now try to apply the rules:

$$\frac{6x^{-4} \times 2x^3}{8x^{-3}} = \dots\dots\dots\dots$$

20

$$\boxed{\tfrac{3}{2}x^2}$$

Because $\dfrac{6x^{-4} \times 2x^3}{8x^{-3}} = \dfrac{12}{8}.\dfrac{x^{-4+3}}{x^{-3}} = \dfrac{12}{8}.\dfrac{x^{-1}}{x^{-3}} = \dfrac{3}{2}x^{-1+3} = \dfrac{3}{2}x^2$

That was easy enough. In the same way:

Simplify $E = (5x^2y^{-\frac{3}{2}}z^{\frac{1}{4}})^2 \times (4x^4y^2z)^{-\frac{1}{2}}$

$E = \dots\dots\dots\dots$

21

$$\boxed{\frac{25x^2}{2y^4}}$$

$$\begin{aligned} E &= 25x^4y^{-3}z^{\frac{1}{2}} \times 4^{-\frac{1}{2}}x^{-2}y^{-1}z^{-\frac{1}{2}} \\ &= 25x^4y^{-3}z^{\frac{1}{2}} \times \frac{1}{2}x^{-2}y^{-1}z^{-\frac{1}{2}} \\ &= \frac{25}{2}x^2y^{-4}z^0 = \frac{25}{2}x^2y^{-4}.1 = \frac{25x^2}{2y^4} \end{aligned}$$

And one more:

Simplify $F = \sqrt[3]{a^6b^3} \div \sqrt{\dfrac{1}{9}a^4b^6} \times \left(4\sqrt{a^6b^2}\right)^{-\frac{1}{2}}$ giving the result without fractional indices.

$F = \dots\dots\dots\dots$

22

$$\boxed{\frac{3}{2ab^2\sqrt{ab}}}$$

$$\begin{aligned} F &= a^2b \div \frac{1}{3}a^2b^3 \times \frac{1}{(4a^3b)^{\frac{1}{2}}} = a^2b \times \frac{3}{a^2b^3} \times \frac{1}{2a^{\frac{3}{2}}b^{\frac{1}{2}}} \\ &= \frac{3}{b^2} \times \frac{1}{2a^{\frac{3}{2}}b^{\frac{1}{2}}} = \frac{3}{b^2.2a(ab)^{\frac{1}{2}}} = \frac{3}{2ab^2\sqrt{ab}} \end{aligned}$$

Logarithms

23 Powers

Any real number can be written as another number raised to a power. For example:

$9 = 3^2$ and $27 = 3^3$

By writing numbers in the form of a number raised to a power some of the arithmetic operations can be performed in an alternative way. For example:

$$\begin{aligned} 9 \times 27 &= 3^2 \times 3^3 \\ &= 3^{2+3} \\ &= 3^5 \\ &= 243 \end{aligned}$$

Here the process of multiplication is replaced by the process of relating numbers to powers and then adding the powers.

If there were a simple way of relating numbers such as 9 and 27 to powers of 3 and then relating powers of 3 to numbers such as 243, the process of multiplying two numbers could be converted to the simpler process of adding two powers. In the past a system based on this reasoning was created. It was done using tables that were constructed of numbers and their respective powers.

In this instance:

Number	*Power of 3*
1	0
3	1
9	2
27	3
81	4
243	5
...	...

They were not called tables of powers but tables of *logarithms*. Nowadays, calculators have superseded the use of these tables but the logarithm remains an essential concept.

Let's just formalize this

Logarithms

24

If a, b and c are three real numbers where:

$a = b^c$ and $b > 1$

the power c is called the *logarithm* of the number a to the base b and is written:

$c = \log_b a$ spoken as *c is the log of a to the base b*

For example, because

$25 = 5^2$

the power 2 is the logarithm of 25 to the base 5. That is:

$2 = \log_5 25$

So in each of the following what is the value of x, remembering that if $a = b^c$ then $c = \log_b a$?

(a) $x = \log_2 16$
(b) $4 = \log_x 81$
(c) $2 = \log_7 x$

The answers are in the next frame

25

(a) $x = 4$
(b) $x = 3$
(c) $x = 49$

Because

(a) If $x = \log_2 16$ then $2^x = 16 = 2^4$ and so $x = 4$
(b) If $4 = \log_x 81$ then $x^4 = 81 = 3^4$ and so $x = 3$
(c) If $2 = \log_7 x$ then $7^2 = x = 49$

Move on to the next frame

26 Rules of logarithms

Since logarithms are powers, the rules that govern the manipulation of logarithms closely follow the rules of powers.

(a) If $x = a^b$ so that $b = \log_a x$ and
$y = a^c$ so that $c = \log_a y$ then:

$xy = a^b a^c = a^{b+c}$ hence $\log_a xy = b + c = \log_a x + \log_a y$. That is:

$\log_a xy = \log_a x + \log_a y$ *The log of a product equals the sum of the logs*

(b) Similarly $x \div y = a^b \div a^c = a^{b-c}$ so that $\log_a(x \div y) = b - c = \log_a x - \log_a y$
That is:

$\log_a(x \div y) = \log_a x - \log_a y$ *The log of a quotient equals the difference of the logs*

(c) Because $x^n = (a^b)^n = a^{bn}$, $\log_a(x^n) = bn = n \log_a x$. That is:

$\log_a(x^n) = n \log_a x$ *The log of a number raised to a power is the product of the power and the log of the number*

The following important results are also obtained from these rules:

(d) $\log_a 1 = 0$ because, from the laws of powers $a^0 = 1$. Therefore, from the definition of a logarithm $\log_a 1 = 0$

(e) $\log_a a = 1$ because $a^1 = a$ so that $\log_a a = 1$

(f) $\log_a a^x = x$ because $\log_a a^x = x \log_a a = x.1$ so that $\log_a a^x = x$

(g) $a^{\log_a x} = x$ because if we take the log of the left-hand side of this equation:

$\log_a a^{\log_a x} = \log_a x \log_a a = \log_a x$ so that $a^{\log_a x} = x$

(h) $\log_a b = \dfrac{1}{\log_b a}$ because, if $\log_b a = c$ then $b^c = a$ and so $b = \sqrt[c]{a} = a^{\frac{1}{c}}$

Hence, $\log_a b = \dfrac{1}{c} = \dfrac{1}{\log_b a}$. That is $\log_a b = \dfrac{1}{\log_b a}$

So, cover up the results above and complete the following

(a) $\log_a(x \times y) =$ ....
(b) $\log_a(x \div y) =$ ....
(c) $\log_a(x^n) =$ ....
(d) $\log_a 1 =$ ....
(e) $\log_a a =$ ....
(f) $\log_a a^x =$ ....
(g) $a^{\log_a x} =$ ....
(h) $\dfrac{1}{\log_b a} =$ ....

27

(a) $\log_a x + \log_a y$	(e) 1
(b) $\log_a x - \log_a y$	(f) x
(c) $n \log_a x$	(g) x
(d) 0	(h) $\log_a b$

Now try it with numbers. Complete the following:

(a) $\log_a(6{\cdot}788 \times 1{\cdot}043) = \dots\dots\dots\dots$
(b) $\log_a(19{\cdot}112 \div 0{\cdot}054) = \dots\dots\dots\dots$
(c) $\log_a(5{\cdot}889^{1\cdot2}) = \dots\dots\dots\dots$
(d) $\log_8 1 = \dots\dots\dots\dots$
(e) $\log_7 7 = \dots\dots\dots\dots$
(f) $\log_3 27 = \dots\dots\dots\dots$
(g) $12^{\log_{12} 4} = \dots\dots\dots\dots$
(h) $\dfrac{1}{\log_3 4} = \dots\dots\dots\dots$

28

(a) $\log_a 6{\cdot}788 + \log_a 1{\cdot}043$	(e) 1
(b) $\log_a 19{\cdot}112 - \log_a 0{\cdot}054$	(f) 3
(c) $12 \log_a 5{\cdot}889$	(g) 4
(d) 0	(h) $\log_4 3$

Next frame

Base 10 and base *e*

29

On a typical calculator there are buttons that provide access to logarithms to two different bases, namely 10 and the exponential number $e = 2{\cdot}71828\dots$.

Logarithms to base 10 were commonly used in conjunction with tables for arithmetic calculations – they are called *common logarithms* and are written without indicating the base. For example:

$\log_{10} 1{\cdot}2345$ is normally written simply as $\log 1{\cdot}2345$

You will see it on your calculator as $\boxed{\log}$.

The logarithms to base e are called *natural logarithms* and are important for their mathematical properties. These also are written in an alternative form:

$\log_e 1{\cdot}2345$ is written as $\ln 1{\cdot}2345$

You will see it on your calculator as $\boxed{\ln}$.

So, use your calculator and complete the following (to 3 dp):

(a) $\log 5{\cdot}321 = \dots\dots\dots\dots$
(b) $\log 0{\cdot}278 = \dots\dots\dots\dots$
(c) $\log 1 = \dots\dots\dots\dots$
(d) $\log(-1{\cdot}005) = \dots\dots\dots\dots$
(e) $\ln 13{\cdot}45 = \dots\dots\dots\dots$
(f) $\ln 0{\cdot}278 = \dots\dots\dots\dots$
(g) $\ln 0{\cdot}00001 = \dots\dots\dots\dots$
(h) $\ln(-0{\cdot}001) = \dots\dots\dots\dots$

The answers are in the next frame

30

(a) 0·726	(e) 2·599
(b) −0·556	(f) −1·280
(c) 0	(g) −11·513
(d) E	(h) E

Notice that for any base the:

logarithm of 1 is zero
logarithm of 0 is not defined
logarithm of a number greater than 1 is positive
logarithm of a number between 0 and 1 is negative
logarithm of a negative number cannot be evaluated as a real number.

Move to the next frame

31

Change of base

In the previous two frames you saw that $\log 0{\cdot}278 \neq \ln 0{\cdot}278$, i.e. logarithms with different bases have different values. The different values are, however, related to each other as can be seen from the following:

Let $a = b^c$ so that $c = \log_b a$ and let $x = a^d$ so that $d = \log_a x$. Now:

$x = a^d = (b^c)^d = b^{cd}$ so that $cd = \log_b x$. That is:

$$\log_b a \log_a x = \log_b x$$

This is the change of base formula which relates the logarithms of a number relative to two different bases. For example:

$\log_e 0{\cdot}278 = -1{\cdot}280$ to 3 dp and
$\log_e 10 \times \log_{10} 0{\cdot}278 = 2{\cdot}303 \times (-0{\cdot}556) = -1{\cdot}280$ which confirms that:

$$\log_e 10 \log_{10} 0{\cdot}278 = \log_e 0{\cdot}278$$

Now, use your calculator to complete each of the following (to 3 dp):

(a) $\log_2 3{\cdot}66 =$ ....
(b) $\log_{3{\cdot}4} 0{\cdot}293 =$ ....
(c) $\log_{9{\cdot}9} 6{\cdot}35 =$ ....
(d) $\log_{7{\cdot}34} 7{\cdot}34 =$ ....

32

(a) 1·872	(c) 0·806
(b) − 1·003	(d) 1

Because

(a) $(\log_{10} 2) \times (\log_2 3{\cdot}66) = \log_{10} 3{\cdot}66$ so that

$$\log_2 3{\cdot}66 = \frac{\log_{10} 3{\cdot}66}{\log_{10} 2} = \frac{0{\cdot}563\ldots}{0{\cdot}301\ldots} = 1{\cdot}872$$

(b) $(\log_{10} 3{\cdot}4) \times (\log_{3\cdot4} 0{\cdot}293) = \log_{10} 0{\cdot}293$ so that

$$\log_{3\cdot4} 0{\cdot}293 = \frac{\log_{10} 0{\cdot}293}{\log_{10} 3{\cdot}4} = \frac{-0{\cdot}533\ldots}{0{\cdot}531\ldots} = -1{\cdot}003$$

(c) $(\log_{10} 9{\cdot}9) \times (\log_{9\cdot9} 6{\cdot}35) = \log_{10} 6{\cdot}35$ so that

$$\log_{9\cdot9} 6{\cdot}35 = \frac{\log_{10} 6{\cdot}35}{\log_{10} 9{\cdot}9} = \frac{0{\cdot}802\ldots}{0{\cdot}995\ldots} = 0{\cdot}806$$

(d) $\log_{7\cdot34} 7{\cdot}34 = 1$ because for any base a, $\log_a a = 1$.

Logarithmic equations

33

The following four examples serve to show you how logarithmic expressions and equations can be manipulated.

Example 1

Simplify the following:

$$\log_a x^2 + 3\log_a x - 2\log_a 4x$$

Solution

$$\begin{aligned}\log_a x^2 + 3\log_a x - 2\log_a 4x &= \log_a x^2 + \log_a x^3 - \log_a (4x)^2\\ &= \log_a\left(\frac{x^2x^3}{16x^2}\right)\\ &= \log_a\left(\frac{x^3}{16}\right)\end{aligned}$$

Example 2

Solve the following for x:

$$2\log_a x - \log_a(x-1) = \log_a(x-2)$$

Solution

$$\begin{aligned}\text{LHS} &= 2\log_a x - \log_a(x-1)\\ &= \log_a x^2 - \log_a(x-1)\\ &= \log_a\left(\frac{x^2}{x-1}\right)\\ &= \log_a(x-2) \text{ so that } \frac{x^2}{x-1} = x-2. \text{ That is:}\end{aligned}$$

$$x^2 = (x-2)(x-1) = x^2 - 3x + 2 \text{ so that } -3x + 2 = 0 \text{ giving } x = \frac{2}{3}$$

▶

Example 3

Find y in terms of x:

$$5\log_a y - 2\log_a(x+4) = 2\log_a y + \log_a x$$

Solution

$5\log_a y - 2\log_a(x+4) = 2\log_a y + \log_a x$ so that
$5\log_a y - 2\log_a y = \log_a x + 2\log_a(x+4)$ that is
$\log_a y^5 - \log_a y^2 = \log_a x + \log_a(x+4)^2$ that is
$\log_a\left(\dfrac{y^5}{y^2}\right) = \log_a y^3 = \log_a x(x+4)^2$ so that $y^3 = x(x+4)^2$ hence
$y = \sqrt[3]{x(x+4)^2}$

Example 4

For what values of x is $\log_a(x-3)$ valid?
Solution

$\log_a(x-3)$ is valid for $x-3>0$, that is $x>3$

Now you try some

34

1 Simplify $2\log_a x - 3\log_a 2x + \log_a x^2$

2 Solve the following for x:

$$4\log_a\sqrt{x} - \log_a 3x = \log_a x^{-2}$$

3 Find y in terms of x where:

$$2\log_a y - 3\log_a(x^2) = \log_a\sqrt{y} + \log_a x$$

Next frame for the answers

35

1 $2\log_a x - 3\log_a 2x + \log_a x^2 = \log_a x^2 - \log_a(2x)^3 + \log_a x^2$

$$= \log_a\left(\frac{x^2x^2}{8x^3}\right)$$

$$= \log_a\left(\frac{x}{8}\right)$$

2 LHS $= 4\log_a\sqrt{x} - \log_a 3x$

$$= \log_a(\sqrt{x})^4 - \log_a 3x$$

$$= \log_a x^2 - \log_a 3x$$

$$= \log_a\left(\frac{x^2}{3x}\right)$$

$$= \log_a\left(\frac{x}{3}\right)$$

$= \log_a x^{-2}$ the right-hand side of the equation.

So that:

$x^{-2} = \dfrac{x}{3}$, that is $x^3 = 3$ giving $x = \sqrt[3]{3}$

▶

3 $2\log_a y - 3\log_a(x^2) = \log_a \sqrt{y} + \log_a x$, that is
$\log_a y^2 - \log_a(x^2)^3 = \log_a y^{\frac{1}{2}} + \log_a x$ so that
$\log_a y^2 - \log_a y^{\frac{1}{2}} = \log_a(x^2)^3 + \log_a x$ giving
$\log_a \dfrac{y^2}{y^{\frac{1}{2}}} = \log_a x^6.x$. Consequently $y^{\frac{3}{2}} = x^7$ and so $y = \sqrt[3]{x^{14}}$

At this point let us pause and summarize the main facts so far on powers and logarithms

Revision summary

36

1 Rules of powers:

(a) $a^m \times a^n = a^{m+n}$ (b) $a^{-m} = \dfrac{1}{a^m}$

(c) $a^m \div a^n = a^{m-n}$ (d) $a^{\frac{1}{m}} = \sqrt[m]{a}$

(e) $(a^m)^n = a^{mn}$ (f) $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ or $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$

(g) $a^0 = 1$

2 Rules of logarithms:

(a) $\log_a xy = \log_a x + \log_a y$ *The log of a product equals the sum of the logs*

(b) $\log_a(x \div y) = \log_a x - \log_a y$ *The log of a quotient equals the difference of the logs*

(c) $\log_a(x^n) = n\log_a x$ *The log of a number raised to a power is the product of the power and the log of the number*

(d) $\log_a 1 = 0$

(e) $\log_a a = 1$ and $\log_a a^x = x$

(f) $a^{\log_a x} = x$

(g) $\log_a b = \dfrac{1}{\log_b a}$

3 Logarithms to base 10 are called *common logarithms* and are written as $\log x$.

4 Logarithms to base $e = 2{\cdot}71828\ldots$ are called *natural logarithms* and are written as $\ln x$.

37

Revision exercise

1 Simplify each of the following:

(a) $a^6 \times a^5$ (b) $x^7 \div x^3$

(c) $(w^2)^m \div (w^m)^3$ (d) $s^3 \div t^{-4} \times (s^{-3}t^{-2})^3$

(e) $\dfrac{8x^{-3} \times 3x^2}{6x^{-4}}$ (f) $(4a^3b^{-1}c)^2 \times (a^{-2}b^4c^{-2})^{\frac{1}{2}} \div \left[64(a^6b^4c^2)^{-\frac{1}{2}}\right]$

(g) $\sqrt[3]{8a^3b^6} \div \sqrt{\dfrac{1}{25}a^4b^7} \times \left(16\sqrt{a^4b^6}\right)^{-\frac{1}{2}}$

2 Express the following without logs:

(a) $\log K = \log P - \log T + 1{\cdot}3 \log V$ (b) $\ln A = \ln P + rn$

3 Rewrite $R = r\sqrt{\dfrac{f+P}{f-P}}$ in log form.

4 Evaluate by calculator or by change of base where necessary (to 3 dp):

(a) $\log 5{\cdot}324$ (b) $\ln 0{\cdot}0023$ (c) $\log_4 1{\cdot}2$

38

1 (a) $a^6 \times a^5 = a^{6+5} = a^{11}$

(b) $x^7 \div x^3 = x^{7-3} = x^4$

(c) $(w^2)^m \div (w^m)^3 = w^{2m} \div w^{3m} = w^{2m} \times w^{-3m} = w^{-m}$

(d) $s^3 \div t^{-4} \times (s^{-3}t^{-2})^3 = s^3 \times t^4 \times s^{-9}t^{-6} = s^{-6}t^{-2}$

(e) $\dfrac{8x^{-3} \times 3x^2}{6x^{-4}} = \dfrac{24x^{-1}}{6x^{-4}} = 4x^3$

(f) $(4a^3b^{-1}c)^2 \times (a^{-2}b^4c^{-2})^{\frac{1}{2}} \div 64(a^6b^4c^2)^{-\frac{1}{2}}$

$= (16a^6b^{-2}c^2) \times (a^{-1}b^2c^{-1}) \div 64(a^{-3}b^{-2}c^{-1})$

$= (16a^6b^{-2}c^2) \times (a^{-1}b^2c^{-1}) \times 64^{-1}(a^3b^2c^1)$

$= \dfrac{a^8b^2c^2}{4}$

(g) $\sqrt[3]{8a^3b^6} \div \sqrt{\dfrac{1}{25}a^4b^7} \times \left(16\sqrt{a^4b^6}\right)^{-\frac{1}{2}} = (2ab^2) \div \dfrac{a^2b^{\frac{7}{2}}}{5} \times \left(4ab^{\frac{3}{2}}\right)^{-1}$

$= (2ab^2) \times \dfrac{5}{a^2b^{\frac{7}{2}}} \times \dfrac{1}{4ab^{\frac{3}{2}}}$

$= \dfrac{5ab^2}{2a^2b^{\frac{7}{2}}ab^{\frac{3}{2}}}$

$= \dfrac{5}{2a^2b^3}$

2 (a) $K = \dfrac{PV^{1{\cdot}3}}{T}$ (b) $A = Pe^{rn}$

3 $\log R = \log r + \dfrac{1}{2}(\log(f+P) - \log(f-P))$

4 (a) 0·726 (b) −6·075 (c) 0·132

Now move to the next topic

Multiplication of algebraic expressions of a single variable

39

Example 1

$(2x + 5)(x^2 + 3x + 4)$

Each term in the second expression is to be multiplied by $2x$ and then by 5 and the results added together, so we set it out thus:

	$x^2 + 3x + 4$
	$2x + 5$
Multiply throughout by $2x$	$2x^3 + 6x^2 + 8x$
Multiply by 5	$5x^2 + 15x + 20$
Add the two lines	$2x^3 + 11x^2 + 23x + 20$

$$\text{So } (2x + 5)(x^2 + 3x + 4) = 2x^3 + 11x^2 + 23x + 20$$

Be sure to keep the same powers of the variable in the same column.

Next frame

40

Now look at this one.

Example 2

Determine $(2x + 6)(4x^3 - 5x - 7)$

You will notice that the second expression is a cubic (highest power x^3), but that there is no term in x^2. In this case, we insert $0x^2$ in the working to keep the columns complete, that is:

$$\begin{array}{l} 4x^3 + 0x^2 - 5x - 7 \\ \underline{2x \;\; + 6 \qquad\qquad} \end{array}$$

which gives ....

Finish it

41

$$\boxed{8x^4 + 24x^3 - 10x^2 - 44x - 42}$$

Here it is set out:

$$\begin{array}{l} 4x^3 + 0x^2 - 5x - 7 \\ 2x + 6 \\ \hline 8x^4 + 0x^3 - 10x^2 - 14x \\ \phantom{8x^4 +{}} 24x^3 + 0x^2 - 30x - 42 \\ \hline 8x^4 + 24x^3 - 10x^2 - 44x - 42 \\ \hline \end{array}$$

They are all done in the same way, so here is one more for practice.

Example 3

Determine the product $(3x - 5)(2x^3 - 4x^2 + 8)$

You can do that without any trouble. The product is ....

42

$$\boxed{6x^4 - 22x^3 + 20x^2 + 24x - 40}$$

All very straightforward:

$$\begin{array}{l} 2x^3 - 4x^2 + 0x + 8 \\ 3x - 5 \\ \hline 6x^4 - 12x^3 + 0x^2 + 24x \\ - 10x^3 + 20x^2 + 0x - 40 \\ \hline 6x^4 - 22x^3 + 20x^2 + 24x - 40 \\ \hline \end{array}$$

Fractions

43

Algebraic fractions

A numerical fraction is represented by one integer divided by another. Division of symbols follows the same rules to create *algebraic fractions*. For example:

$5 \div 3$ can be written as the fraction $\dfrac{5}{3}$ so $a \div b$ can be written as $\dfrac{a}{b}$

▶

Addition and subtraction

The addition and subtraction of algebraic fractions follow the same rules as the addition and subtraction of numerical fractions – the operations can only be performed when the denominators are the same. For example, just as:

$$\frac{4}{5}+\frac{3}{7}=\frac{4\times 7}{5\times 7}+\frac{3\times 5}{7\times 5}=\frac{4\times 7+3\times 5}{5\times 7}$$

$$=\frac{43}{35} \qquad \text{(where 35 is the LCM of 5 and 7)}$$

so:

$$\frac{a}{b}+\frac{c}{d}=\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}=\frac{ad+cb}{bd} \text{ provided } b\neq 0 \text{ and } d\neq 0$$

(where bd is the LCM of b and d)

So that:

$$\frac{a}{b}-\frac{c}{d^2}+\frac{d}{a}=\dots\dots\dots\dots$$

Answer in the next frame

44

$$\boxed{\frac{a^2d^2-abc+bd^3}{abd^2}}$$

Because

$$\frac{a}{b}-\frac{c}{d^2}+\frac{d}{a}=\frac{aad^2}{bad^2}-\frac{cab}{d^2ab}+\frac{dd^2b}{ad^2b}$$

$$=\frac{a^2d^2-abc+bd^3}{abd^2}$$

where abd^2 is the LCM of a, b and d^2.

On now to the next frame

45 Multiplication and division

Fractions are multiplied by multiplying their numerators and denominators separately. For example, just as:

$$\frac{5}{4} \times \frac{3}{7} = \frac{5 \times 3}{4 \times 7} = \frac{15}{28}$$

so:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

The *reciprocal* of a number is unity divided by the number. For example, the reciprocal of a is $1/a$ and the reciprocal of $\frac{a}{b}$ is:

$$\frac{1}{a/b} = 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}$$ *the numerator and denominator in the divisor are interchanged*

To divide by an algebraic fraction we multiply by its reciprocal. For example:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

So that $\frac{2a}{3b} \div \frac{a^2b}{6} = \ldots\ldots\ldots\ldots$

Check with the next frame

46

$$\boxed{\frac{4}{ab^2}}$$

Because

$$\frac{2a}{3b} \div \frac{a^2b}{6} = \frac{2a}{3b} \times \frac{6}{a^2b} = \frac{4}{ab^2}$$

Try another one:

$$\frac{2a}{3b} \div \frac{a^2b}{6} \times \frac{ab}{2} = \ldots\ldots\ldots\ldots$$

The answer is in the next frame

47

$$\boxed{\frac{2}{b}}$$

Because

$$\frac{2a}{3b} \div \frac{a^2b}{6} \times \frac{ab}{2} = \frac{2a}{3b} \times \frac{6}{a^2b} \times \frac{ab}{2} = \frac{4}{ab^2} \times \frac{ab}{2} = \frac{2}{b}$$

Remember that by the rules of precedence we work through the expression from the left to the right so we perform the division before we multiply. If we were to multiply before dividing in the above expression we should obtain:

$$\begin{aligned}\frac{2a}{3b} \div \frac{a^2b}{6} \times \frac{ab}{2} &= \frac{2a}{3b} \div \frac{a^3b^2}{12} \\ &= \frac{2a}{3b} \times \frac{12}{a^3b^2} \\ &= \frac{24a}{3a^3b^3} = \frac{8}{a^2b^3}\end{aligned}$$

and this would be wrong.

In the following frames we extend this idea

Division of one expression by another

48

Let us consider $(12x^3 - 2x^2 - 3x + 28) \div (3x + 4)$. The result of this division is called the *quotient* of the two expressions and we find the quotient by setting out the division in the same way as we do for the long division of numbers:

$$3x + 4 \overline{\big)\, 12x^3 \;-\; 2x^2 \;-3x\; +28}$$

To make $12x^3$, $3x$ must be multiplied by $4x^2$, so we insert this as the first term in the quotient, multiply the divisor $(3x + 4)$ by $4x^2$, and subtract this from the first two terms:

$$\begin{array}{rl} & 4x^2 \\ 3x+4 & \overline{\big)\, 12x^3 \;-\; 2x^2 \;-3x\; +28} \\ & \underline{12x^3 \;+16x^2} \\ & \quad\; -18x^2 \;-3x \end{array}$$

Bring down the next term $(-3x)$ and repeat the process

To make $-18x^2$, $3x$ must be multiplied by $-6x$, so do this and subtract as before, not forgetting to enter the $-6x$ in the quotient. Do this and we get ....

49

$$\begin{array}{r|l} & 4x^2 \quad - \quad 6x \\ 3x+4 & 12x^3 - 2x^2 - 3x + 28 \\ & \underline{12x^3 + 16x^2} \\ & \quad -18x^2 - 3x \\ & \quad \underline{-18x^2 - 24x} \\ & \qquad\qquad 21x \end{array}$$

Now bring down the next term and continue in the same way and finish it off.

So $(12x^3 - 2x^2 - 3x + 28) \div (3x + 4) = \ldots\ldots\ldots\ldots$

50

$$\boxed{4x^2 - 6x + 7}$$

As before, if an expression has a power missing, insert the power with zero coefficient. Now you can determine $(4x^3 + 13x + 33) \div (2x + 3)$

Set it out as before and check the result with the next frame

51

$$\boxed{2x^2 - 3x + 11}$$

Here it is:

$$\begin{array}{r|l} & 2x^2 \quad -3x \quad +11 \\ 2x+3 & 4x^3 - 0x^2 + 13x + 33 \\ & \underline{4x^3 + 6x^2} \\ & \quad -6x^2 + 13x \\ & \quad \underline{-6x^2 - 9x} \\ & \qquad\quad 22x + 33 \\ & \qquad\quad \underline{22x + 33} \\ & \qquad\quad \underline{\;\cdot \quad\; \cdot\;} \end{array}$$

So $(4x^3 + 13x + 33) \div (2x + 3) = 2x^2 - 3x + 11$

And one more. Determine $(6x^3 - 7x^2 + 1) \div (3x + 1)$

Setting out as before, the quotient is $\ldots\ldots\ldots\ldots$

52

$$\boxed{2x^2 - 3x + 1}$$

After inserting the x term with zero coefficient, the rest is straightforward.

At this point let us pause and summarize the main facts so far for multiplication and division of algebraic expressions

Revision summary 53

1 *Multiplication of algebraic expressions*
Two algebraic expressions are multiplied together by successively multiplying the second expression by each term of the first expression.

Long division of algebraic expressions
Two algebraic expressions are divided by setting out the division in the same way as we do for the long division of numbers.

2 The manipulation of algebraic fractions follows identical principles as those for arithmetic fractions.

3 Only fractions with identical denominators can be immediately added or subtracted.

4 Two fractions are multiplied by multiplying their respective numerators and denominators separately.

5 Two fractions are divided by multiplying the numerator fraction by the reciprocal of the divisor fraction.

Revision exercise 54

1 Perform the following multiplications and simplify your results:

(a) $(2a+4b)(a-3b)$ (b) $(8x-4)(4x^2-3x+2)$

(c) $(9s^2+3)(s^2-4)$ (d) $(2x+3)(5x^3+3x-4)$

2 Simplify each of the following into a single algebraic fraction:

(a) $\dfrac{ab}{c}+\dfrac{cb}{a}$ (b) $\dfrac{ab}{c}-1$ (c) $\left(\dfrac{ab}{c}+\dfrac{ac}{b}\right)+\dfrac{bc}{a}$

3 Perform the following divisions:

(a) $(x^2+5x-6)\div(x-1)$ (b) $(x^2-x-2)\div(x+1)$

(c) $(12x^3-11x^2-25)\div(3x-5)$ (d) $\dfrac{a^3+8b^3}{a+2b}$

55

1 (a) $(2a+4b)(a-3b)=2a(a-3b)+4b(a-3b)=2a^2-2ab-12b^2$

2 (b) $(8x-4)(4x^2-3x+2)=8x(4x^2-3x+2)-4(4x^2-3x+2)$
$$=32x^3-24x^2+16x-16x^2+12x-8$$
$$=32x^3-40x^2+28x-8$$

(c) $(9s^2+3)(s^2-4)=9s^2(s^2-4)+3(s^2-4)$
$$=9s^4-36s^2+3s^2-12$$
$$=9s^4-33s^2-12$$

(d) $(2x+3)(5x^3+3x-4)=2x(5x^3+3x-4)+3(5x^3+3x-4)$
$$=10x^4+6x^2-8x+15x^3+9x-12$$
$$=10x^4+15x^3+6x^2+x-12$$

▶

2 (a) $\dfrac{ab}{c}+\dfrac{cb}{a}=\dfrac{aab}{ac}+\dfrac{cbc}{ac}=\dfrac{b(a^2+c^2)}{ac}$

(b) $\dfrac{ab}{c}-1=\dfrac{ab}{c}-\dfrac{c}{c}=\dfrac{ab-c}{c}$

(c) $\left(\dfrac{ab}{c}+\dfrac{ac}{b}\right)+\dfrac{bc}{a}=\dfrac{ab}{c}+\dfrac{ac}{b}+\dfrac{bc}{a}=\dfrac{a^2b^2+a^2c^2+b^2c^2}{abc}$

3 (a)

$$
(x^2+5x-6)\div(x-1)=
\begin{array}{r|l}
 & \phantom{x^2+{}}x+6 \\ \cline{2-2}
x-1 & x^2+5x-6 \\
 & x^2-x \\ \cline{2-2}
 & \phantom{x^2+{}}6x-6 \\
 & \phantom{x^2+{}}6x-6 \\ \cline{2-2}
 & \phantom{x^2+{}}\bullet \quad \bullet \\ \cline{2-2}
\end{array}
$$

(b)

$$
(x^2-x-2)\div(x+1)=
\begin{array}{r|l}
 & \phantom{x^2-{}}x-2 \\ \cline{2-2}
x+1 & x^2-x-2 \\
 & x^2+x \\ \cline{2-2}
 & -2x-2 \\
 & -2x-2 \\ \cline{2-2}
 & \phantom{x^2-{}}\bullet \quad \bullet \\ \cline{2-2}
\end{array}
$$

(c)

$$
(12x^3-11x^2-25)\div(3x-5)=
\begin{array}{r|l}
 & \phantom{12x^3-{}}4x^2+3x+5 \\ \cline{2-2}
3x-5 & 12x^3-11x^2+0x-25 \\
 & 12x^3-20x^2 \\ \cline{2-2}
 & \phantom{12x^3-{}}9x^2+0x \\
 & \phantom{12x^3-{}}9x^2-15x \\ \cline{2-2}
 & \phantom{12x^3-11x^2+{}}15x-25 \\
 & \phantom{12x^3-11x^2+{}}15x-25 \\ \cline{2-2}
 & \phantom{12x^3-11x^2+{}}\bullet \quad \bullet \\ \cline{2-2}
\end{array}
$$

(d)

$$
\frac{a^3+8b^3}{a+2b}=
\begin{array}{r|l}
 & a^2-2ab+4b^2 \\ \cline{2-2}
a+2b & a^3 \qquad\qquad\qquad +8b^3 \\
 & a^3+2a^2b \\ \cline{2-2}
 & -2a^2b \\
 & -2a^2b-4ab^2 \\ \cline{2-2}
 & \phantom{a^3-2a^2b-{}}4ab^2+8b^3 \\
 & \phantom{a^3-2a^2b-{}}4ab^2+8b^3 \\ \cline{2-2}
 & \phantom{a^3-2a^2b-{}}\bullet \quad \bullet \\ \cline{2-2}
\end{array}
$$

Factorization of algebraic expressions

56

An algebraic fraction can often be simplified by writing the numerator and denominator in terms of their factors and cancelling where possible.

For example $\dfrac{25ab^2 - 15a^2b}{40ab^2 - 24a^2b} = \dfrac{5ab(5b - 3a)}{8ab(5b - 3a)} = \dfrac{5}{8}$

This is an obvious example, but there are many uses for factorization of algebraic expressions in advanced processes.

1 Common factors

The simplest form of factorization is the extraction of highest common factors (HCF) from an expression. For example, $(10x + 8)$ can clearly be written $2(5x + 4)$.

Similarly with $(35x^2y^2 - 10xy^3)$:

the HCF of the coefficients 35 and 10 is 5

the HCF of the powers of x is x

the HCF of the powers of y is y^2

So $(35x^2y^2 - 10xy^3) = 5xy^2(7x - 2y)$

In the same way: (a) $8x^4y^3 + 6x^3y^2 = \ldots\ldots\ldots\ldots$

and (b) $15a^3b - 9a^2b^2 = \ldots\ldots\ldots\ldots$

57

(a) $2x^3y^2(4xy + 3)$

(b) $3a^2b(5a - 3b)$

2 Common factors by grouping

Four-termed expressions can sometimes be factorized by grouping into two binomial expressions and extracting common factors from each.

For example: $2ac + 6bc + ad + 3bd$

$= (2ac + 6bc) + (ad + 3bd) = 2c(a + 3b) + d(a + 3b)$

$= (a + 3b)(2c + d)$

Similarly: $x^3 - 4x^2y + xy^2 - 4y^3 = \ldots\ldots\ldots\ldots$

58

$$\boxed{(x-4y)(x^2+y^2)}$$

Because

$$\begin{aligned} x^3 - 4x^2y + xy^2 - 4y^3 &= (x^3 - 4x^2y) + (xy^2 - 4y^3) \\ &= x^2(x-4y) + y^2(x-4y) \\ &= (x-4y)(x^2+y^2) \end{aligned}$$

In some cases it might be necessary to rearrange the order of the original four terms. For example:

$$\begin{aligned} 12x^2 - y^2 + 3x - 4xy^2 &= 12x^2 + 3x - y^2 - 4xy^2 \\ &= (12x^2 + 3x) - (y^2 + 4xy^2) = 3x(4x+1) - y^2(1+4x) \\ &= (4x+1)(3x-y^2) \end{aligned}$$

Likewise, $20x^2 - 3y^2 + 4xy^2 - 15x = \dots\dots\dots\dots$

59

$$\boxed{(4x-3)(5x+y^2)}$$

Rearranging terms:

$$(20x^2 - 15x) + (4xy^2 - 3y^2) = 5x(4x-3) + y^2(4x-3) = (4x-3)(5x+y^2)$$

60

Useful products of two simple factors

A number of standard results are well-worth remembering for the products of simple factors of the form $(a+b)$ and $(a-b)$. These are:

(a) $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$

i.e. $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2$

i.e. $(a-b)^2 = a^2 - 2ab + b^2$

(c) $(a-b)(a+b) = a^2 + ab - ba - b^2$

i.e. $(a-b)(a+b) = a^2 - b^2$ *the difference of two squares*

For our immediate purpose, these results can be used in reverse:

$$\begin{aligned} a^2 + 2ab + b^2 &= (a+b)^2 \\ a^2 - 2ab + b^2 &= (a-b)^2 \\ a^2 - b^2 &= (a-b)(a+b) \end{aligned}$$

If an expression can be seen to be one of these forms, its factors can be obtained at once.

These expressions that involve the variables raised to the power 2 are examples of what are called *quadratic* expressions. If a quadratic expression can be seen to be one of these forms, its factors can be obtained at once.

On to the next frame

61

Remember

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$
$$a^2 - b^2 = (a-b)(a+b)$$

Example 1

$$x^2 + 10x + 25 = (x)^2 + 2(x)(5) + (5)^2,\text{ like } a^2 + 2ab + b^2$$
$$= (x+5)^2$$

So $x^2 + 10x + 25 = (x+5)^2$

Example 2

$$4a^2 - 12a + 9 = (2a)^2 - 2(2a)(3) + (3)^2,\text{ like } a^2 - 2ab + b^2$$
$$= (2a-3)^2$$

So $4a^2 - 12a + 9 = (2a-3)^2$

Example 3

$$25x^2 - 16y^2 = (5x)^2 - (4y)^2$$
$$= (5x-4y)(5x+4y)$$

So $25x^2 - 16y^2 = (5x-4y)(5x+4y)$

Now can you factorize the following:

(a) $16x^2 + 40xy + 25y^2 = \ldots\ldots\ldots\ldots$

(b) $9x^2 - 12xy + 4y^2 = \ldots\ldots\ldots\ldots$

(c) $(2x+3y)^2 - (x-4y)^2 = \ldots\ldots\ldots\ldots$

62

(a) $(4x+5y)^2$
(b) $(3x-2y)^2$
(c) $(x+7y)(3x-y)$

Quadratic expressions as the product of two simple factors

1 $(x+g)(x+k) = x^2 + (g+k)x + gk$

The coefficient of the middle term is the sum of the two constants g and k and the last term is the product of g and k.

2 $(x-g)(x-k) = x^2 - (g+k)x + gk$

The coefficient of the middle term is minus the sum of the two constants g and k and the last term is the product of g and k.

3 $(x+g)(x-k) = x^2 + (g-k)x - gk$

The coefficient of the middle term is the difference of the two constants g and k and the last term is minus the product of g and k.

Now let's try some specific types of quadratic

63

Factorization of a quadratic expression, $ax^2 + bx + c$ when $a = 1$

If $a = 1$, the quadratic expression is similar to those you have just considered, that is $x^2 + bx + c$. From rules **1–3** in the previous frame you can see that the values of f_1 and f_2 in $(x+f_1)$ and $(x+f_2)$, the factors of the quadratic expression, will depend upon the signs of b and c. Notice that b, c, f_1 and f_2 can be positive or negative. Notice that:

If c is positive (a) f_1 and f_2 are factors of c and both have the sign of b
(b) the sum of f_1 and f_2 is b

If c is negative (a) f_1 and f_2 are factors of c and have opposite signs, the numerically larger having the sign of b
(b) the difference between f_1 and f_2 is b

There are examples of this in the next frame

64

Example 1

$x^2 + 5x + 6$

(a) Possible factors of 6 are (1, 6) and (2, 3), so $(\pm 1, \pm 6)$ and $(\pm 2, \pm 3)$ are possible choices for f_1 and f_2.
(b) c is positive so the required factors add up to b, that is 5.
(c) c is positive so the required factors have the sign of b, that is positive, therefore $(2, 3)$.

$$\text{So } x^2 + 5x + 6 = (x+2)(x+3)$$

▶

Example 2

$x^2 - 9x + 20$

(a) Possible factors of 20 are $(1, 20)$, $(2, 10)$ and $(4, 5)$, so $(\pm 1, \pm 20)$, $(\pm 2, \pm 10)$ and $(\pm 4, \pm 5)$ are possible choices for f_1 and f_2.
(b) c is positive so the required factors add up to b, that is -9.
(c) c is positive so the required factors have the sign of b, that is negative, therefore $(-4, -5)$.

$$\text{So } x^2 - 9x + 20 = (x - 4)(x - 5)$$

Example 3

$x^2 + 3x - 10$

(a) Possible factors of 10 are $(1, 10)$ and $(2, 5)$, so $(\pm 1, \pm 10)$ and $(\pm 2, \pm 5)$ are possible choices for f_1 and f_2.
(b) c is negative so the required factors differ by b, that is 3.
(c) c is negative so the required factors differ in sign, the numerically larger having the sign of b, that is positive, therefore $(-2, 5)$.

$$\text{So } x^2 + 3x - 10 = (x - 2)(x + 5)$$

Example 4

$x^2 - 2x - 24$

(a) Possible factors of 24 are (1, 24), (2, 12), (3, 8) and (4, 6), so $(\pm 1, \pm 24)$, $(\pm 2, \pm 12)$, $(\pm 3, \pm 8)$ and $(\pm 4, \pm 6)$ are possible choices for f_1 and f_2.
(b) c is negative so the required factors differ by b, that is -2.
(c) c is negative so the required factors differ in sign, the numerically larger having the sign of b, that is negative, therefore $(4, -6)$.

$$\text{So } x^2 - 2x - 24 = (x + 4)(x - 6)$$

Now, here is a short exercise for practice. Factorize each of the following into two linear factors:

(a) $x^2 + 7x + 12$	(d) $x^2 + 2x - 24$
(b) $x^2 - 11x + 28$	(e) $x^2 - 2x - 35$
(c) $x^2 - 3x - 18$	(f) $x^2 - 10x + 16$

Finish all six and then check with the next frame

65

(a) $(x + 3)(x + 4)$	(d) $(x + 6)(x - 4)$
(b) $(x - 4)(x - 7)$	(e) $(x - 7)(x + 5)$
(c) $(x - 6)(x + 3)$	(f) $(x - 2)(x - 8)$

66

Factorization of a quadratic expression $ax^2 + bx + c$ when $a \neq 1$

If $a \neq 1$, the factorization is slightly more complicated, but still based on the same considerations as for the simpler examples already discussed.

To factorize such an expression into its linear factors, if they exist, we carry out the following steps.

(a) We obtain $|ac|$, i.e. the numerical value of the product ac ignoring the sign of the product.
(b) We write down all the possible pairs of factors of $|ac|$.
(c) (i) *If c is positive,* we select the two factors of $|ac|$ whose sum is equal to $|b|$: both of these factors have the same sign as b.
   (ii) *If c is negative,* we select the two factors of $|ac|$ which differ by the value of $|b|$: the numerically larger of these two factors has the same sign as that of b and the other factor has the opposite sign.
   (iii) In each case, denote the two factors so obtained by f_1 and f_2.
(d) Then $ax^2 + bx + c$ is now written $ax^2 + f_1x + f_2x + c$ and this is factorized by finding common factors by grouping – as in the previous work.

Example 1

To factorize $6x^2 + 11x + 3 \qquad (ax^2 + bx + c)$

In this case, $a = 6$; $b = 11$; $c = 3$. Therefore $|ac| = 18$

Possible factors of $18 = (1, 18)$, $(2, 9)$ and $(3, 6)$

c is positive. So required factors, f_1 and f_2, add up to ....

67

$\boxed{|b|, \text{ i.e. } 11}$

So required factors are (2, 9).

c is positive. Both factors have the same sign as b, i.e. positive.

So $f_1 = 2$; $f_2 = 9$; and $6x^2 + 11x + 3 = 6x^2 + 2x + 9x + 3$

which factorizes by grouping into ....

68

$\boxed{(2x + 3)(3x + 1)}$

Because

$$\begin{aligned}6x^2 + 11x + 3 &= 6x^2 + 2x + 9x + 3\\ &= (6x^2 + 9x) + (2x + 3)\\ &= 3x(2x + 3) + 1(2x + 3) = (2x + 3)(3x + 1)\end{aligned}$$

Now this one.

▶

Example 2

To factorize $3x^2 - 14x + 8$ $\quad (ax^2 + bx + c)$

$a = 3$; $b = -14$; $c = 8$; $|ac| = 24$

Possible factors of 24 =(1, 24), (2, 12), (3, 8) and (4, 6)

c is positive. So required factors total $|b|$, i.e. 14. Therefore (2, 12)

c is positive. So factors have same sign as b, i.e. negative, $f_1 = -2$; $f_2 = -12$

$$\text{So } 3x^2 - 14x + 8 = 3x^2 - 2x - 12x + 8$$
$$= \ldots\ldots\ldots\ldots$$

Finish it off

69

$$\boxed{(x-4)(3x-2)}$$

Because

$$\begin{aligned} 3x^2 - 2x - 12x + 8 &= (3x^2 - 12x) - (2x - 8) \\ &= 3x(x-4) - 2(x-4) \\ 3x^2 - 14x + 8 &= (x-4)(3x-2) \end{aligned}$$

And finally, this one.

Example 3

To factorize $8x^2 + 18x - 5$ $\quad (ax^2 + bx + c)$

Follow the routine as before and all will be well

So $8x^2 + 18x - 5 = \ldots\ldots\ldots\ldots$

70

$$\boxed{(2x+5)(4x-1)}$$

In this case, $a = 8$; $b = 18$; $c = -5$; $|ac| = 40$

Possible factors of 40 = (1, 40), (2, 20), (4, 10) and (5, 8)

c is negative. So required factors differ by $|b|$, i.e. 18. Therefore (2, 20)

c is negative. So numerically larger factor has sign of b, i.e. positive.

c is negative. So signs of f_1 and f_2 are different, $f_1 = 20$; $f_2 = -2$

$$\begin{aligned} \text{So } 8x^2 + 18x - 5 &= 8x^2 + 20x - 2x - 5 \\ &= 4x(2x+5) - 1(2x+5) \\ &= (2x+5)(4x-1) \end{aligned}$$

Next frame

71

Test for simple factors

Some quadratic equations are not capable of being written as the product of *simple factors* – that is, factors where all the coefficients are integers. To save time and effort, a quick test can be applied before the previous routine is put into action.

To determine whether $ax^2 + bx + c$ can be factorized into two simple factors, first evaluate the expression $(b^2 - 4ac)$.

If $(b^2 - 4ac)$ *is a perfect square*, that is it can be written as k^2 for some integer k, $ax^2 + bx + c$ can be factorized into two simple factors.

If $(b^2 - 4ac)$ *is not a perfect square*, no simple factors of $ax^2 + bx + c$ exist.

Example 1

$3x^2 - 4x + 5 \qquad a = 3;\ b = -4;\ c = 5$

$b^2 - 4ac = 16 - 4 \times 3 \times 5 = 16 - 60 = -44$ (not a perfect square)

There are no simple factors of $3x^2 - 4x + 5$

Now test in the same way

Example 2

$2x^2 + 5x - 3 \qquad a = 2;\ b = 5;\ c = -3$

$b^2 - 4ac = 25 - 4 \times 2 \times (-3) = 25 + 24 = 49 = 7^2$ (perfect square)

$2x^2 + 5x - 3$ can be factorized into simple factors.

Now as an exercise, determine whether or not each of the following could be expressed as the product of two simple factors:

(a) $4x^2 + 3x - 4$ (c) $3x^2 + x - 4$
(b) $6x^2 + 7x + 2$ (d) $7x^2 - 3x - 5$

72

(a) No simple factors	(c) Simple factors
(b) Simple factors	(d) No simple factors

Now we can link this test with the previous work. Work through the following short exercise: it makes useful revision.

Test whether each of the following could be expressed as the product of two simple factors and, where possible, determine those factors:

(a) $2x^2 + 7x + 3$ (c) $7x^2 - 5x - 4$
(b) $5x^2 - 4x + 6$ (d) $8x^2 + 2x - 3$

Check the results with the next frame

73

(a) $(2x+1)(x+3)$	(c) No simple factors
(b) No simple factors	(d) $(2x-1)(4x+3)$

Here is the working:

(a) $2x^2+7x+3 \qquad a=2;\ b=7;\ c=3$

$b^2-4ac=49-4\times 2\times 3=49-24=25=5^2$. Factors exist.

$|ac|=6$; possible factors of 6 are $(1,6)$ and $(2,3)$

c is positive. Factors add up to 7, i.e.$(1,6)$

Both factors have the same sign as b, i.e. positive.

So $f_1=1$ and $f_2=6$

$$2x^2+7x+3=2x^2+x+6x+3$$
$$=(2x^2+x)+(6x+3)=x(2x+1)+3(2x+1)$$
$$=(2x+1)(x+3)$$

(b) $5x^2-4x+6 \qquad a=5;\ b=-4;\ c=6$

$b^2-4ac=16-4\times 5\times 6=16-120=-104$. Not a complete square. Therefore, no simple factors exist.

(c) $7x^2-5x-4 \qquad a=7;\ b=-5;\ c=-4$

$b^2-4ac=25-4\times 7\times(-4)=25+112=137$. Not a complete square. Therefore, no simple factors exist.

(d) $8x^2+2x-3 \qquad a=8;\ b=2;\ c=-3$

$b^2-4ac=4-4\times 8\times(-3)=4+96=100=10^2$. Factors exist.

$|ac|=24$; possible factors of 24 are $(1,24)$, $(2,12)$, $(3,8)$ and $(4,6)$

c is negative. Factors differ by $|b|$, i.e. 2. So $(4,6)$

f_1 and f_2 of opposite signs. Larger factor has the same sign as b, i.e. positive. $f_1=6$; $f_2=-4$.

$$8x^2+2x-3=8x^2+6x-4x-3$$
$$=(8x^2-4x)+(6x-3)=4x(2x-1)+3(2x-1)$$
$$\text{So } 8x^2+2x-3=(2x-1)(4x+3)$$

At this point let us pause and summarize the main facts so far on factorization

74

Revision summary

Factorization of algebraic expressions

(a) Common factors of binomial expressions.

(b) Common factors of expressions by grouping.

Useful standard factors:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 - b^2 = (a - b)(a + b) \quad \text{(Difference of two squares)}$$

Factorization of quadratic expressions of the form $ax^2 + bx + c$.
Test for possibility of simple factors: $(b^2 - 4ac)$ is a complete square.
Determination of factors of $ax^2 + bx + c$:

(a) Evaluate $|ac|$.

(b) Write down all possible factors of $|ac|$.

(c) *If c is positive*, select two factors of $|ac|$ with sum equal to $|b|$.
If c is positive, both factors have the same sign as b.

(d) *If c is negative*, select two factors of $|ac|$ that differ by $|b|$.
If c is negative, the factors have opposite signs, the numerically larger having the same sign as b.

(e) Let the required two factors be f_1 and f_2.
Then $ax^2 + bx + c = ax^2 + f_1x + f_2x + c$ and factorize this by the method of common factors by grouping.

75

Revision exercise

1 Factorize the following:

(a) $18xy^3 - 8x^3y$
(b) $x^3 - 6x^2y - 2xy + 12y^2$
(c) $16x^2 - 24xy - 18x + 27y$
(d) $(x - 2y)^2 - (2x - y)^2$
(e) $x^2 + 7x - 30$
(f) $4x^2 - 36$
(g) $x^2 + 10x + 25$
(h) $3x^2 - 11x - 4$

76

1 (a) $18xy^3 - 8x^3y = 2xy(9y^2 - 4x^2)$
$= 2xy(3y - 2x)(3y + 2x)$

(b) $x^3 - 6x^2y - 2xy + 12y^2 = x^2(x - 6y) - 2y(x - 6y)$
$= (x^2 - 2y)(x - 6y)$

(c) $16x^2 - 24xy - 18x + 27y = (16x^2 - 24xy) - (18x - 27y)$
$= 8x(2x - 3y) - 9(2x - 3y)$
$= (8x - 9)(2x - 3y)$

(d) $(x - 2y)^2 - (2x - y)^2 = x^2 - 4xy + 4y^2 - 4x^2 + 4xy - y^2$
$= 3y^2 - 3x^2$
$= 3(y^2 - x^2)$
$= 3(y - x)(y + x)$

(e) $x^2 + 7x - 30 = x^2 + (10 - 3)x + (10) \times (-3)$
$= (x + 10)(x - 3)$

(f) $4x^2 - 36 = (2x)^2 - (6)^2$
$= (2x - 6)(2x + 6)$

(g) $x^2 + 10x + 25 = x^2 + (2 \times 5)x + 5^2$
$= (x + 5)^2$

(h) $3x^2 - 11x - 4 = 3x(x - 4) + (x - 4)$
$= (3x + 1)(x - 4)$

77

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise.** *Work through these at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

78 Checklist F.2

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Use alphabetic symbols to supplement the numerals and to combine these symbols using all the operations of arithmetic? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 7
• Simplify algebraic expressions by collecting like terms and by abstracting common factors from similar terms? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	8 to 11
• Remove brackets and so obtain alternative algebraic expressions? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	12 to 14
• Manipulate expressions involving powers and logarithms? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	18 to 35
• Multiply two algebraic expressions together? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	39 to 42
• Manipulate algebraic fractions and divide one expression by another? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	43 to 52
• Factorize algebraic expressions using standard factorizations? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	56 to 65
• Factorize quadratic algebraic expressions? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	66 to 73

Test exercise F.2

79 Frames

1 Simplify each of the following:

(a) $2ab - 4ac + ba - 2cb + 3ba$

(b) $3x^2yz - zx^2y + 4yxz^2 - 2x^2zy + 3z^2yx - 3zy^2$

(c) $c^p \times c^{-q} \div c^{-2}$

(d) $\dfrac{\left(x^{\frac{1}{2}}\right)^{-\frac{2}{3}} \div \left(y^{\frac{3}{4}}\right)^2 \times \left(x^{\frac{3}{5}}\right)^{-\frac{5}{3}}}{\left(x^{\frac{1}{4}}\right)^{-1} \times \left(y^{\frac{1}{3}}\right)^6}$ **1** to **22**

2 Remove the brackets in each of the following:

(a) $2f(3g - 4h)(g + 2h)$

(b) $(5x - 6y)(2x + 6y)(5x - y)$

(c) $4\{3[p - 2(q - 3)] - 3(r - 2)\}$ **12** to **17**

3 Evaluate by calculator or by change of base where necessary:

(a) $\log 0{\cdot}0270$ (b) $\ln 47{\cdot}89$ (c) $\log_7 126{\cdot}4$ **23** to **32**

4 Rewrite the following in log form:

(a) $V = \dfrac{\pi h}{4}(D - h)(D + h)$ (b) $P = \dfrac{1}{16}(2d - 1)^2 N\sqrt{S}$

5 Express the following without logarithms:

(a) $\log x = \log P + 2\log Q - \log K - 3$

(b) $\log R = 1 + \dfrac{1}{3}\log M + 3\log S$

(c) $\ln P = \dfrac{1}{2}\ln(Q + 1) - 3\ln R + 2$ **33** to **35**

6 Perform the following multiplications and simplify your results:

(a) $(3x - 1)(x^2 - x - 1)$

(b) $(a^2 + 2a + 2)(3a^2 + 4a + 4)$ **39** to **43**

7 Simplify each of the following:

(a) $\dfrac{x^3}{y^2} \div \dfrac{x}{y^3}$

(b) $\dfrac{ab}{c} \div \dfrac{ac}{b} \times \dfrac{bc}{a}$ **43** to **47**

8 Perform the following divisions:

(a) $(x^2 + 2x - 3) \div (x - 1)$

(b) $\dfrac{n^3 + 27}{n + 3}$

(c) $\dfrac{3a^3 + 2a^2 + 1}{a + 1}$ **48** to **55**

Frames

9 Factorize the following:

(a) $36x^3y^2 - 8x^2y$ (e) $x^2 + 10x + 24$

(b) $x^3 + 3x^2y + 2xy^2 + 6y^3$ (f) $x^2 - 10x + 16$

(c) $4x^2 + 12x + 9$ (g) $x^2 - 5x - 36$

(d) $(3x + 4y)^2 - (2x - y)^2$ (h) $6x^2 + 5x - 6$

56 to 76

Further problems F.2

80

1 Determine the following:

(a) $(2x^2 + 5x - 3)(4x - 7)$

(b) $(4x^2 - 7x + 3)(5x + 6)$

(c) $(5x^2 - 3x - 4)(3x - 5)$

(d) $(6x^3 - 5x^2 - 14x + 12) \div (2x - 3)$

(e) $(15x^3 + 46x^2 - 49) \div (5x + 7)$

(f) $(18x^3 + 13x + 14) \div (3x + 2)$

2 Simplify the following, giving the result without fractional indices

$(x^2 - 1)^2 \times \sqrt{x+1} \div (x - 1)^{\frac{3}{2}}$

3 Simplify:

(a) $\sqrt{a^{\frac{7}{3}}b^5c^{\frac{2}{3}}} \div \sqrt[3]{a^{\frac{1}{2}}b^3c^{-1}}$

(b) $\sqrt[3]{x^9y^{\frac{1}{3}}z^{\frac{1}{2}}} \times y^{\frac{8}{9}} \times (2^{-8}x^6y^2z^{\frac{1}{3}})^{-\frac{1}{2}}$

(c) $(6x^3y^{\frac{5}{2}}z^{\frac{1}{4}})^2 \div (9x^6y^4z^3)^{\frac{1}{2}}$

(d) $(x^2 - y^2)^{\frac{1}{2}} \times (x - y)^{\frac{3}{2}} \times (x + y)^{-\frac{1}{2}}$

4 Evaluate:

(a) $\log 0{\cdot}008472$ (b) $\ln 25{\cdot}47$ (c) $\log_8 387{\cdot}5$

5 Express in log form:

(a) $f = \dfrac{1}{\pi d\sqrt{LC}}$ (b) $K = \dfrac{a^3 \times \sqrt{b}}{c^{\frac{1}{6}}d^{\frac{2}{5}}}$

6 Rewrite the following without logarithms:

(a) $\log W = 2(\log A + \log w) - (\log 32 + 2\log \pi + 2\log r + \log c)$

(b) $\log S = \log K - \log 2 + 2\log \pi + 2\log n + \log y + \log r + 2\log L - 2\log h - \log g$

(c) $\ln I = \ln(2V) - \{\ln(KR + r) - \ln K + KL\}$

▶

7 Factorize the following:

(a) $15x^2y^2 + 20xy^3$

(b) $14a^3b - 12a^2b^2$

(c) $2x^2 + 3xy - 10x - 15y$

(d) $4xy - 7y^2 - 12x + 21y$

(e) $15x^2 + 8y + 20xy + 6x$

(f) $6xy - 20 + 15x - 8y$

(g) $9x^2 + 24xy + 16y^2$

(h) $16x^2 - 40xy + 25y^2$

(i) $25x^3y^4 - 16xy^2$

(j) $(2x + 5y)^2 - (x - 3y)^2$

8 Find simple factors of the following where possible:

(a) $5x^2 + 13x + 6$

(b) $2x^2 - 11x + 12$

(c) $6x^2 - 5x - 6$

(d) $3x^2 + 7x - 4$

(e) $5x^2 - 19x + 12$

(f) $4x^2 - 6x + 9$

(g) $6x^2 - 5x - 7$

(h) $9x^2 - 18x + 8$

(i) $10x^2 + 11x - 6$

(j) $15x^2 - 19x + 6$

(k) $8x^2 + 2x - 15$

Programme F.3

Expressions and equations

Frames 1 to 63

Learning outcomes

When you have completed this Programme you will be able to:

- Numerically evaluate an algebraic expression by substituting numbers for variables
- Recognize the different types of equation
- Evaluate an independent variable
- Change the subject of an equation by transposition
- Evaluate polynomial expressions by 'nesting'
- Use the remainder and factor theorems to factorize polynomials
- Factorize quartic polynomials

If you already feel confident about these why not try the short quiz over the page? You can check your answers at the end of the book.

Quiz F.3

	Frames
1 Given $P = A\left(1 + \frac{r}{100}\right)^n$ find P to 2 dp given that $A = 12\,345{\cdot}66$, $r = 4{\cdot}65$ and $n = 6\frac{255}{365}$.	1 to 4
2 Given $T = 2\pi\sqrt{\frac{l^2 + 4t^2}{3g(r - t)}}$ find: (a) l in terms of T, t, r and g (b) r in terms of T, t, l and g.	7 to 18
3 Write $f(x) = 7x^3 - 6x^2 + 4x + 1$ in nested form and find the value of $f(-2)$.	25 to 29
4 Without dividing in full, determine the remainder of: $(4x^4 + 3x^3 - 2x^2 - x + 7) \div (x + 3)$	30 to 32
5 Factorize $6x^4 + 5x^3 - 39x^2 + 4x + 12$.	33 to 56

Expressions and equations

1

Evaluating expressions

When numerical values are assigned to the variables and constants in an expression, the expression itself assumes a numerical value that is obtained by following the usual precedence rules. This process is known as *evaluating* the expression. For example, if $l = 2$ and $g = 9{\cdot}81$ then the expression:

$$2\pi\sqrt{\frac{l}{g}}$$

is evaluated as:

$$2\pi\sqrt{\frac{2}{9{\cdot}81}} = 2{\cdot}84 \text{ to 2 dp where } \pi = 3{\cdot}14159\ldots$$

So let's look at three examples:

Example 1

If $V = \frac{\pi h}{6}(3R^2 + h^2)$, determine the value of V when $h = 2{\cdot}85$, $R = 6{\cdot}24$ and $\pi = 3{\cdot}142$.

Substituting the given values:

$$\begin{aligned} V &= \frac{3{\cdot}142 \times 2{\cdot}85}{6}(3 \times 6{\cdot}24^2 + 2{\cdot}85^2) \\ &= \frac{3{\cdot}142 \times 2{\cdot}85}{6}(3 \times 38{\cdot}938 + 8{\cdot}123) \\ &= \ldots\ldots\ldots\ldots \end{aligned}$$

Finish it off

2

$$\boxed{V = 186{\cdot}46}$$

Example 2

If $R = \frac{R_1 R_2}{R_1 + R_2}$, evaluate R when $R_1 = 276$ and $R_2 = 145$.

That is easy enough, $R = \ldots\ldots\ldots\ldots$

3

$$\boxed{R = 95{\cdot}06}$$

Now let us deal with a more interesting one.

Example 3

If $V = \dfrac{\pi b}{12}(D^2 + Dd + d^2)$ evaluate V to 3 sig fig when $b = 1{\cdot}46$, $D = 0{\cdot}864$, $d = 0{\cdot}517$ and $\pi = 3{\cdot}142$.

Substitute the values in the expressions and then apply the rules carefully. Take your time with the working: there are no prizes for speed!

$V = \dots\dots\dots\dots$

4

$$\boxed{V = 0{\cdot}558 \text{ to 3 sig fig}}$$

Here it is:

$$\begin{aligned}
V &= \frac{3{\cdot}142 \times 1{\cdot}46}{12}(0{\cdot}864^2 + 0{\cdot}864 \times 0{\cdot}517 + 0{\cdot}517^2)\\
&= \frac{3{\cdot}142 \times 1{\cdot}46}{12}(0{\cdot}7465 + 0{\cdot}864 \times 0{\cdot}517 + 0{\cdot}2673)\\
&= \frac{3{\cdot}142 \times 1{\cdot}46}{12}(0{\cdot}7465 + 0{\cdot}4467 + 0{\cdot}2673)\\
&= \frac{3{\cdot}142 \times 1{\cdot}46}{12}(1{\cdot}4605) = 0{\cdot}5583\dots
\end{aligned}$$

$\therefore V = 0{\cdot}558$ to 3 sig fig

5

Equations

Because different values of the variables and constants produce different values for the expression, we assign these expression values to another variable and so form an *equation*. For example, the equation:

$$r = 2s^3 + 3t$$

states that the variable r can be assigned values by successively assigning values to s and to t, each time evaluating $2s^3 + 3t$. The variable r is called the *dependent* variable and *subject* of the equation whose value *depends* on the values of the *independent* variables s and t.

An *equation* is a statement of the equality of two expressions but there are different types of equation:

Conditional equation

A *conditional equation*, usually just called an *equation*, is true only for certain values of the symbols involved. For example, the equation:

$$x^2 = 4$$

is an equation that is only true for each of the two values $x = +2$ and $x = -2$.

Identity

An *identity* is a statement of equality of two expressions that is true for all values of the symbols for which both expressions are defined. For example, the equation:

$$2(5-x) \equiv 10 - 2x$$

is true no matter what value is chosen for x – it is an *identity*. The expression on the left is not just equal to the expression on the right, it is *equivalent* to it – one expression is an alternative form of the other. Hence the symbol $\equiv$ which stands for 'is equivalent to'.

Defining equation

A *defining equation* is a statement of equality that defines an expression. For example:

$$a^2 \triangleq a \times a$$

Here the symbolism a^2 is defined to mean $a \times a$ where $\triangleq$ means 'is defined to be'.

Assigning equation

An *assigning equation* is a statement of equality that assigns a specific value to a variable. For example:

$$p := 4$$

Here, the value 4 is assigned to the variable p.

Formula

A *formula* is a statement of equality that expresses a mathematical fact where all the variables, dependent and independent, are well-defined. For example, the equation:

$$A = \pi r^2$$

expresses the fact that the area A of a circle of radius r is given as πr^2.

The uses of $\equiv$, $\triangleq$ and $:=$ as connectives are often substituted by the $=$ sign. While it is not strictly correct to do so, it is acceptable.

So what type of equation is each of the following?

(a) $l = \dfrac{T^2 g}{4\pi^2}$ where T is the periodic time and l the length of a simple pendulum and where g is the acceleration due to gravity

(b) $v = 23{\cdot}4$

(c) $4n = 4 \times n$

(d) $x^2 - 2x = 0$

(e) $\dfrac{r^3 - s^3}{r - s} = r^2 + rs + r^2$ where $r \neq s$

The answers are in the next frame

6

(a) Formula
(b) Assigning equation
(c) Defining equation
(d) Conditional equation
(e) Identity

Because

(a) It is a statement of a mathematical fact that relates the values of the variables T and l and the constant g where T, l and g represent well-defined entities.

(b) It assigns the value 23·4 to the variable v.

(c) It defines the notation $4n$ whereby the multiplication sign is omitted.

(d) It is only true for certain values of the variables, namely $x = 0$ and $x = 2$.

(e) The left-hand side is an alternative form of the right-hand side, as can be seen by performing the division. Notice that the expression on the left is not defined when $r = s$ whereas the expression on the right is. We say that the equality only holds true for numerical values when *both* expressions are defined.

Now to the next frame

7

Evaluating independent variables

Sometimes, the numerical values assigned to the variables and constants in a formula include a value for the dependent variable and exclude a value of one of the independent variables. In this case the exercise is to find the corresponding value of the independent variable. For example, given that:

$$T = 2\pi\sqrt{\frac{l}{g}} \text{ where } \pi = 3{\cdot}14 \text{ and } g = 9{\cdot}81$$

what is the length l that corresponds to $T = 1{\cdot}03$? That is, given:

$$1{\cdot}03 = 6{\cdot}28\sqrt{\frac{l}{9{\cdot}81}}$$

find l. We do this by isolating l on one side of the equation.

So we first divide both sides by 6·28 to give: $\dfrac{1{\cdot}03}{6{\cdot}28} = \sqrt{\dfrac{l}{9{\cdot}81}}$

An equation is like a balance, so if any arithmetic operation is performed on one side of the equation the identical operation must be performed on the other side to maintain the balance.

▶

Square both sides to give: $\left(\frac{1\cdot03}{6\cdot28}\right)^2 = \frac{l}{9\cdot81}$ and now multiply both sides by 9·81 to give:

$$9\cdot81\left(\frac{1\cdot03}{6\cdot28}\right)^2 = l = 0\cdot264 \text{ to 3 sig fig}$$

So that if:

$$I = \frac{nE}{R + nr}$$

and $n = 6$, $E = 2\cdot01$, $R = 12$ and $I = 0\cdot98$, the corresponding value of r is ....

Next frame

8

$$\boxed{r = 0\cdot051}$$

Because

Given that $0\cdot98 = \frac{6 \times 2\cdot01}{12 + 6r} = \frac{12\cdot06}{12 + 6r}$ we see, by taking the reciprocal of each side, that:

$$\frac{1}{0\cdot98} = \frac{12 + 6r}{12\cdot06} \text{ and hence } \frac{12\cdot06}{0\cdot98} = 12 + 6r$$

after multiplying both sides by 12·06. Subtracting 12 from both sides yields:

$$\frac{12\cdot06}{0\cdot98} - 12 = 6r$$

and, dividing both sides by 6 gives:

$$\frac{1}{6}\left(\frac{12\cdot06}{0\cdot98} - 12\right) = r, \text{ giving } r = 0\cdot051 \text{ to 2 sig fig}$$

By this process of arithmetic manipulation the independent variable r in the original equation has been *transposed* to become the dependent variable of a new equation, so enabling its value to be found.

You will often encounter the need to transpose a variable in an equation so it is essential that you acquire the ability to do so. Furthermore, you will also need to transpose variables to obtain a new equation rather than just to find the numerical value of the transposed variable as you have done so far. In what follows we shall consider the transposition of variables algebraically rather then arithmetically.

9 Transposition of formulas

The formula for the period of oscillation, T seconds, of a pendulum is given by:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum measured in metres, g is the gravitational constant ($9{\cdot}81\ \mathrm{m\,s^{-2}}$) and $\pi = 3{\cdot}142$ to 4 sig fig. The single symbol on the left-hand side (LHS) of the formula – the dependent variable – is often referred to as the *subject of the formula*. We say that *T is given in terms of l*. What we now require is a new formula where l is the subject. That is, where l is given in terms of T. To effect this transposition, keep in mind the following:

The formula is an equation, or balance. Whatever is done to one side must be done to the other.

To remove a symbol from the right-hand side (RHS) we carry out the opposite operation to that which the symbol is at present involved in. The 'opposites' are – *addition* and *subtraction*, *multiplication* and *division*, *powers* and *roots*.

In this case we start with:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

To isolate l we start by removing the 2π by dividing both sides by 2π. This gives:

$$\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$

We next remove the square root sign by squaring both sides to give:

$$\frac{T^2}{4\pi^2} = \frac{l}{g}$$

Next we remove the g on the RHS by multiplying both sides by g to give:

$$\frac{gT^2}{4\pi^2} = l$$

Finally, we interchange sides to give:

$$l = \frac{gT^2}{4\pi^2}$$

because it is more usual to have the subject of the formula on the LHS.

Now try a few examples

10

Example 1

Transpose the formula $a = \dfrac{2(ut - s)}{t^2}$ to make u the subject.

(a) u is part of the numerator on the RHS. Therefore first multiply both sides by t^2:

$at^2 = 2(ut - s)$

(b) We can now multiply out the bracket:

$at^2 = 2ut - 2s$

(c) Now we isolate the term containing u by adding $2s$ to each side:

$at^2 + 2s = 2ut$

(d) u is here multiplied by $2t$, therefore we divide each side by $2t$:

$\dfrac{at^2 + 2s}{2t} = u$

(e) Finally, write the transposed formula with the new subject on the LHS, i.e. $u = \dfrac{at^2 + 2s}{2t}$

Apply the procedure carefully and take one step at a time.

Example 2

Transpose the formula $d = 2\sqrt{h(2r - h)}$ to make r the subject.

(a) First we divide both sides by 2:

$\dfrac{d}{2} = \sqrt{h(2r - h)}$

(b) To open up the expression under the square root sign, we ....

11

square both sides

So $\dfrac{d^2}{4} = h(2r - h)$

(c) At present, the bracket expression is multiplied by h. Therefore, we ....

12

divide both sides by h

So $\dfrac{d^2}{4h} = 2r - h$

(d) Next, we ....

13

$$\boxed{\text{add } h \text{ to both sides}}$$

So $\dfrac{d^2}{4h} + h = 2r$

Finish it off

14

$$\boxed{r = \frac{1}{2}\left\{\frac{d^2}{4h} + h\right\}}$$

Of course, this could be written in a different form:

$$r = \frac{1}{2}\left\{\frac{d^2}{4h} + h\right\} = \frac{1}{2}\left\{\frac{d^2 + 4h^2}{4h}\right\} = \frac{d^2 + 4h^2}{8h}$$

All these forms are equivalent to each other.

Now, this one.

Example 3

Transpose $V = \dfrac{\pi h(3R^2 + h^2)}{6}$ to make R the subject.

First locate the symbol R in its present position and then take the necessary steps to isolate it. Do one step at a time.

$R = \ldots\ldots\ldots\ldots$

15

$$\boxed{R = \sqrt{\frac{2V}{\pi h} - \frac{h^2}{3}}}$$

Because

$$V = \frac{\pi h(3R^2 + h^2)}{6}$$

$$6V = \pi h(3R^2 + h^2)$$

$$\frac{6V}{\pi h} = 3R^2 + h^2$$

$$\frac{6V}{\pi h} - h^2 = 3R^2 \qquad \text{So } \frac{2V}{\pi h} - \frac{h^2}{3} = R^2$$

$$\text{Therefore } \sqrt{\frac{2V}{\pi h} - \frac{h^2}{3}} = R, \qquad R = \sqrt{\frac{2V}{\pi h} - \frac{h^2}{3}}$$

16

Example 4

This one is slightly different.

Transpose the formula $n = \dfrac{IR}{E - Ir}$ to make I the subject.

In this case, you will see that the symbol I occurs twice on the RHS. Our first step, therefore, is to move the denominator completely by multiplying both sides by $(E - Ir)$:

$$n(E - Ir) = IR$$

Then we can free the I on the LHS by multiplying out the bracket:

$$nE - nIr = IR$$

Now we collect up the two terms containing I on to the RHS:

$$\begin{aligned} nE &= IR + nIr \\ &= I(R + nr) \end{aligned}$$

So $I = \dfrac{nE}{R + nr}$

Move on to the next frame

17

Example 5

Here is one more, worked in very much the same way as the previous example, so you will have no trouble.

Transpose the formula $\dfrac{R}{r} = \sqrt{\dfrac{f + P}{f - P}}$ to make f the subject.

Work right through it, using the rules and methods of the previous examples.

$f = \ldots\ldots\ldots\ldots$

18

$$\boxed{f = \frac{(R^2 + r^2)P}{R^2 - r^2}}$$

Here it is:

$$\frac{R^2}{r^2} = \frac{f + P}{f - P}$$

$$\frac{R^2}{r^2}(f - P) = f + P$$

$$R^2(f - P) = r^2(f + P)$$

$$R^2 f - R^2 P = r^2 f + r^2 P$$

$$R^2 f - r^2 f = R^2 P + r^2 P$$

$$f(R^2 - r^2) = P(R^2 + r^2) \qquad \text{So } f = \frac{(R^2 + r^2)P}{R^2 - r^2}$$

At this point let us pause and summarize the main facts so far

19

Revision summary

1 An algebraic expression is evaluated by substituting numbers for the variables and constants in the expression and then using the arithmetic precedence rules.

2 Values so obtained can be assigned to a variable to form an equation. This variable is called the *subject* of the equation.

3 The subject of an equation is called the *dependent variable* and the variables within the expression are called the *independent variables*.

4 *There is more than one type of equation:*

Conditional equation
Identity
Defining equation
Assigning equation
Formula

5 Any one of the independent variables in a formula can be made the subject of a new formula obtained by transposing it with the dependent variable of the original formula.

6 Transposition is effected by performing identical arithmetic operations on both sides of the equation.

20

Revision exercise

1 Evaluate each of the following to 3 sig fig:

(a) $I = \dfrac{nE}{R + nr}$ where $n = 4$, $E = 1{\cdot}08$, $R = 5$ and $r = 0{\cdot}04$

(b) $A = P\left(1 + \dfrac{r}{100}\right)^n$ where $P = 285{\cdot}79$, $r = 5{\cdot}25$ and $n = 12$

(c) $P = A\dfrac{(nv/u)^{\frac{3}{2}}}{1 + (nv/u)^3}$ where $A = 40$, $u = 30$, $n = 2{\cdot}5$ and $v = 42{\cdot}75$

2 Transpose the formula $f = \dfrac{S(M - m)}{M + m}$ to make m the subject.

21

1 (a) $I = \dfrac{4 \times 1{\cdot}08}{5 + 4 \times 0{\cdot}04} = \dfrac{4{\cdot}32}{5 + 0{\cdot}16} = 0{\cdot}837$

(b) $A = 285{\cdot}79\left(1 + \dfrac{5{\cdot}25}{100}\right)^{12} = 285{\cdot}79(1{\cdot}0525)^{12}$

$= 285{\cdot}79 \times 1{\cdot}84784\ldots = 528$

(c) $P = 40\dfrac{(2{\cdot}5 \times 42{\cdot}75 \div 30)^{\frac{3}{2}}}{1 + (2{\cdot}5 \times 42{\cdot}75 \div 30)^3} = 40\dfrac{3{\cdot}5625^{\frac{3}{2}}}{1 + 3{\cdot}5625^3} = 40\dfrac{3{\cdot}5625^{\frac{3}{2}}}{1 + 3{\cdot}5625^3}$

$= 40\dfrac{6{\cdot}7240\ldots}{46{\cdot}2131\ldots} = 5{\cdot}82$

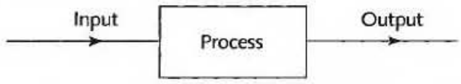
2 $f = \dfrac{S(M - m)}{M + m}$ so $f(M + m) = S(M - m)$ thus $fM + fm = SM - Sm$, that is $fm + Sm = SM - fM$. Factorizing yields $m(f + S) = M(S - f)$ giving $m = \dfrac{M(S - f)}{f + S}$.

And now to the next topic

The evaluation process

22 Systems

A *system* is a process that is capable of accepting an *input*, *processing* the input and producing an *output*:



We can use this idea of a system to describe the way we evaluate an algebraic expression. For example, given the expression:

$3x - 4$

we evaluate it for $x = 5$, say, by multiplying 5 by 3 and then subtracting 4 to obtain the value 11; we *process* the *input* 5 to produce the *output* 11:

If we use the letter x to denote the input and the letter f to denote the process we denote the output as:

$f(x)$, that is 'f acting on x'

where the process f, represented by the box in the diagram, is:

multiply x by 3 and then subtract 4

How the evaluation is actually done, whether mentally, by pen and paper or by using a calculator is not important. What is important is that the prescription for evaluating it is given by the expression $3x - 4$ and that we can represent the *process* of executing this prescription by the label f.

The advantage of this notion is that it permits us to tabulate the results of evaluation in a meaningful way. For example, if:

$f(x) = 3x - 4$

then:

$f(5) = 15 - 4 = 11$

and in this way the corresponding values of the two variables are recorded.

So that, if $f(x) = 4x^3 - \dfrac{6}{2x}$ then:

(a) $f(3) =$ ....

(b) $f(-4) =$ ....

(c) $f(2/5) =$ ....

(d) $f(-3{\cdot}24) =$ .... (to 5 sig fig)

Answers are in the next frame

23

(a) $f(3) = 107$
(b) $f(-4) = -255{\cdot}25$
(c) $f(2/5) = -7{\cdot}244$
(d) $f(-3{\cdot}24) = -135{\cdot}12$

Because

(a) $f(3) = 4 \times 3^3 - \dfrac{6}{2 \times 3} = 108 - 1 = 107$

(b) $f(-4) = 4 \times (-4)^3 - \dfrac{6}{2 \times (-4)} = -256 + 0{\cdot}75 = -255{\cdot}25$

(c) $f(2/5) = 4 \times (2/5)^3 - \dfrac{6}{2 \times (2/5)} = 0{\cdot}256 - 7{\cdot}5 = -7{\cdot}244$

(d) $f(-3{\cdot}24) = 4 \times (-3{\cdot}24)^3 - \dfrac{6}{2 \times (-3{\cdot}24)} = -136{\cdot}05 + 0{\cdot}92593 = -135{\cdot}12$

Polynomial equations

Polynomial expressions

24

A *polynomial* in x is an expression involving powers of x, normally arranged in descending (or sometimes ascending) powers. The degree of the polynomial is given by the highest power of x occurring in the expression, for example:

$5x^4 + 7x^3 + 3x - 4$ is a polynomial of the 4th degree

and $2x^3 + 4x^2 - 2x + 7$ is a polynomial of the 3rd degree.

Polynomials of low degree often have alternative names:

$2x - 3$ is a polynomial of the 1st degree – or a *linear* expression.

$3x^2 + 4x + 2$ is a polynomial of the 2nd degree – or a *quadratic* expression.

A polynomial of the 3rd degree is often referred to as a *cubic* expression.

A polynomial of the 4th degree is often referred to as a *quartic* expression.

Evaluation of polynomials

If $f(x) = 3x^4 - 5x^3 + 7x^2 - 4x + 2$, then evaluating $f(3)$ would involve finding the values of each term before finally totalling up the five individual values. This would mean recording the partial values – with the danger of including errors in the process.

This can be avoided by using the method known as *nesting* – so move to the next frame to see what it entails.

25

Evaluation of a polynomial by nesting

Consider the polynomial $f(x) = 5x^3 + 2x^2 - 3x + 6$. To express this in *nested* form, write down the coefficient and one factor x from the first term and add on the coefficient of the next term:

i.e. $5x + 2$

Enclose these in brackets, multiply by x and add on the next coefficient:

i.e. $(5x + 2)x - 3$

Repeat the process: enclose the whole of this in square brackets, multiply by x and add on the next coefficient:

i.e. $[(5x + 2)x - 3]x + 6$

So $f(x) = 5x^3 + 2x^2 - 3x + 6$

$= [(5x + 2)x - 3]x + 6$ in nested form.

Starting with the innermost brackets, we can now substitute the given value of x and carry on in a linear fashion. No recording is required:

$f(4) = [(22)4 - 3]4 + 6$

$= [85]4 + 6 = 346 \qquad$ So $f(4) = 346$

Note: The working has been set out here purely by way of explanation. Normally it would be carried out mentally.

So, in the same way, $f(2) = \dots\dots\dots\dots$ and $f(-1) = \dots\dots\dots\dots$

26

$\boxed{48;\ 6}$

Notes: (a) The terms of the polynomial must be arranged in descending order of powers.

(b) If any power is missing from the polynomial, it must be included with a zero coefficient before nesting is carried out.

Therefore, if $f(x) = 3x^4 + 2x^2 - 4x + 5$

(a) $f(x)$ in nested form $= \dots\dots\dots\dots$

(b) $f(2) = \dots\dots\dots\dots$

27

(a) $f(x) = \{[(3x + 0)x + 2]x - 4\}x + 5$
(b) $f(2) = 53$

On to the next frame

28

Now a short exercise. In each of the following cases, express the polynomial in nested form and evaluate the function for the given value of x:

(a) $f(x) = 4x^3 + 3x^2 + 2x - 4$ $[x = 2]$

(b) $f(x) = 2x^4 + x^3 - 3x^2 + 5x - 6$ $[x = 3]$

(c) $f(x) = x^4 - 3x^3 + 2x - 3$ $[x = 5]$

(d) $f(x) = 2x^4 - 5x^3 - 3x^2 + 4$ $[x = 4]$

Results in the next frame

29

(a) $[(4x+3)x+2]x-4$	$f(2) = 44$
(b) $\{[(2x+1)x-3]x+5\}x-6$	$f(3) = 171$
(c) $\{[(x-3)x+0]x+2\}x-3$	$f(5) = 257$
(d) $\{[(2x-5)x-3]x+0\}x+4$	$f(4) = 148$

This method for evaluating polynomials will be most useful in the following work, so let us now move on to the next topic.

Remainder theorem

30

The *remainder theorem* states that if a polynomial $f(x)$ is divided by $(x-a)$, the quotient will be a polynomial $g(x)$ of one degree less than the degree of $f(x)$, together with a remainder R still to be divided by $(x-a)$ [see Frame 24]. That is:

$$\frac{f(x)}{x-a} = g(x) + \frac{R}{x-a}$$

$$\text{So } f(x) = (x-a).g(x) + R$$

When $x = a$ $\quad f(a) = 0.g(a) + R \quad$ i.e. $R = f(a)$

That is:

If $f(x)$ were to be divided by $(x-a)$, the remainder would be $f(a)$.

So, $(x^3 + 3x^2 - 13x - 10) \div (x-3)$ would give a remainder $R = f(3) = \ldots\ldots\ldots\ldots$

31

$\boxed{5}$

Because $f(x) = x^3 + 3x^2 - 13x - 10 = [(x+3)x - 13]x - 10$

so $f(3) = 5$

We can verify this by actually performing the long division:

$(x^3 + 3x^2 - 13x - 10) \div (x-3) = \ldots\ldots\ldots\ldots$

32

$\boxed{x^2 + 6x + 5 \text{ with remainder } 5}$

Here it is:

$$
\begin{array}{r|l}
 & x^2 + 6x + 5 \\
\hline
x-3 & x^3 + 3x^2 - 13x - 10 \\
 & x^3 - 3x^2 \\
\hline
 & \quad 6x^2 - 13x \\
 & \quad 6x^2 - 18x \\
\hline
 & \qquad 5x - 10 \\
 & \qquad 5x - 15 \\
\hline
 & \qquad\quad 5 \leftarrow \text{Remainder}
\end{array}
$$

Now as an exercise, apply the remainder theorem to determine the remainder in each of the following cases.

(a) $(5x^3 - 4x^2 - 3x + 6) \div (x - 2)$

(b) $(4x^3 - 3x^2 + 5x - 3) \div (x - 4)$

(c) $(x^3 - 2x^2 - 3x + 5) \div (x - 5)$

(d) $(2x^3 + 3x^2 - x + 4) \div (x + 2)$

(e) $(3x^3 - 11x^2 + 10x - 12) \div (x - 3)$

Finish all five and then check with the next frame

33

(a) 24	(d) 2
(b) 225	(e) 0
(c) 65	

Factor theorem

If $f(x)$ is a polynomial and substituting $x = a$ gives a remainder of zero, i.e. $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

For example, if $f(x) = x^3 + 2x^2 - 14x + 12 = [(x + 2)x - 14]x + 12$ and we substitute $x = 2$, $f(2) = 0$, so that division of $f(x)$ by $(x - 2)$ gives a zero remainder, i.e. $(x - 2)$ is a factor of $f(x)$. The remaining factor can be found by long division of $f(x)$ by $(x - 2)$.

$f(x) = (x - 2)(\dots\dots\dots\dots)$

34

$\boxed{x^2 + 4x - 6}$

$$\text{So } f(x) = (x - 2)(x^2 + 4x - 6)$$

The quadratic factor so obtained can sometimes be factorized further into two simple factors, so we apply the $(b^2 - 4ac)$ test – which we have used before. In this particular case $(b^2 - 4ac) = \dots\dots\dots\dots$

35

$\boxed{40}$

because $(b^2 - 4ac) = 16 - [4 \times 1 \times (-6)] = 16 + 24 = 40$. This is not a perfect square, so no simple factors exist. Therefore, we cannot factorize further.

So $f(x) = (x - 2)(x^2 + 4x - 6)$

As an example, test whether $(x - 3)$ is a factor of $f(x) = x^3 - 5x^2 - 2x + 24$ and, if so, determine the remaining factor (or factors).

$$f(x) = x^3 - 5x^2 - 2x + 24 = [(x - 5)x - 2]x + 24$$

$$\therefore\ f(3) = \ldots\ldots\ldots\ldots$$

36

$\boxed{0}$

There is no remainder. So $(x - 3)$ is a factor of $f(x)$. Long division now gives the remaining quadratic factor, so that $f(x) = (x - 3)(\ldots\ldots\ldots\ldots)$

37

$\boxed{x^2 - 2x - 8}$

$$
\begin{array}{r|l}
 & \qquad\quad x^2 - 2x - 8 \\ \cline{2-2}
x - 3 & x^3 - 5x^2 - 2x + 24 \\
 & \underline{x^3 - 3x^2} \\
 & \quad - 2x^2 - 2x \\
 & \quad \underline{- 2x^2 + 6x} \\
 & \qquad\quad - 8x + 24 \\
 & \qquad\quad \underline{- 8x + 24} \\
 & \qquad\qquad \underline{\bullet \quad \bullet}
\end{array}
$$

$f(x) = (x - 3)(x^2 - 2x - 8)$

Now test whether $x^2 - 2x - 8$ can be factorized further.

$b^2 - 4ac = \ldots\ldots\ldots\ldots$

38

$\boxed{36, \text{ i.e. } 6^2}$

$(b^2 - 4ac) = 6^2$. Therefore there are simple factors of $x^2 - 2x - 8$. We have previously seen how to factorize a quadratic expression when such factors exist and, in this case:

$$x^2 - 2x - 8 = (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

39

$$\boxed{(x-4)(x+2)}$$

Collecting our results together:

$$\begin{aligned} f(x) &= x^3 - 5x^2 - 2x + 24 \\ &= (x-3)(x^2 - 2x - 8) \\ &= (x-3)(x-4)(x+2) \end{aligned}$$

And now another example. Show that $(x-4)$ is a factor of $f(x) = x^3 - 6x^2 - 7x + 60$ and, as far as possible, factorize $f(x)$ into simple factors.

Work through it just as before and then check with the next frame

40

$$\boxed{f(x) = (x-4)(x+3)(x-5)}$$

Here it is: $f(x) = x^3 - 6x^2 - 7x + 60 = [(x-6)x - 7]x + 60$

$f(4) = 0$, so $(x-4)$ is a factor of $f(x)$.

$$\begin{array}{r|rrrr} & x^2 & -\ 2x & -\ 15 & \\ \hline x-4 & x^3 - 6x^2 & -\ 7x & +\ 60 & \\ & x^3 - 4x^2 & & & \\ \hline & -2x^2 & -\ 7x & & \\ & -2x^2 & +\ 8x & & \\ \hline & & -15x & +\ 60 & \\ & & -15x & +\ 60 & \\ \hline & & \bullet & \bullet & \quad f(x) = (x-4)(x^2 - 2x - 15) \end{array}$$

Now we attend to the quadratic factor. $(b^2 - 4ac) = 64$, i.e. 8^2. This is a complete square. Simple factors exist.

$$x^2 - 2x - 15 = (x+3)(x-5)$$

$$\text{so } f(x) = (x-4)(x+3)(x-5)$$

But how do we proceed if we are not given the first factor? We will attend to that in the next frame.

41

If we are not given the first factor, we proceed as follows:

(a) We write the cubic function in nested form.

(b) By trial and error, we substitute values of x, e.g. $x = 1$, $x = -1$, $x = 2$, $x = -2$ etc. until we find a substitution $x = k$ that gives a zero remainder. Then $(x - k)$ is a factor of $f(x)$.

After that, of course, we can continue as in the previous examples.

So, to factorize $f(x) = x^3 + 5x^2 - 2x - 24$ as far as possible, we first write $f(x)$ in nested form, i.e. ....

42

$$\boxed{f(x) = [(x+5)x-2]x-24}$$

Now substitute values $x = k$ for x until $f(k) = 0$:

$f(1) = -20$	so $(x-1)$ is not a factor
$f(-1) = -18$	so $(x+1)$ is not a factor
$f(2) = 0$	so $(x-2)$ *is* a factor of $f(x)$

Now you can work through the rest of it, giving finally that

$f(x) = \dots\dots\dots\dots$

43

$$\boxed{f(x) = (x-2)(x+3)(x+4)}$$

Because the long division gives $f(x) = (x-2)(x^2+7x+12)$ and factorizing the quadratic expression finally gives

$f(x) = (x-2)(x+3)(x+4)$

And now one more, entirely on your own:

Factorize $f(x) = 2x^3 - 9x^2 + 7x + 6$

There are no snags. Just take your time. Work through the same steps as before and you get

$f(x) = \dots\dots\dots\dots$

44

$$\boxed{f(x) = (x-2)(x-3)(2x+1)}$$

$$f(x) = 2x^3 - 9x^2 + 7x + 6 = [(2x-9)x+7]x+6$$

$x = 2$ is the first substitution to give $f(x) = 0$. So $(x-2)$ is a factor.

Long division then leads to $f(x) = (x-2)(2x^2-5x-3)$.

$(b^2-4ac) = 49$, i.e. 7^2, showing that simple factors exist for the quadratic.

In fact $2x^2 - 5x - 3 = (x-3)(2x+1)$

so $f(x) = (x-2)(x-3)(2x+1)$

Factorization of quartic polynomials

45

The same method can be applied to polynomials of the fourth degree, provided that the given function has at least one simple factor.

For example, to factorize $f(x) = 2x^4 - x^3 - 8x^2 + x + 6$:

In nested form, $f(x) = \{[(2x - 1)x - 8]x + 1\}x + 6$

$f(1) = 0$. so $(x - 1)$ *is* a factor.

$$
\begin{array}{r|l}
 & \quad 2x^3 + x^2 - 7x - 6 \\
x - 1 & 2x^4 - x^3 - 8x^2 + x + 6 \\
 & 2x^4 - 2x^3 \\
\hline
 & \quad x^3 - 8x^2 \\
 & \quad x^3 - x^2 \\
\hline
 & \quad -7x^2 + x \\
 & \quad -7x^2 + 7x \\
\hline
 & \quad -6x + 6 \\
 & \quad -6x + 6 \\
\hline
 & \quad \cdot \quad \cdot
\end{array}
$$

So $f(x) = 2x^4 - x^3 - 8x^2 + x + 6$

$= (x - 1)(2x^3 + x^2 - 7x - 6) = (x - 1).g(x)$

Now we can proceed to factorize $g(x) = (2x^3 + x^2 - 7x - 6)$ as we did with previous cubics:

$g(x) = [(2x + 1)x - 7]x - 6$

$g(1) = -10$ so $(x - 1)$ is not a factor of $g(x)$

$g(-1) = 0$ so $(x + 1)$ *is* a factor of $g(x)$

Long division shows that $g(x) = (x + 1)(2x^2 - x - 6)$

so $f(x) = (x - 1)(x + 1)(2x^2 - x - 6)$

Attending to the quadratic $(2x^2 - x - 6)$:

$(b^2 - 4ac) = 1 + 48 = 49 = 7^2$ There are simple factors.

In fact $2x^2 - x - 6 = (2x + 3)(x - 2)$

Finally, then, $f(x) = (x - 1)(x + 1)(x - 2)(2x + 3)$

On to the next frame for another example

46

Factorize $f(x) = x^4 + x^3 - 9x^2 + x + 10$.

First, in nested form, $f(x) = \dots\dots\dots\dots$

$$\boxed{f(x) = \{[(x+1)x - 9]x + 1\}x + 10}$$ 47

Now we substitute $x = 1, -1, 2, \ldots$ from which we get

$f(1) = 4$ so $(x-1)$ is not a factor

$f(-1) = 0$ so $(x+1)$ *is* a factor

$$\begin{array}{r|llll}
 & x^3 & +0x^2 & -\ 9x & +10 \\ \hline
x+1 & x^4 + x^3 & -9x^2 & +\ \ x & +10 \\
 & x^4 + x^3 & & & \\ \cline{2-2}
 & \bullet\quad\bullet & -9x^2 & +\ \ x & \\
 & & -9x^2 & -\ 9x & \\ \cline{3-4}
 & & & 10x & +10 \\
 & & & 10x & +10 \\ \cline{4-5}
 & & & \bullet & \bullet \\ \cline{4-5}
\end{array}$$

So $f(x) = (x+1)(x^3 + 0x^2 - 9x + 10) = (x+1).g(x)$

Then in nested form, $g(x) = \ldots\ldots\ldots\ldots$

$$\boxed{g(x) = [(x+0)x - 9]x + 10}$$ 48

Now we hunt for factors by substituting $x = 1, -1, 2 \ldots$ in $g(x)$

$g(1) = \ldots\ldots\ldots\ldots;\ g(-1) = \ldots\ldots\ldots\ldots;\ g(2) = \ldots\ldots\ldots\ldots$

$$\boxed{g(1) = 2;\ g(-1) = 18;\ g(2) = 0}$$ 49

$g(2) = 0$ so $5(x-2)$ *is* a factor of $g(x)$

Long division $(x^3 + 0x^2 - 9x + 10) \div (x-2)$ gives the quotient $\ldots\ldots\ldots\ldots$

$$\boxed{x^2 + 2x - 5}$$ 50

$f(x) = (x+1)(x-2)(x^2 + 2x - 5)$, so we finally test the quadratic factor for simple factors and finish it off.

There are no linear factors of the quadratic

so $f(x) = (x+1)(x-2)(x^2 + 2x - 5)$

One stage of long division can be avoided if we can find two factors of the original polynomial. 51

For example, factorize $f(x) = 2x^4 - 5x^3 - 15x^2 + 10x + 8$.

In nested form, $f(x) \ldots\ldots\ldots\ldots$

52

$$\boxed{f(x) = \{[(2x-5)x-15]x+10\}x+8}$$

$f(1) = 0$ so $(x-1)$ *is* a factor of $f(x)$
$f(-1) = -10$ so $(x+1)$ is not a factor of $f(x)$
$f(2) = -40$ so $(x-2)$ is not a factor of $f(x)$
$f(-2) = \ldots\ldots\ldots\ldots$

53

$$\boxed{f(-2) = 0 \text{ so } (x+2) \textit{ is} \text{ a factor of } f(x)}$$

$$\begin{aligned} f(x) &= (x-1)(x+2)(ax^2+bx+c) \\ &= (x^2+x-2)(ax^2+bx+c) \end{aligned}$$

We can now find the quadratic factor by dividing $f(x)$ by (x^2+x-2):

$$\begin{array}{r|rrrrr}
 & 2x^2 & -7x & -4 & & \\ \hline
x^2+x-2 & 2x^4 & -5x^3 & -15x^2 & +10x & +8 \\
 & 2x^4 & +2x^3 & -4x^2 & & \\ \hline
 & & -7x^3 & -11x^2 & +10x & \\
 & & -7x^3 & -7x^2 & +14x & \\ \hline
 & & & -4x^2 & -4x & +8 \\
 & & & -4x^2 & -4x & +8 \\ \hline
 & & & \cdot & \cdot & \cdot \\ \hline
\end{array}$$

So $f(x) = (x-1)(x+2)(2x^2-7x-4)$

Finally test the quadratic factor for simple factors and finish it off.

$f(x) = \ldots\ldots\ldots\ldots$

54

$$\boxed{f(x) = (x-1)(x+2)(x-4)(2x+1)}$$

For the quadratic, $(2x^2-7x-4)$, $b^2-4ac = 81 = 9^2$ so factors exist.

In fact, $2x^2-7x-4 = (x-4)(2x+1)$

so $f(x) = (x-1)(x+2)(x-4)(2x+1)$

Next frame

55

Now one further example that you can do on your own. It is similar to the previous one.

Factorize $f(x) = 2x^4-3x^3-14x^2+33x-18$.

Follow the usual steps and you will have no trouble.

$f(x) = \ldots\ldots\ldots\ldots$

56

$$\boxed{f(x) = (x-1)(x-2)(x+3)(2x-3)}$$

Here is the working:

$$\begin{aligned} f(x) &= 2x^4 - 3x^3 - 14x^2 + 33x - 18 \\ &= \{[(2x-3)x - 14]x + 33\}x - 18 \end{aligned}$$

$$\begin{aligned} f(1) &= 0 && \text{so } (x-1) \text{ is a factor of } f(x) \\ f(-1) &= -60 && \text{so } (x+1) \text{ is not a factor of } f(x) \\ f(2) &= 0 && \text{so } (x-2) \text{ is a factor of } f(x) \end{aligned}$$

$$\begin{aligned} \text{So } f(x) &= (x-1)(x-2)(ax^2 + bx + c) \\ &= (x^2 - 3x + 2)(ax^2 + bx + c) \end{aligned}$$

$$\begin{array}{r|rrrrr} & & & 2x^2 & +3x & -9 \\ \hline x^2 - 3x + 2 & 2x^4 & -3x^3 & -14x^2 & +33x & -18 \\ & 2x^4 & -6x^3 & +4x^2 & & \\ \hline & & 3x^3 & -18x^2 & +33x & \\ & & 3x^3 & -9x^2 & +6x & \\ \hline & & & -9x^2 & +27x & -18 \\ & & & -9x^2 & +27x & -18 \\ \hline & & & \cdot & \cdot & \cdot \end{array}$$

So $f(x) = (x-1)(x-2)(2x^2 + 3x - 9)$

For $2x^2 + 3x - 9$, $(b^2 - 4ac) = 81 = 9^2$ Simple factors exist.

So $f(x) = (x-1)(x-2)(x+3)(2x-3)$

At this point let us pause and summarize the main facts so far on evaluating polynomial equations, the remainder theorem and the factor theorem.

Revision summary 57

1 The process whereby we evaluate an algebraic expression can be described as a system f which accepts input x, processes the input and produces an output $f(x)$.

2 *Remainder theorem*
If a polynomial $f(x)$ is divided by $(x-a)$, the quotient will be a polynomial $g(x)$ of one degree less than that of $f(x)$, together with a remainder R such that $R = f(a)$.

3 *Factor theorem*
If $f(x)$ is a polynomial and substituting $x = a$ gives a remainder of zero, i.e. $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.

58

Revision exercise

1 Rewrite $f(x) = 6x^3 - 5x^2 + 4x - 3$ in nested form and determine the value of $f(2)$.

2 Without dividing in full, determine the remainder of $(x^4 - 2x^3 + 3x^2 - 4) \div (x - 2)$.

3 Factorize $6x^4 + x^3 - 25x^2 - 4x + 4$.

59

1 $f(x) = 6x^3 - 5x^2 + 4x - 3 = (((6x - 5)x + 0)x + 4)x - 3$ so that
$f(2) = (((12 - 5)2 + 0)2 + 4)2 - 3 = 61$

2 $f(x) = x^4 - 2x^3 + 3x^2 - 4 = (((x - 2)x + 3)x + 0)x - 4$ so that
$f(2) = (((2 - 2)2 + 3)2 + 0)2 - 4 = 8 =$ remainder

3 $f(x) = 6x^4 + x^3 - 25x^2 - 4x + 4 = (((6x + 1)x - 25)x - 4)x + 4$
$f(2) = 0$ so $x - 2$ *is* a factor, $f(-2) = 0$ so $x + 2$ *is* a factor.
$(x - 2)(x + 2) = x^2 - 4$ and:

$$
\begin{array}{r|lllll}
 & 6x^2 & +\ x & & -\ 1 & \\
\hline
x^2 - 4 & 6x^4 & +x^3 & -25x^2 & -4x & +4 \\
 & 6x^4 & & -24x^2 & & \\
\cline{2-4}
 & & x^3 & -\ x^2 & & \\
 & & x^3 & & -4x & \\
\cline{3-5}
 & & & -\ x^2 & & +4 \\
 & & & -\ x^2 & & +4 \\
\cline{4-6}
 & & & \bullet & & \bullet \\
\cline{4-6}
\end{array}
$$

Furthermore, $6x^2 + x - 1 = (2x + 1)(3x - 1)$ so that:

$$6x^4 + x^3 - 25x^2 - 4x + 4 = (x + 2)(x - 2)(2x + 1)(3x - 1)$$

60

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through them at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

Checklist F.3

61

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Numerically evaluate an algebraic expression by substituting numbers for variables? 1 to 4
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognize the different types of equation? 5 to 6
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate an independent variable? 7 to 8
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Change the subject of an equation by transposition? 9 to 18
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate polynomial expressions by 'nesting'? 25 to 29
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the remainder and factor theorems to factorize polynomials? 30 to 44
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Factorize quartic polynomials? 45 to 56
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.3

62

1 Evaluate to 3 sig fig: Frames

$\sqrt{\dfrac{2V}{\pi h}-\dfrac{h^2}{3}}$ where

(a) $V = 23{\cdot}05$ and $h = 2{\cdot}69$

(b) $V = 85{\cdot}67$ and $h = 5{\cdot}44$. 1 to 4

2 Given $q = \sqrt{(5p^2-1)^2-2}$ find p in terms of q. 7 to 18

▶

Frames

3 Express each of the following functions in nested form and determine the value of the function for the stated value of x:

(a) $f(x) = 2x^3 - 3x^2 + 5x - 4$ $f(3) = \ldots\ldots\ldots\ldots$

(b) $f(x) = 4x^3 + 2x^2 - 7x - 2$ $f(2) = \ldots\ldots\ldots\ldots$

(c) $f(x) = 3x^3 - 4x^2 + 8$ $f(4) = \ldots\ldots\ldots\ldots$

(d) $f(x) = 2x^3 + x - 5$ $f(5) = \ldots\ldots\ldots\ldots$ 25 to 29

4 Determine the remainder that would occur if $(4x^3 - 5x^2 + 7x - 3)$ were divided by $(x - 3)$ 30 to 32

5 Test whether $(x - 2)$ is a factor of $f(x) = 2x^3 + 2x^2 - 17x + 10$. If so, factorize $f(x)$ as far as possible. 33 to 44

6 Rewrite $f(x) = 2x^3 + 7x^2 - 14x - 40$ as the product of three linear factors. 33 to 44

7 Express the quartic $f(x) = 2x^4 - 7x^3 - 2x^2 + 13x + 6$ as the product of four linear factors. 45 to 56

Further problems F.3

63

1 Evaluate the following, giving the results to 4 sig fig:

(a) $K = 14{\cdot}26 - 6{\cdot}38 + \sqrt{136{\cdot}5} \div (8{\cdot}72 + 4{\cdot}63)$

(b) $P = (21{\cdot}26 + 3{\cdot}74) \div 1{\cdot}24 + 4{\cdot}18^2 \times 6{\cdot}32$

(c) $Q = \dfrac{3{\cdot}26 + \sqrt{12{\cdot}13}}{14{\cdot}192 - 2{\cdot}4 \times 1{\cdot}63^2}$

2 Transpose each of the following formulas to make the symbol in the square brackets the subject:

(a) $f = \dfrac{1}{2\pi\sqrt{LC}}$ [L]

(b) $P = \dfrac{S(C - F)}{C}$ [C]

(c) $S = \sqrt{\dfrac{3D(L - D)}{8}}$ [L]

(d) $T = \dfrac{M - m}{1 + Mm}$ [M]

(e) $A = \pi r\sqrt{r^2 + h^2}$ [h]

(f) $V = \dfrac{\pi h}{6}(3R^2 + h^2)$ [R]

▶

3 Rewrite the following in nested form and, in each case, determine the value of the function for the value of x stated:

(a) $f(x) = 5x^3 - 4x^2 + 3x - 12$ $[x = 2]$

(b) $f(x) = 3x^3 - 2x^2 - 5x + 3$ $[x = 4]$

(c) $f(x) = 4x^3 + x^2 - 6x + 2$ $[x = -3]$

(d) $f(x) = 2x^4 - 4x^3 + 2x^2 - 3x + 6$ $[x = 3]$

(e) $f(x) = x^4 - 5x^3 + 3x - 8$ $[x = 6]$

4 Without dividing in full, determine the remainder in each of the following cases:

(a) $(5x^3 + 4x^2 - 6x + 3) \div (x - 4)$

(b) $(3x^3 - 5x^2 + 3x - 4) \div (x - 5)$

(c) $(4x^3 + x^2 - 7x + 2) \div (x + 3)$

(d) $(2x^3 + 3x^2 - 4x + 5) \div (x + 4)$

(e) $(3x^4 - 2x^3 - 10x - 5) \div (x - 4)$

5 Factorize the following cubics as completely as possible:

(a) $x^3 + 6x^2 + 5x - 12$

(b) $2x^3 + 9x^2 - 11x - 30$

(c) $3x^3 - 4x^2 - 28x - 16$

(d) $3x^3 - x^2 + x + 5$

(e) $6x^3 - 5x^2 - 34x + 40$

(f) $4x^3 - 39x + 35$

(g) $2x^3 - x^2 - 10x + 8$

(h) $15x^3 + 53x^2 + 8x - 48$

(i) $x^3 + x^2 - 2x - 8$

(j) $6x^3 + 37x^2 + 67x + 30$

6 Factorize the following quartics, expressing the results as the products of linear factors where possible:

(a) $f(x) = 2x^4 - 5x^3 - 15x^2 + 10x + 8$

(b) $f(x) = 3x^4 - 7x^3 - 25x^2 + 63x - 18$

(c) $f(x) = 4x^4 - 4x^3 - 35x^2 + 45x + 18$

(d) $f(x) = x^4 + 2x^3 - 6x^2 - 2x + 5$

(e) $6x^4 - 11x^3 - 35x^2 + 34x + 24$

(f) $2x^4 + 5x^3 - 20x^2 - 20x + 48$

Programme F.4

Graphs

Frames 1 to 37

Learning outcomes

When you have completed this Programme you will be able to:

- Construct a collection of ordered pairs of numbers from an equation
- Plot points associated with ordered pairs of numbers against Cartesian axes and generate graphs
- Appreciate the existence of asymptotes to curves and discontinuities
- Use a spreadsheet to draw Cartesian graphs of equations
- Describe regions of the x–y plane that are represented by inequalities

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.4

Frames

1 Given the equation:

$$y^2 - x^2 = 1$$

(a) Transpose the equation to make y the subject of the transposed equation.
(b) Construct ordered pairs of numbers corresponding to the even integer values of x where $-10 \le x \le 10$.
(c) Plot the ordered pairs of numbers on a Cartesian graph and join the points plotted with a continuous curve. 1 to 7

2 Plot the graph of:

(a) $y = \dfrac{1}{x^2}$ for $-3 \le x \le 3$ with intervals of 0·5

(b) $y = \begin{cases} -x : x \le 0 \\ x \quad : x > 0 \end{cases}$ for $-3 \le x \le 4$ with intervals of 0·5 8 to 15

3 Use a spreadsheet to draw the Cartesian graphs of:
(a) $y = 2x^2 - 7x - 4$ for $-2 \le x \le 4{\cdot}9$ with step value 0·3.
(b) $y = x^3 - x^2 + x - 1$ for $-2 \le x \le 4{\cdot}9$ with step value 0·3. 16 to 27

4 Describe the region of the x–y plane that corresponds to each of the following:
(a) $y > -x$ (b) $y \le x - 3x^3$ (c) $x^2 + y^2 \le 1$ 28 to 33

Graphs of equations

1

Equations

A conditional equation is a statement of the equality of two expressions that is only true for restricted values of the symbols involved. For example, consider the equation:

$$x - y = 1$$

Transposing this equation to make y the subject of a new equation gives:

$$y = x - 1$$

We evaluate this equation by freely selecting a value of the independent variable x and then calculating the corresponding value of the dependent variable y. The value of y is *restricted* to being the value obtained from the right-hand side of the equation.

We could equally have transposed the original equation to make y the independent variable:

$$x = y + 1$$

but for historic reasons the variable y is normally selected to be the subject of the transposed equation rather than x, so we shall concentrate on equations of the form $y =$ *some expression in* x.

So how are the values of y related to the values of x in the equation $x^2 + y^2 = 1$?

See the next frame

2

$$\boxed{y = \pm\sqrt{1 - x^2}}$$

Because

$y^2 = 1 - x^2$ subtracting x^2 from both sides

so that:

$y = \pm\sqrt{1 - x^2}$ (the symbol $\sqrt{x}$ refers to the *positive* square root of x whereas $x^{\frac{1}{2}}$ refers to both the positive and the negative square roots of x)

Here we see that not only are the y-values restricted by the equation but our choice of value of the independent variable x is also restricted. The value of $1 - x^2$ must not be negative otherwise we would be unable to find the square root.

The permitted values of x are ....

The answer is in the next frame

3

$$\boxed{-1 \le x \le 1}$$

Because we demand that $1 - x^2$ must not be negative, it must be greater than or equal to zero. That is, $1 - x^2 \ge 0$ so that $1 \ge x^2$ (which can also be written as $x^2 \le 1$). This inequality is only satisfied if $-1 \le x \le 1$. Notice also, that for each permitted value of x there are two values of y, namely $y = +\sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.

Move to the next frame

4

Ordered pairs of numbers

Evaluating an equation of a single independent variable enables a collection of *ordered pairs of numbers* to be constructed. For example, consider the equation:

$$y = x^2$$

If we select a value for the independent variable x, say $x = 2$, the corresponding value of the dependent variable y is found to be $2^2 = 4$. From these two values the ordered pair of numbers (2, 4) can be constructed – it is called an *ordered* pair because the first number of the pair is always the value of the independent variable and the second number is the corresponding value of the dependent variable.

So the collection of ordered pairs constructed from $y = x^2$ using successive integer values of x from −5 to 5 is ....

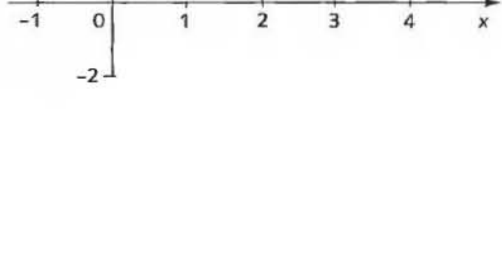
See the next frame

5

$$\boxed{\begin{array}{l}(-5, 25), (-4, 16), (-3, 9), (-2, 4), (-1, 1),\\ (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\end{array}}$$

Cartesian axes

If, on a sheet of graph paper, we draw two straight lines perpendicular to each other and on each line mark off the integers so that the two lines intersect at their common zero points, we can then *plot* the ordered pair of numbers (2, 4) as a point in the plane referenced against the integers on the two lines thus:



▶

The arrangement of numbered lines is called the *Cartesian coordinate frame* and each line is called an *axis* (plural *axes*). The horizontal line is always taken to be the independent variable axis (here the x-axis) and the vertical line the independent variable axis (here the y-axis). Notice that the scales of each axis need not be the same – the scales are chosen to make optimum use of the sheet of graph paper.

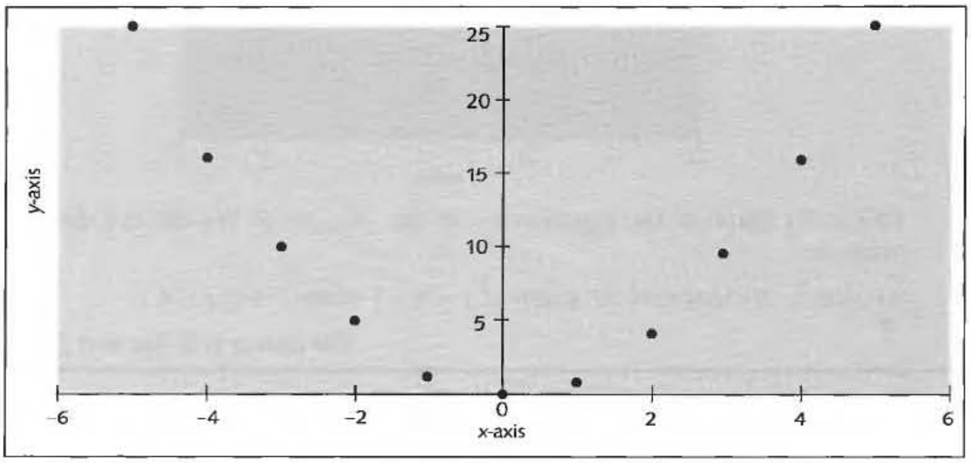
Drawing a graph

If, for an equation in a single independent variable, we construct a collection of ordered pairs and plot each pair as a point in the same Cartesian coordinate frame we obtain a collection of isolated points.

On a sheet of graph paper plot the points (–5, 25), (–4, 16), (–3, 9), (–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4), (3, 9), (4, 16) and (5, 25) obtained from the equation $y = x^2$.

Check your graph with the next frame

6

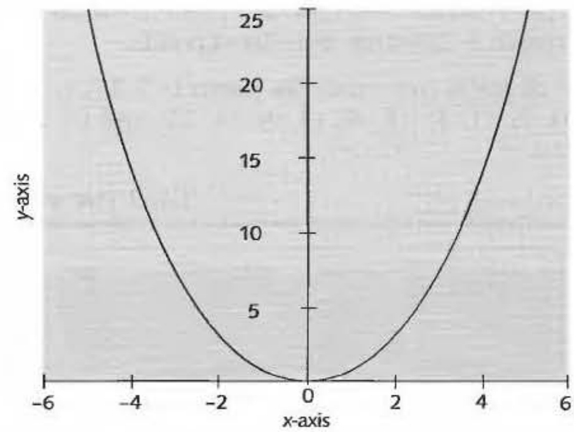


▶

There is an infinite number of possible choices of x values and, hence, points. If we were able to plot this infinity of all possible points they would merge together to form a continuous line known as the *graph of the equation*. In practice, what we do is to plot a collection of isolated points and then join them up with a continuous line. For example, if we were to plot all the ordered pairs of numbers that could be constructed from the equation:

$$y = x^2 \text{ for } -5 \le x \le 5$$

we would end up with the shape given below:



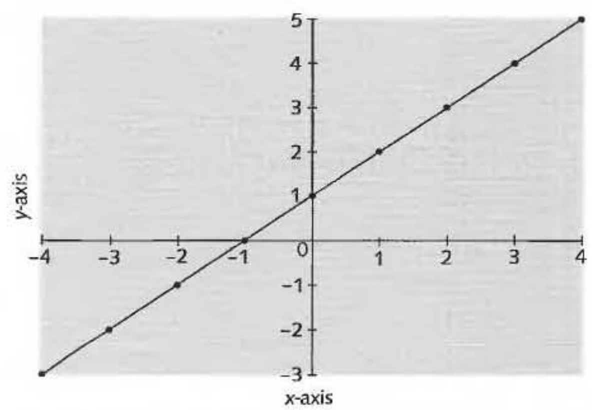
This is the graph of the equation $y = x^2$ for $-5 \le x \le 5$. We call the shape a *parabola*.

So what is the shape of the graph of $y = x + 1$ where $-4 \le x \le 4$?

The answer is in the next frame

7

A straight line



Because, if we construct the following table:

x-value	−4	−3	−2	−1	0	1	2	3	4
y-value	−3	−2	−1	0	1	2	3	4	5

this gives rise to the following ordered pairs: (−4, −3), (−3, −2), (−2, −1), (−1, 0), (0, 1), (1, 2), (2, 3), (3, 4) and (4, 5) which can be plotted as shown.

The plot we have obtained is the plot of just those ordered pairs of numbers that we have constructed, joined together by a continuous line.

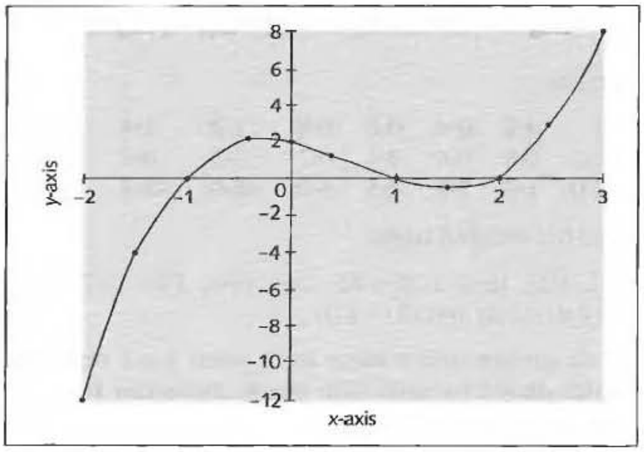
Next frame

Try another one. Construct the graph of: 8

$y = x^3 - 2x^2 - x + 2$ for $-2 \leq x \leq 3$ with intervals of 0·5

Check the next frame for the answer

9



Because not all equations are polynomial equations we shall construct the ordered pairs by making use of a table rather than by *nesting* as we did in Programme F.3. Here is the table:

x	**−2**	**−1·5**	**−1**	**−0·5**	**0**	**0·5**	**1**	**1·5**	**2**	**2·5**	**3**
x^3	−8	−3·4	−1	−0·1	0	0·1	1	3·4	8	15·6	27
$-2x^2$	−8	−4·5	−2	−0·5	0	−0·5	−2	−4·5	−8	−12·5	−18
$-x$	2	1·5	1	0·5	0	−0·5	−1	−1·5	−2	−2·5	−3
2	2	2	2	2	2	2	2	2	2	2	2
y	**−12**	**−4·4**	**0**	**1·9**	**2**	**1·1**	**0**	**−0·6**	**0**	**2·6**	**8**

Pairs: (−2, −12), (−1·5, −4·4), (−1, 0), (−0·5, 1·9), (0, 2), (0·5, 1·1), (1, 0), (1·5, −0·6), (2, 0), (2·5, 2·6), (3, 8)

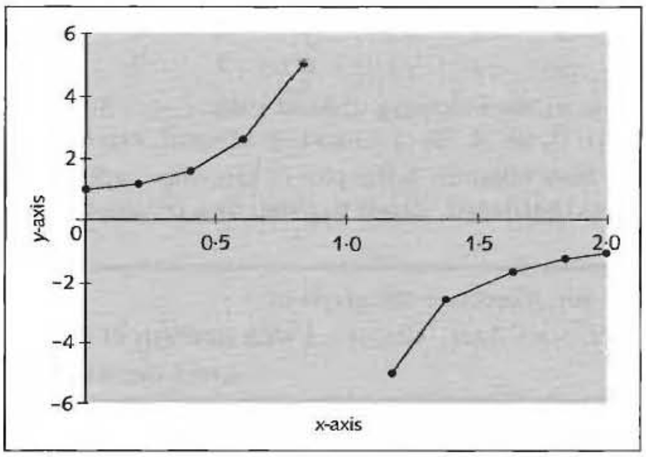
How about the graph of:

$$y = \frac{1}{1-x} \text{ for } 0 \leq x \leq 2?$$

Select values of x ranging from 0 to 2 with intervals of 0·2 and take care here with the values that are near to 1.

The answer is in the next frame

10



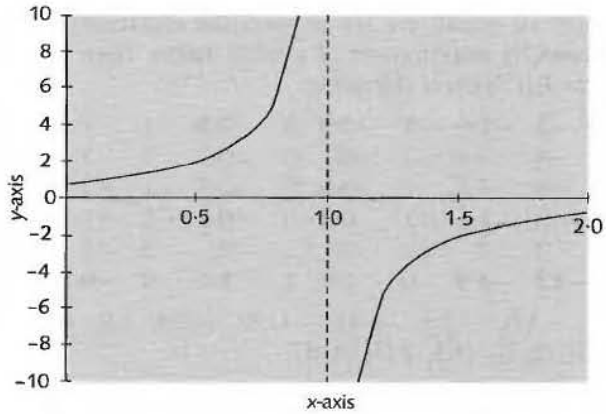
Here is the table:

x	**0**	**0·2**	**0·4**	**0·6**	**0·8**	**1·2**	**1·4**	**1·6**	**1·8**	**2·0**
$1-x$	1·0	0·8	0·6	0·4	0·2	−0·2	−0·4	−0·6	−0·8	−1·0
y	**1·0**	**1·3**	**1·7**	**2·5**	**5·0**	**−5·0**	**−2·5**	**−1·7**	**−1·3**	**−1·0**

Giving rise to the ordered pairs:

(0, 1·0), (0·2, 1·3), (0·4, 1·7), (0·6, 2·5), (0·8, 5·0), (1·2, −5·0), (1·4, −2·5), (1·6, −1·7), (1·8, −1·3) and (2, −1·0)

Notice that we cannot find a value for y when $x = 1$ because then $1 - x = 0$ and we cannot divide by zero. The graph above can be improved upon by plotting more points – see below:



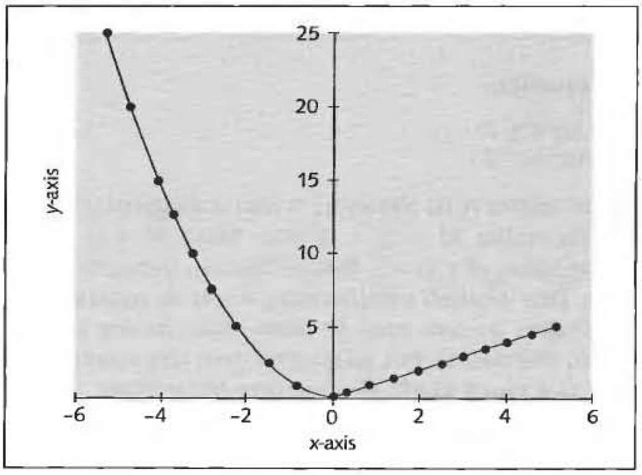
As x approaches the value 1 from the left, the graph rises and approaches (but never crosses) the vertical line through $x = 1$. Also as x approaches the value 1 from the right, the graph falls and approaches (but never crosses) the same vertical line. The vertical line through $x = 1$ is called a vertical *asymptote* to the graph. Not all asymptotes are vertical or even straight lines. Indeed, whenever a curve approaches a second curve without actually meeting or crossing it, the second curve is called an asymptote to the first curve.

Now, try finding the graph of the following:

$$y = \begin{cases} x^2 \text{ for } & -5 \le x < 0 \\ x \text{ for } & x \ge 0 \end{cases} \quad -5 \le x \le 5 \text{ with intervals of } 0{\cdot}5$$

Check the next frame

11



Because the equation:

$$y = \begin{cases} x^2 \text{ for } & -5 \le x < 0 \\ x \text{ for } & x \ge 0 \end{cases}$$

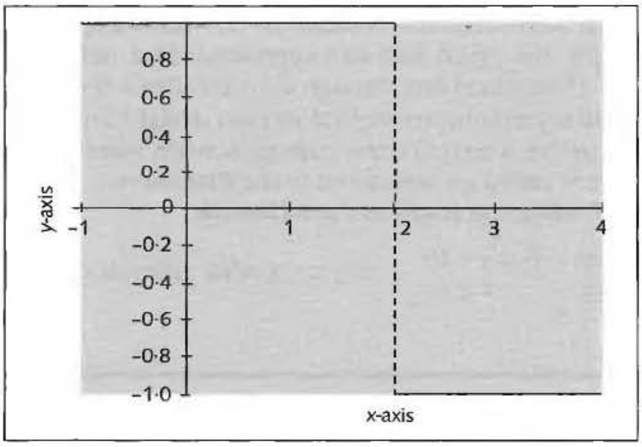
means that if x is chosen so that its value lies between -5 and 0 then $y = x^2$ and that part of the graph is the parabola shape. If x is greater than or equal to zero then $y = x$ and that part of the graph is the straight line. Notice that not all equations are of the simple form $y =$ *some expression in x.*

And finally, how about the graph of:

$$y = \begin{cases} 1 & \text{for } x \le 2 \\ -1 & \text{for } x > 2 \end{cases} \quad \text{for } -1 \le x \le 4$$

Next frame

12



Because the equation:

$$y = \begin{cases} 1 & \text{for } x \leq 2 \\ -1 & \text{for } x > 2 \end{cases}$$

means that no matter what the value is that is assigned to x, if it is less than or equal to 2 the value of y is 1. If the value of x is greater than 2 the corresponding value of y is -1. Notice the gap between the two continuous straight lines. This is called a *discontinuity* – not all equations produce smooth continuous shapes so *care must be taken when joining points together with a continuous line.* The dashed line joining the two end points of the straight lines is only there as a visual guide, *it is not part of the graph.*

At this point let us pause and summarize the main facts so far

13

Revision summary

1 Ordered pairs of numbers can be generated from an equation involving a single independent variable.

2 Ordered pairs of numbers generated from an equation can be plotted against a Cartesian coordinate frame.

3 The graph of the equation is produced when the plotted points are joined by smooth curves.

4 Some equations have graphs that are given specific names, such as straight lines and parabolas.

5 Not all equations are of the simple form $y =$ *some expression in* x.

6 Not all graphs are smooth, unbroken lines. Some graphs consist of breaks called discontinuities.

14

Revision exercise

1 Given the equation:

$$x^2 + y^3 = 1$$

(a) Transpose the equation to make y the subject of the transposed equation.

(b) Construct ordered pairs of numbers corresponding to the integer values of x where $-5 \leq x \leq 5$.

(c) Plot the ordered pairs of numbers on a Cartesian graph and join the points plotted with a continuous curve.

2 Plot the graph of:

(a) $y = x^2 + \dfrac{1}{x}$ for $-3 \leq x \leq 3$ with intervals of 0·5.

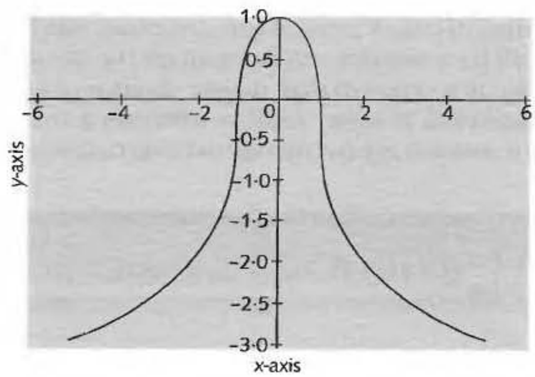
(b) $y = \begin{cases} x^2 + x + 1 : x \leq 1 \\ 3 - x \quad\;\; : x > 1 \end{cases}$ for $-3 \leq x \leq 4$ with intervals of 0·5.

15

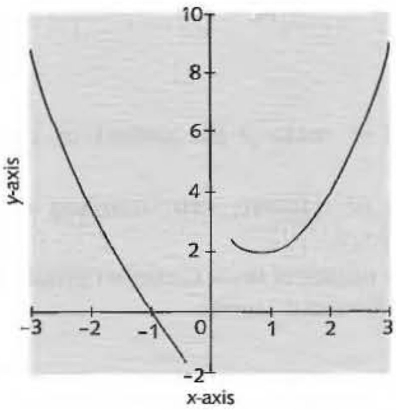
1 (a) $y = (1 - x^2)^{\frac{1}{3}}$

(b) (−5, −2·9), (−4, −2·5), (−3, −2), (−2, −1·4), (−1, 0), (0, 1), (1, 0), (2, −1·4), (3, −2), (4, −2·5), (5, −2·9)

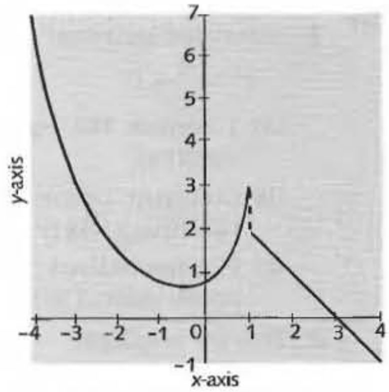
(c)



2 (a)



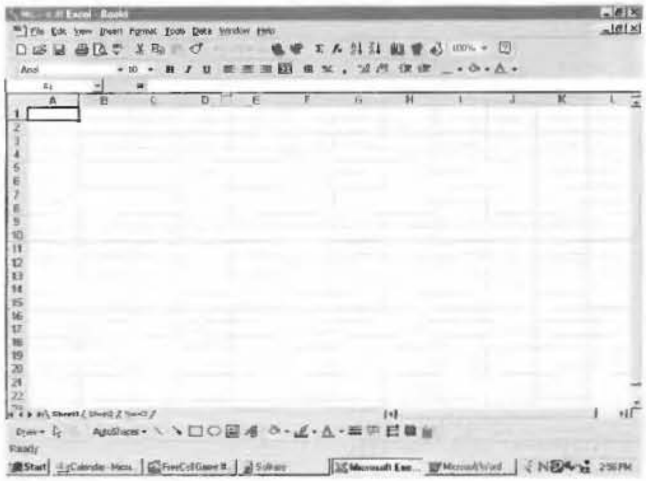
(b)



Using a spreadsheet

16 Spreadsheets

Electronic spreadsheets provide extensive graphing capabilities and their use is widespread. Because *Microsoft* products are the most widely used software products on a PC, all the examples will be based on the *Microsoft* spreadsheet *Excel 97* for *Windows*. It is expected that all later versions of *Excel* will support the handling characteristics of earlier versions with only a few minor changes. *If you have access to a computer algebra package you might ask your tutor how to use it to draw graphs.*



The features displayed are common to many spreadsheets but there may be minor differences in style between different products.

Rows and columns 17

Every electronic spreadsheet consists of a collection of cells arranged in a regular array of columns and rows. To enable the identification of individual cells each cell has an address given by the *column label* followed by the *row label*. In an *Excel* spreadsheet the columns are labelled alphabetically from **A** onwards and the rows are numbered from **1** onwards.

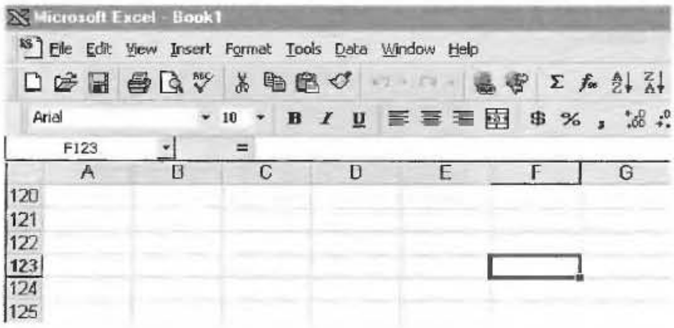
So that the cell with address **F123** is on the .... column of the .... row.

Check with the next frame

18

F (6th) column of the **123**rd row

Because the address consists of the column label **F** (6th letter of the alphabet) followed by the row number **123**.



At any time one particular cell boundary is highlighted with a *cursor* and this cell is called the *active cell*. An alternative cell can become the active cell by pressing the cursor movement keys on the keyboard (←, ↑, → and ↓) or, alternatively, by pointing at a particular cell with the *mouse pointer* (⇖) and then clicking the mouse button. Try it.

Next frame

Text and number entry 19

Every cell on the spreadsheet is capable of having numbers or text entered into it via the keyboard. Make the cell with the address **B10** the active cell and type in the text:

Text

and then press Enter (↵). Now make cell **C15** the active cell and type in the number 12 followed by Enter.

Next frame

20 Formulas

As well as text and numbers, each cell is capable of containing a formula. In an *Excel* spreadsheet every formula begins with the = (equals) sign when it is being entered at the keyboard. Move the cursor to cell **C16** and enter at the keyboard:

= 3*C15

followed by Enter. The * represents multiplication (×) and the formula states that the contents of cell **C16** will be three times the contents of cell **C15**. You will notice that the number displayed in cell **C16** is three times the number displayed in **C15**. Now change the number in **C15** by overwriting to see the effect. The number in **C16** also changes to reflect the change in the number in **C15**.

Next frame

21 Clearing entries

To clear an entry, point and click at the cell to be cleared. This makes it the active cell. Click the **Edit** command on the Command Bar to reveal a drop-down menu.

Select from this menu the option **Clear** to reveal a further drop-down menu. In this second menu select **All** and the cell contents are then cleared. Now make sure that all entries on the spreadsheet have been cleared because we want to use the spreadsheet to construct a graph.

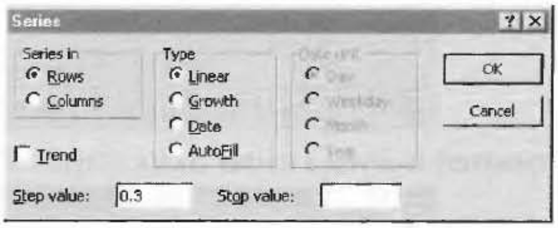
Let's now put all this together in the next frame

Construction of a Cartesian graph

Follow these instructions to plot the graph of $y = (x - 2)^3$:

Enter the number -1 in **A1**
Highlight the cells **A1** to **A21** by pointing at **A1**, holding down the mouse button, dragging the pointer to **A21** and then releasing the mouse button (all the cells from **A2** to **A21** turn black to indicate that they have been selected)

Select the **Edit-Fill-Series** commands from the Command Bar and then in the *Series* window change the **Step value** from 1 to 0·3 and Click the **OK** button:



Cells **A2** to **A21** fill with single place decimals ranging from -1 to $+5$ with step value intervals of 0·3. These are the x-values, where $-1 \leq x \leq 5$.

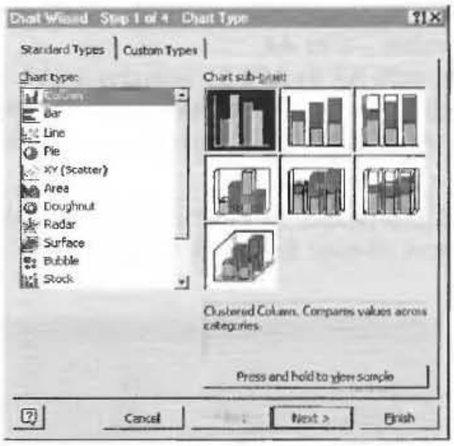
In cell **B1**, type in the formula =(A1−2)^3 and then press Enter (the symbol ^ represents raising to a power). The number -27 appears in cell **B1** – that is $(-1-2)^3 = -27$ where -1 is the content of **A1**.

Activate cell **B1** and select the **Edit-Copy** commands
Highlight cells **B2** to **B21** and select **Edit-Paste**

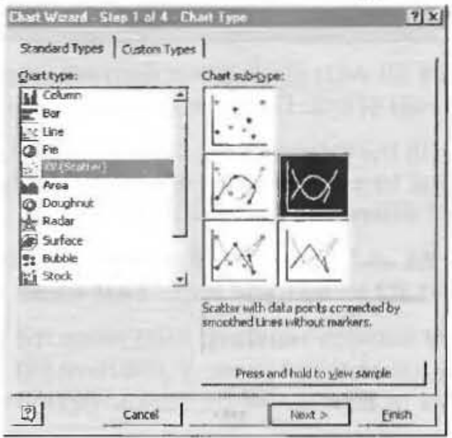
Cells **B2** to **B21** fill with numbers, each being the number in the adjoining cell minus 2, all raised to the power 3; you have just copied the formula in **B1** into the cells **B2** to **B21**. These are the corresponding y-values.

Highlight the two columns of numbers – cells **A1:B21**
Click the *Chart Wizard* button

The *Chart Wizard Step 1 of 4* window has appeared, requesting a choice of chart type:

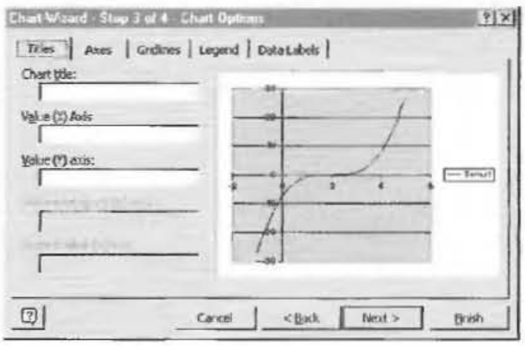


Click **XY (Scatter)** to reveal a further choice of types of XY Scatter charts:



Click the type in the top right-hand corner to select it (*Scatter with data points connected by smooth lines without markers*). Press the **Next** button in this window to reveal the *Chart Wizard Step 2 of 4* window.

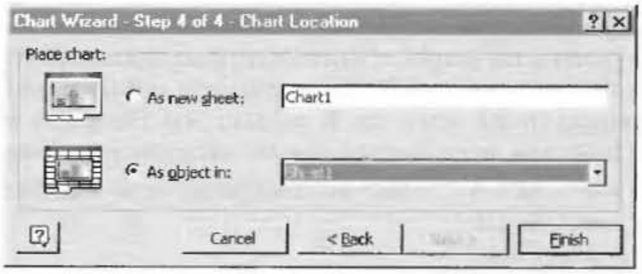
Click the **Next** button to reveal the *Chart Wizard Step 3 of 4* window:



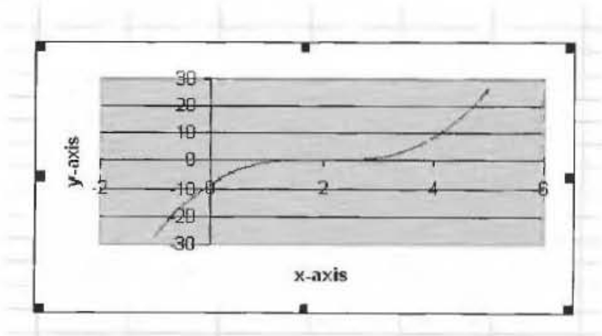
In the *Chart Wizard Step 3 of 4 window:*

Click the *Legend* tab
Clear the tick in the **Show legend** square (just Click the square)
Click the *Titles* tab
Enter in the **Value (X) Axis**
x-axis
Enter in the **Value (Y) Axis**
y-axis
Click the **Next** button to reveal the *Chart Wizard Step 4 of 4* window

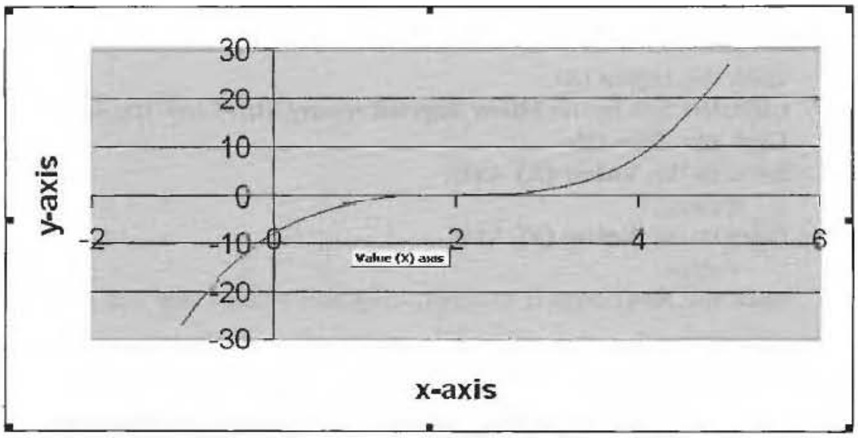
In the *Chart Wizard Step 4 of 4* window ensure that the lower radio button is selected (it will have a black spot inside it when it is selected):



Click the **Finish** button to reveal the chart:



The small black squares (called *Handles*) around the Chart can be used to change the size of the Chart. Place the cursor over a Handle, hold down the mouse button and Drag. Try it:

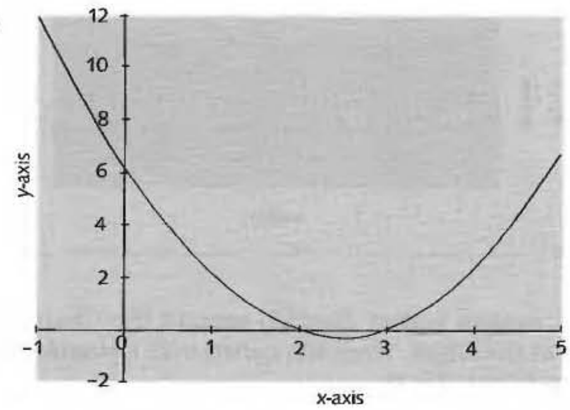


Now produce the graphs of the following equations. All you need to do is to change the formula in cell **B1** by activating it and then overtyping. Copy the new formula in **B1** down the **B** column and the graph will automatically update itself (*you do not have to clear the old graph, just change the formula*):

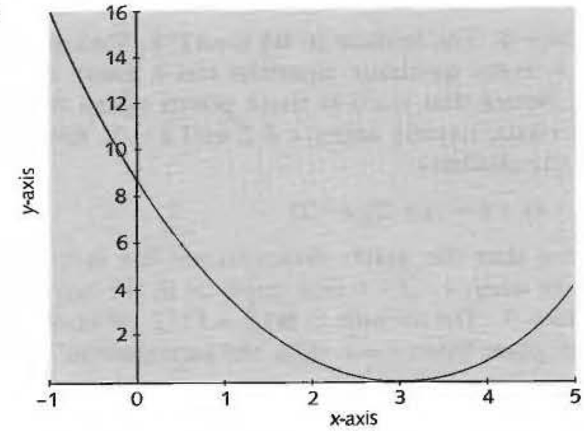
(a) $y = x^2 - 5x + 6$ Use * for multiplication so that $5x$ is entered as $5*x$
(b) $y = x^2 - 6x + 9$
(c) $y = x^2 - x + 1$
(d) $y = x^3 - 6x^2 + 11x - 6$

23

(a)

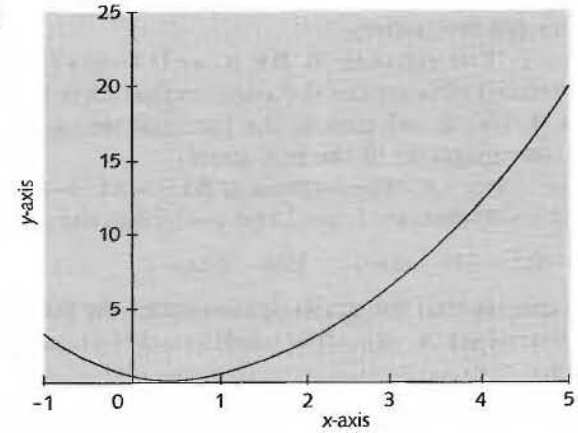


(b)



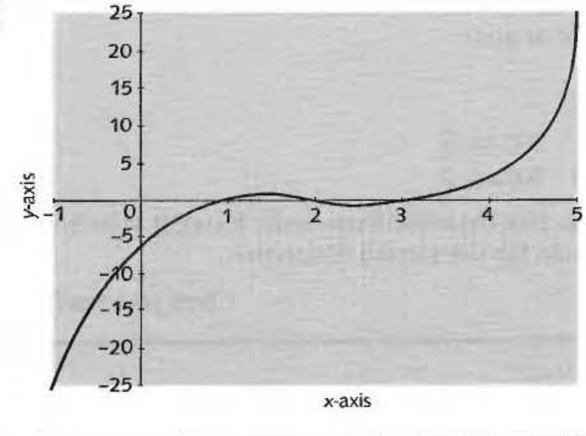
16
14
12
10
8
6
4
2
y-axis
-1
0
1
2
3
4
5
x-axis

(c)



25
20
15
10
5
y-axis
-1
0
1
2
3
4
5
x-axis

(d)



25
20
15
10
5
-5
-10
-15
-20
-25
y-axis
-1
0
1
2
3
4
5
x-axis

Because:

(a) $y = x^2 - 5x + 6$ The formula in **B1** is =A1^2–5*A1+6. This shape is the parabola – every quadratic equation has a graph in the shape of a parabola. Notice that $y = 0$ at those points where the curve coincides with the x-axis, namely when $x = 2$ and $x = 3$. Also, because we can factorize the quadratic:

$$y = x^2 - 5x + 6 = (x - 2)(x - 3)$$

we can see that the graph demonstrates the fact that $y = 0$ when $x - 2 = 0$ or when $x - 3 = 0$ (see graph (a) in the box above).

(b) $y = x^2 - 6x + 9$ The formula in **B1** is =A1^2–6*A1+9. Notice that $y = 0$ at just one point when $x = 3$. Here, the factorization of the quadratic is:

$$y = x^2 - 6x + 9 = (x - 3)(x - 3)$$

so the graph demonstrates the fact that $y = 0$ only when $x = 3$ (see graph (b) in the box above).

(c) $y = x^2 - x + 1$ The formula in **B1** is =A1^2–A1+1. Notice that the curve never touches or crosses the x-axis so that there is no value of x for which $y = 0$. This is reflected in the fact that we cannot factorize the quadratic. (see graph (c) in the box above).

(d) $y = x^3 - 6x^2 + 11x - 6$ The formula in **B1** is =A1^3–6*A1^2+11*A1–6. Notice that $y = 0$ when $x = 1$, $x = 2$ and $x = 3$. Also, the cubic factorizes as:

$$y = x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$$

Again, we can see that the graph demonstrates the fact that $y = 0$ when $(x - 1) = 0$ or when $(x - 2) = 0$ or when $(x - 3) = 0$ (see graph (d) in the box above).

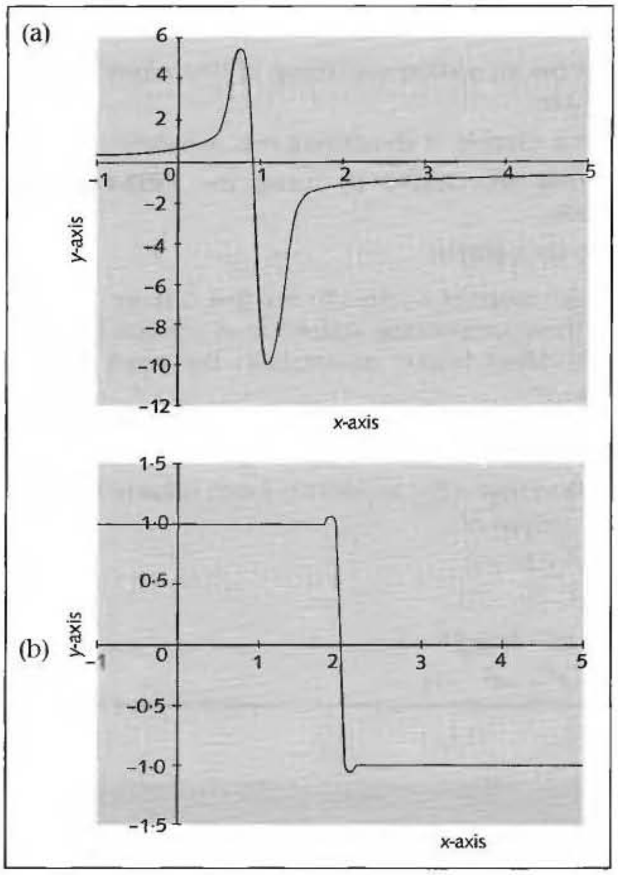
Let's try two graphs that we have already plotted manually. Repeat the same procedure by simply changing the formula in cell **B1** and then copying it into cells **B2** to **B21** to plot:

(a) $y = \dfrac{1}{1 - x}$

(b) $y = \begin{cases} 1 & \text{for } x \leq 2 \\ -1 & \text{for } x > 2 \end{cases}$

You will have to give the second one some thought as to how you are going to enter the formula for the second equation.

Check your results in the next frame



Because:

(a) $y = \dfrac{1}{1-x}$ The formula in **B1** is =1/(1−A1). Notice that the asymptotic behaviour is hidden by the joining of the two inside ends of the graph. See graph (a) in the box above. We can overcome this problem:

Make cell **A8** the active cell
On the Command Bar select **Insert-Rows**

An empty row appears above cell **A8** and the stray line on the graph disappears. (Clear the empty row using **Edit-Delete-Entire row**.)

(b) Here there are two formulas. The first formula in **B1** is =1 copied down to cell **B11** and the second formula in **B12** is =−1 copied down to **B21**.

Notice that the two sides of the graph are joined together when they should not be – see graph (b) in the box above. Make **A12** the active cell and insert a row to remove the stray line just as you did in part (a).

At this point let us pause and summarize the main facts so far on using a spreadsheet

25

Revision summary

1 A spreadsheet consists of an array of cells arranged in regular columns and rows.

2 Each cell has an address consisting of the column letter followed by the row number.

3 Each cell is capable of containing text, a number or a formula.

4 Cell entries are cleared by using the **Edit-Clear-All** sequence of commands.

5 To construct a graph:

Enter the range of x-values in the first column
Enter the corresponding collection of y-values in the second column
Use the *Chart Wizard* to construct the graph using the *XY (Scatter)* option.

26

Revision exercise

1 For x in the range $-2 \leq x \leq 6$ with a step value of 0·4 use a spreadsheet to draw the graphs of:

(a) $y = x^2 - 3x + 2$

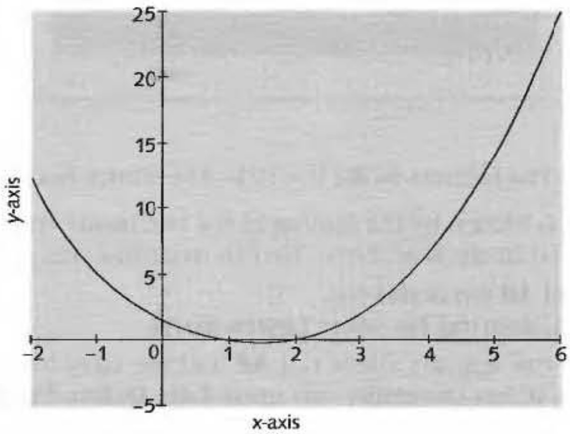
(b) $y = x^2 - 1$

(c) $y = 4x^2 - 3x + 25$

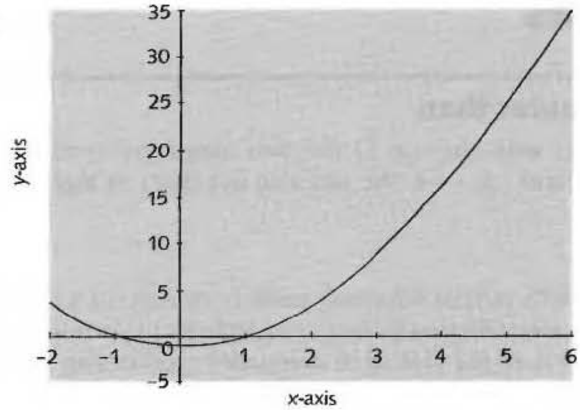
(d) $y = -x^3 + 6x^2 - 8x$

27

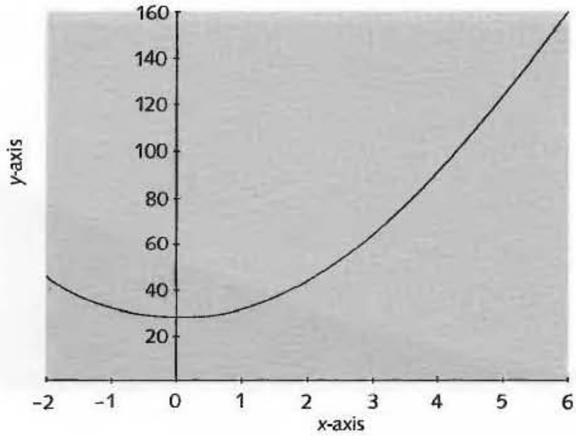
(a)



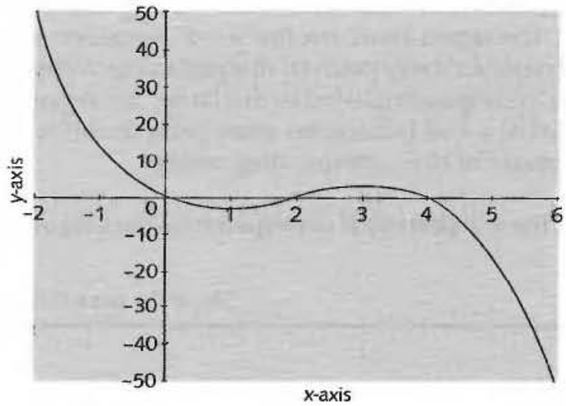
(b)



(c)



(d)



Let's look now at an extension of these ideas

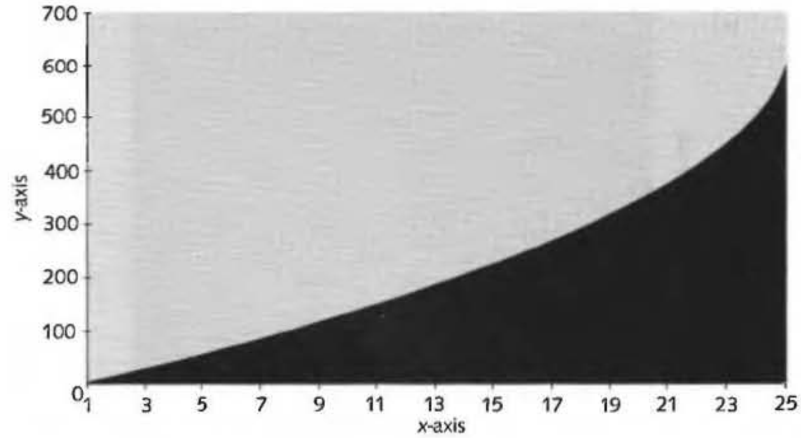
Inequalities

28 Less than or greater than

You are familiar with the use of the two inequality symbols > and <. For example, $3 < 5$ and $-2 > -4$. We can also use them in algebraic expressions. For example:

$y > x$

The inequality tells us that whatever value is chosen for x the corresponding value for y is greater. Obviously there is an infinity of y-values greater than the chosen value for x, so the plot of an inequality produces an area rather than a line. For example, if we were to plot the graph of $y = x^2$ for $1 \leq x \leq 25$ we would obtain the graph shown below:



The graph is in the form of the curve $y = x^2$ acting as a separator for two different regions. The region above the line $y = x^2$ represents the plot of $y > x^2$ for $1 \leq x \leq 25$ because for every point in this region the y-value is greater than the square of the corresponding x-value. Similarly, the region below the line represents the plot of $y < x^2$ because for every point in this region the y-value is less than the square of the corresponding x-value.

So, the region of the x–y plane that corresponds to the inequality $y < x^3 - 2x^2$ is given by ....

Check the next frame for the answer

29

$\boxed{\text{The region of the graph below the curve } y = x^3 - 2x^2}$

Because

For every point in this region the y-value is less than the corresponding x-value.

How about $y \geq 2x - 1$? The region of the x–y plane that this describes is ....

30

$\boxed{\text{The region of the graph on or above the line } y = 2x - 1}$

Because

The inequality $y \geq 2x - 1$ means one of two conditions. Namely, $y > 2x - 1$ *or* $y = 2x - 1$ so that every point on or above the line satisfies one or the other of these two conditions.

At this point let us pause and summarize the main facts so far on inequalities

31

Revision summary

1 The graph of an inequality is a region of the x–y plane rather than a line or a curve.

2 Points above the graph of $y = f(x)$ are in the region described by $y > f(x)$.

3 Points below the graph of $y = f(x)$ are in the region described by $y < f(x)$.

32

Revision exercise

1 Describe the regions of the x–y plane defined by each of the following inequalities:

(a) $y < 3x - 4$

(b) $y > -2x^2 + 1$

(c) $y \leq x^2 - 3x + 2$

33

1 (a) The region below the line $y = 3x - 4$.

(b) The region above the line $y = -2x^2 + 1$.

(c) The region below and on the line $y = x^2 - 3x + 2$.

34

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through them at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

35

Checklist F.4

Frames

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:

- Construct a collection of ordered pairs of numbers from an equation? — 1 to 4

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Plot points associated with ordered pairs of numbers against Cartesian axes and generate graphs? — 5 to 9

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Appreciate the existence of asymptotes to curves and discontinuities? — 10 to 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use a spreadsheet to draw Cartesian graphs of equations? — 16 to 24

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe regions of the x–y plane that are represented by inequalities? — 28 to 30

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.4

36

Frames

1 Given the equation:

$$x^2 + y^2 = 1$$

(a) Transpose the equation to make y the subject of the transposed equation.

(b) Construct ordered pairs of numbers corresponding to the integer values of x where $-1 \leq x \leq 1$ in intervals of 0·2.

(c) Plot the ordered pairs of numbers on a Cartesian graph and join the points plotted with a continuous curve. — 1 to 7

2 Plot the graph of:

(a) $y = x - \dfrac{1}{x}$ for $-3 \leq x \leq 3$.

What is happening near to $x = 0$?

(b) $y = \begin{cases} 2 - x & : x \leq 0 \\ x^2 + 2 & : x > 0 \end{cases}$ for $-4 \leq x \leq 4$ — 8 to 15

Frames

3 Use a spreadsheet to draw the Cartesian graphs of:

(a) $y = 3x^2 + 7x - 6$ for $-4 \leq x \leq 5{\cdot}2$ with step value 0·4.

(b) $y = x^3 + x^2 - x + 1$ for $-2 \leq x \leq 4{\cdot}9$ with step value 0·3. 16 to 27

4 Describe the region of the x–y plane that corresponds to each of the following:

(a) $y < 2 - 3x$ (b) $y > x - \dfrac{2}{x}$ (c) $x + y \leq 1$ 28 to 33

Further problems F.4

37

1 Given $x^2 - y^2 = 0$ find the equation giving y in terms of x and plot the graph of this equation for $-3 \leq x \leq 3$.

2 Using a spreadsheet plot the graph of:

$$y = x^3 + 10x^2 + 10x - 1$$

for $-10 \leq x \leq 2$ with a step value of 0·5.

3 Using a spreadsheet plot the graph of:

$$y = \frac{x^3}{1 - x^2}$$

for $-2 \leq x \leq 2{\cdot}6$ with a step value of 0·2. Draw a sketch of this graph on a sheet of graph paper indicating discontinuities and asymptotic behaviour more accurately.

4 Given the equation $x^2 + y^2 = 4$ transpose it to find y in terms of x. With the aid of a spreadsheet describe the shape that this equation describes.

5 Given the equation:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

transpose it to find y in terms of x. With the aid of a spreadsheet describe the shape that this equation describes.

6 Given the equation $x^2 + y^2 + 2x + 2y + 1 = 0$ transpose it to find y in terms of x. With the aid of a spreadsheet describe the shape that this equation describes.

7 Describe the region of the x–y plane that corresponds to each of the following:

(a) $y \geq 2x + 4$ (b) $y < 3 - x$ (c) $3x - 4y \geq 1$ (d) $x^2 + y^2 > 2$

Programme F.5

Linear equations and simultaneous linear equations

Frames 1 to 27

Learning outcomes

When you have completed this Programme you will be able to:

- Solve any linear equation
- Solve simultaneous linear equations in two unknowns
- Solve simultaneous linear equations in three unknowns

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.5

Frames

1 Solve for x:

(a) $3(x-1)+2(3-2x)=6(x+4)-7$

(b) $\frac{x-2}{3}-\frac{3-x}{2}=1+\frac{4-5x}{4}$

(c) $\frac{4}{x-1}+\frac{2}{x+1}=\frac{6}{x}$ — 1 to 6

2 Solve the following pair of simultaneous equations for x and y:

$$3x-4y=7$$
$$5x-2y=7$$

7 to 11

3 Solve the following set of three equations in three unknowns:

$$x-2y+3z=10$$
$$3x-2y+z=2$$
$$4x+5y+2z=29$$

12 to 20

Linear equations

A linear equation in a single variable (unknown) involves powers of the variable no higher than the first. A linear equation is also referred to as a simple equation. **1**

Solution of simple equations

The solution of simple equations consists essentially of simplifying the expressions on each side of the equation to obtain an equation of the form $ax + b = cx + d$ giving $ax - cx = d - b$ and

hence $x = \dfrac{d-b}{a-c}$ provided $a \neq c$.

Example

$$\begin{aligned} \text{If} \quad 6x + 7 + 5x - 2 + 4x - 1 &= 36 + 7x \\ \text{then} \quad 15x + 4 &= 7x + 36 \\ \text{so } 8x &= 32 \qquad \text{therefore } x = 4 \end{aligned}$$

Similarly:

$$\begin{aligned} \text{if} \quad 5(x-1) + 3(2x+9) - 2 &= 4(3x-1) + 2(4x+3) \\ \text{then} \quad x &= \ldots\ldots\ldots\ldots \end{aligned}$$

2

$$\boxed{x = 2}$$

Because

$$5(x-1) + 3(2x+9) - 2 = 4(3x-1) + 2(4x+3)$$

$$\text{so } 5x - 5 + 6x + 27 - 2 = 12x - 4 + 8x + 6$$

$$11x + 20 = 20x + 2$$

$$\text{so } 18 = 9x \qquad \text{therefore } x = 2$$

Equations which appear not to be simple equations sometimes develop into simple equations during simplification. For example:

$$(4x+1)(x+3) - (x+5)(x-3) = (3x+1)(x-4)$$

$$(4x^2 + 13x + 3) - (x^2 + 2x - 15) = 3x^2 - 11x - 4$$

$$\text{so } 3x^2 + 11x + 18 = 3x^2 - 11x - 4$$

$3x^2$ can now be subtracted from each side, giving:

$$11x + 18 = -11x - 4$$

$$\text{so } 22x = -22 \qquad \text{therefore } x = -1$$

▶

It is always wise to check the result by substituting this value for x in the original equation:

$$\begin{aligned}\text{LHS} &= (4x+1)(x+3)-(x+5)(x-3)\\ &= (-3)(2)-(4)(-4) = -6+16 = 10\\ \text{RHS} &= (3x+1)(x-4) = (-2)(-5) = 10 \quad \therefore\ \text{LHS} = \text{RHS}\end{aligned}$$

So, in the same way, solving the equation:

$$(4x+3)(3x-1)-(5x-3)(x+2) = (7x+9)(x-3)$$

we have $x = \ldots\ldots\ldots\ldots$

3

$$\boxed{x=-3}$$

Simplification gives $\quad 7x^2-2x+3 = 7x^2-12x-27$

Hence $\quad 10x = -30 \quad \therefore\ x = -3$

Where simple equations involve algebraic fractions, the first step is to eliminate the denominators by multiplying throughout by the LCM (Lowest Common Multiple) of the denominators. For example, to solve:

$$\frac{x+2}{2}-\frac{x+5}{3} = \frac{2x-5}{4}+\frac{x+3}{6}$$

The LCM of 2, 3, 4 and 6 is 12:

$$\frac{12(x+2)}{2}-\frac{12(x+5)}{3} = \frac{12(2x-5)}{4}+\frac{12(x+3)}{6}$$

$$6(x+2)-4(x+5) = 3(2x-5)+2(x+3)$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

4

$$\boxed{x=\frac{1}{6}}$$

That was easy enough. Now let us look at this one.

To solve $\dfrac{4}{x-3}+\dfrac{2}{x} = \dfrac{6}{x-5}$

Here, the LCM of the denominators is $x(x-3)(x-5)$. So, multiplying throughout by the LCM, we have

$$\frac{4x(x-3)(x-5)}{x-3}+\frac{2x(x-3)(x-5)}{x} = \frac{6x(x-3)(x-5)}{x-5}$$

After cancelling where possible:

$$4x(x-5)+2(x-3)(x-5) = 6x(x-3)$$

so, finishing it off, $x = \ldots\ldots\ldots\ldots$

5

$$\boxed{x = \frac{5}{3}}$$

Because $4x^2 - 20x + 2(x^2 - 8x + 15) = 6x^2 - 18x$

$$4x^2 - 20x + 2x^2 - 16x + 30 = 6x^2 - 18x$$

$$\therefore\ 6x^2 - 36x + 30 = 6x^2 - 18x$$

Subtracting the $6x^2$ term from both sides:

$-36x + 30 = -18x \quad \therefore\ 30 = 18x \quad \therefore\ x = \frac{5}{3}$

Working in the same way: $\frac{3}{x-2} + \frac{5}{x-3} - \frac{8}{x+3} = 0$

gives the solution $x =$ ....

6

$$\boxed{x = \frac{7}{3}}$$

The LCM of the denominators is $(x-2)(x-3)(x+3)$, so

$$\frac{3(x-2)(x-3)(x+3)}{x-2} + \frac{5(x-2)(x-3)(x+3)}{x-3} - \frac{8(x-2)(x-3)(x+3)}{x+3} = 0$$

$$\therefore\ 3(x-3)(x+3) + 5(x-2)(x+3) - 8(x-2)(x-3) = 0$$

$$3(x^2 - 9) + 5(x^2 + x - 6) - 8(x^2 - 5x + 6) = 0$$

$$3x^2 - 27 + 5x^2 + 5x - 30 - 8x^2 + 40x - 48 = 0$$

$$\therefore\ 45x - 105 = 0$$

$$\therefore\ x = \frac{105}{45} = \frac{7}{3} \quad \therefore\ x = \frac{7}{3}$$

Now on to Frame 7

Simultaneous linear equations with two unknowns

7

A linear equation in two variables has an infinite number of solutions. For example, the two-variable linear equation $y - x = 3$ can be transposed to read:

$$y = x + 3$$

Any one of an infinite number of x-values can be substituted into this equation and each one has a corresponding y-value. However, for two such equations there may be just one pair of x- and y-values that satisfy both equations *simultaneously*.

1 Solution by substitution

To solve the pair of equations

$$5x + 2y = 14 \qquad (1)$$
$$3x - 4y = 24 \qquad (2)$$

From (1): $5x + 2y = 14 \quad \therefore \quad 2y = 14 - 5x \quad \therefore \quad y = 7 - \dfrac{5x}{2}$

If we substitute this for y in (2), we get:

$$3x - 4\left(7 - \frac{5x}{2}\right) = 24$$
$$\therefore \quad 3x - 28 + 10x = 24$$
$$13x = 52 \quad \therefore \quad x = 4$$

If we now substitute this value for x in the other original equation, i.e. (1), we get:

$$5(4) + 2y = 14$$
$$20 + 2y = 14 \quad \therefore \quad 2y = -6 \quad \therefore \quad y = -3$$
$$\therefore \quad \text{We have } x = 4,\ y = -3$$

As a check, we can substitute both these values in (1) and (2):

$$(1) \quad 5x + 2y = 5(4) + 2(-3) = 20 - 6 = 14 \quad ✓$$
$$(2) \quad 3x - 4y = 3(4) - 4(-3) = 12 + 12 = 24 \quad ✓$$
$$\therefore \quad x = 4,\ y = -3 \text{ is the required solution.}$$

Another example:

To solve

$$3x + 4y = 9 \qquad (1)$$
$$2x + 3y = 8 \qquad (2)$$

Proceeding as before, determine the values of x and y and check the results by substitution in the given equations.

$x = \dots\dots\dots\dots, \quad y = \dots\dots\dots\dots$

$$\boxed{x = -5,\ y = 6}$$

Because

(1) $\quad 3x + 4y = 9 \ \therefore\ 4y = 9 - 3x$ so $y = \frac{9}{4} - \frac{3x}{4}$

Substituting this for y in (2) yields:

$$2x + 3\left(\frac{9}{4} - \frac{3x}{4}\right) = 8 \text{ that is } 2x + \frac{27}{4} - \frac{9x}{4} = 8 \ \therefore\ -\frac{x}{4} = \frac{5}{4} \ \therefore\ x = -5$$

Substituting this value for x in (1) yields:

$-15 + 4y = 9 \ \therefore\ 4y = 24 \ \therefore\ y = 6$

2 Solution by equating coefficients

To solve $\quad 3x + 2y = 16 \qquad (1)$

$\qquad\qquad 4x - 3y = 10 \qquad (2)$

If we multiply both sides of (1) by 3 (the coefficient of y in (2)) and we multiply both sides of (2) by 2 (the coefficient of y in (1)) then we have

$9x + 6y = 48$

$8x - 6y = 20$

If we now add these two lines together, the y-term disappears:

$\therefore\ 17x = 68 \quad \therefore\ x = 4$

Substituting this result, $x = 4$, in either of the original equations, provides the value of y.

In (1) $\quad 3(4) + 2y = 16 \quad \therefore\ 12 + 2y = 16 \quad \therefore\ y = 2$

$\qquad \therefore\ x = 4,\ y = 2$

Check in (2): $\quad 4(4) - 3(2) = 16 - 6 = 10 \quad ✓$

Had the y-terms been of the same sign, we should, of course, have subtracted one line from the other to eliminate one of the variables.

Another example:

To solve $\quad 3x + y = 18 \qquad (1)$

$\qquad\qquad 4x + 2y = 21 \qquad (2)$

Working as before: $x = \ldots\ldots\ldots\ldots,\quad y = \ldots\ldots\ldots\ldots$

9

$$\boxed{x = 7{\cdot}5,\ y = -4{\cdot}5}$$

$$\begin{array}{lll} (1) & \times 2 & 6x + 2y = 36 \\ (2) & & \underline{4x + 2y = 21} \\ \text{Subtract:} & & 2x \quad\quad = 15 \quad \therefore\ x = 7{\cdot}5 \end{array}$$

Substitute $x = 7{\cdot}5$ in (1):

$$3(7{\cdot}5) + y = 18 \qquad 22{\cdot}5 + y = 18$$
$$\therefore\ y = 18 - 22{\cdot}5 = -4{\cdot}5 \qquad \therefore\ y = -4{\cdot}5$$

Check in (2): $4x + 2y = 4(7{\cdot}5) + 2(-4{\cdot}5) = 30 - 9 = 21$ ✓

$\therefore\ x = 7{\cdot}5,\ y = -4{\cdot}5$

10

And one more – on your own.

Solve $7x - 4y = 23$ (1)

$4x - 3y = 11$ (2)

Working as before: $x =$ ...., $y =$ ....

11

$$\boxed{x = 5,\ y = 3}$$

$$\begin{array}{lll} (1) & \times 3 & 21x - 12y = 69 \\ (2) & \times 4 & \underline{16x - 12y = 44} \\ \text{Subtract:} & & 5x \quad\quad = 25 \quad \therefore\ x = 5 \end{array}$$

Substitute in (2): $20 - 3y = 11 \quad \therefore\ 3y = 9 \quad \therefore\ y = 3$

$\therefore\ x = 5,\ y = 3$

Check in (1): $7x - 4y = 35 - 12 = 23$ ✓

Simultaneous linear equations with three unknowns

12

With three unknowns and three equations the method of solution is just an extension of the work with two unknowns.

Example 1

To solve

$$3x + 2y - z = 19 \quad (1)$$
$$4x - y + 2z = 4 \quad (2)$$
$$2x + 4y - 5z = 32 \quad (3)$$

We take a pair of equations and eliminate one of the variables using the method in Frame 7:

$$3x + 2y - z = 19 \quad (1)$$
$$4x - y + 2z = 4 \quad (2)$$

(1) × 2	$6x + 4y - 2z = 38$
(2)	$4x - y + 2z = 4$
Add:	$10x + 3y = 42$ (4)

Now take another pair, e.g. (1) and (3):

(1) × 5	$15x + 10y - 5z = 95$
(3)	$2x + 4y - 5z = 32$
Subtract:	$13x + 6y = 63$ (5)

We can now solve equations (4) and (5) for values of x and y in the usual way.

$x = \dots\dots\dots\dots$, $y = \dots\dots\dots\dots$

13

$$\boxed{x = 3,\ y = 4}$$

Because $\quad 10x + 3y = 42 \qquad (4)$

$\qquad\qquad 13x + 6y = 63 \qquad (5)$

$(4) \times 2 \quad 20x + 6y = 84$

$(5) \quad 13x + 6y = 63$

Subtract: $\quad 7x \quad = 21 \quad \therefore\ x = 3$

Substitute in (4): $30 + 3y = 42 \quad \therefore\ 3y = 12 \quad \therefore\ y = 4$

Then we substitute these values in one of the original equations to obtain the value of z:

e.g. (2) $4x - y + 2z = 12 - 4 + 2z = 4$

$\therefore\ 2z = -4 \quad \therefore\ z = -2$

$\therefore\ x = 3,\ y = 4,\ z = -2$

Finally, substitute all three values in the other two original equations as a check procedure:

(1) $3x + 2y - z = 9 + 8 + 2 = 19$

(3) $2x + 4y - 5z = 6 + 16 + 10 = 32$ – so all is well.

The working is clearly longer than with only two unknowns, but the method is no more difficult.

Here is another.

Example 2

Solve $\quad 5x - 3y - 2z = 31 \qquad (1)$

$\qquad 2x + 6y + 3z = 4 \qquad (2)$

$\qquad 4x + 2y - z = 30 \qquad (3)$

Work through it in just the same way and, as usual, check the results.

$x = \ldots\ldots\ldots\ldots,\quad y = \ldots\ldots\ldots\ldots,\quad z = \ldots\ldots\ldots\ldots$

14

$$\boxed{x = 5,\ y = 2,\ z = -6}$$

$$\begin{array}{lrcll}
(1) \times 3 & 15x - 9y - 6z & = & 93 & \\
(2) \times 2 & 4x + 12y + 6z & = & 8 & \\ \hline
 & 19x + 3y & = & 101 & (4) \\
(1) & 5x - 3y - 2z & = & 31 & \\
(3) \times 2 & 8x + 4y - 2z & = & 60 & \\ \hline
 & 3x + 7y & = & 29 & (5)
\end{array}$$

Solving (4) and (5) and substituting back gives the results above:

$x = 5, \quad y = 2, \quad z = -6$

15

Sometimes, the given equations need to be simplified before the method of solution can be carried out.

Pre-simplification

Example 1

Solve the pair of equations:

$$2(x + 2y) + 3(3x - y) = 38 \qquad (1)$$
$$4(3x + 2y) - 3(x + 5y) = -8 \qquad (2)$$

$$2x + 4y + 9x - 3y = 38 \qquad \therefore\ 11x + y = 38 \qquad (3)$$
$$12x + 8y - 3x - 15y = -8 \qquad \therefore\ 9x - 7y = -8 \qquad (4)$$

The pair of equations can now be solved in the usual manner.

$x = \ldots\ldots\ldots, \quad y = \ldots\ldots\ldots\ldots$

16

$$\boxed{x = 3,\ y = 5}$$

Because (3) × 7 $\quad 77x + 7y = 266$

(4) $\quad 9x - 7y = -8$

$86x \quad = 258 \quad \therefore\ x = 3$

Substitute in (3): $\quad 33 + y = 38 \quad \therefore\ y = 5$

Check in (4): $\quad 9(3) - 7(5) = 27 - 35 = -8$ ✓

$\therefore\ x = 3,\ y = 5$

Example 2

$$\frac{2x-1}{5} + \frac{x-2y}{10} = \frac{x+1}{4} \qquad (1)$$

$$\frac{3y+2}{3} + \frac{4x-3y}{2} = \frac{5x+4}{4} \qquad (2)$$

For (1) LCM = 20 $\quad \therefore\ \dfrac{20(2x-1)}{5} + \dfrac{20(x-2y)}{10} = \dfrac{20(x+1)}{4}$

$$4(2x-1) + 2(x-2y) = 5(x+1)$$

$$8x - 4 + 2x - 4y = 5x + 5$$

$$\therefore\ 5x - 4y = 9 \qquad (3)$$

Similarly for (2), the simplified equation is

....

17

$$\boxed{9x - 6y = 4}$$

Because

$$\frac{3y+2}{3} + \frac{4x-3y}{2} = \frac{5x+4}{4} \qquad \text{LCM} = 12$$

$$\therefore\ \frac{12(3y+2)}{3} + \frac{12(4x-3y)}{2} = \frac{12(5x+4)}{4}$$

$$4(3y+2) + 6(4x-3y) = 3(5x+4)$$

$$12y + 8 + 24x - 18y = 15x + 12$$

$$\therefore\ 24x - 6y + 8 = 15x + 12$$

$$\therefore\ 9x - 6y = 4 \qquad (4)$$

So we have $\quad 5x - 4y = 9 \qquad (3)$

$\quad 9x - 6y = 4 \qquad (4)$

Finishing off, we get

$x =$ ...., $\quad y =$ ....

18

$$\boxed{x = -19/3,\ y = -61/6}$$

Because

$$9 \times (5x - 4y = 9) \text{ gives } 45x - 36y = 81$$
$$5 \times (9x - 6y = 4) \text{ gives } 45x - 30y = 20$$

Subtracting gives $-6y = 61$ so that $y = -\dfrac{61}{6}$. Substitution then gives:

$$5x = 9 + 4y = 9 - \frac{244}{6} = -\frac{190}{6}, \text{ so } x = -\frac{190}{30} = -\frac{19}{3}$$

That was easy enough.

Example 3

$$5(x + 2y) - 4(3x + 4z) - 2(x + 3y - 5z) = 16$$
$$2(3x - y) + 3(x - 2z) + 4(2x - 3y + z) = -16$$
$$4(y - 2z) + 2(2x - 4y - 3) - 3(x + 4y - 2z) = -62$$

Simplifying these three equations, we get

....

19

$$\boxed{\begin{aligned} -9x + 4y - 6z &= 16 \\ 17x - 14y - 2z &= -16 \\ x - 16y - 2z &= -56 \end{aligned}}$$

Solving this set of three by equating coefficients, we obtain

$x =$ ...., $y =$ ...., $z =$ ....

20

$$\boxed{x = 2,\ y = 4,\ z = -3}$$

Here is the working as a check:

$$\begin{aligned} -9x + 4y - 6z &= 16 && (1)\\ 17x - 14y - 2z &= -16 && (2)\\ x - 16y - 2z &= -56 && (3) \end{aligned}$$

(1)	$-9x + 4y - 6z = 16$		
(2) $\times$ 3	$51x - 42y - 6z = -48$		
Subtracting:	$60x - 46y = -64$	$\therefore\ 30x - 23y = -32$	(4)
(2)	$17x - 14y - 2z = -16$		
(3)	$x - 16y - 2z = -56$		
Subtracting:	$16x + 2y = 40$	$\therefore\ 8x + y = 20$	(5)
$\therefore$ (4)	$30x - 23y = -32$		
(5) $\times$ 23	$184x + 23y = 460$		
Add:	$214x = 428$	$\therefore\ x = 2$	
Substitute in (5):	$16 + y = 20$	$\therefore\ y = 4$	

Substitute for x and y in (3):

$$\begin{aligned} 2 - 64 - 2z &= -56 \\ -62 - 2z &= -56 \quad \therefore\ z + 31 = 28 \\ &\qquad\quad\ \therefore\ z = -3 \end{aligned}$$

Check by substituting all values in (1):

$$-9(2) + 4(4) - 6(-3) = -18 + 16 + 18 = 16 \quad ✓$$

$$\therefore\ x = 2,\ y = 4,\ z = -3$$

21

Revision summary

1 A *linear equation* involves powers of the variables no higher than the first. A linear equation in a single variable is also referred to as a *simple equation*.

2 Simple equations are solved by isolating the variable on one side of the equation.

3 A linear equation in two variables has an infinite number of solutions. For two such equations there may be only one solution that satisfies both of them simultaneously.

4 Simultaneous equations can be solved:

(a) by substitution

(b) by elimination.

22

Revision exercise

1 Solve the following linear equations:

(a) $2(x-1)-4(x+2)=3(x+5)+(x-1)$

(b) $\dfrac{x-1}{2}-\dfrac{x+1}{2}=5-\dfrac{x+2}{4}$

(c) $\dfrac{3}{x+2}-\dfrac{5}{x}=-\dfrac{2}{x-1}$

2 Solve the following pairs of simultaneous equations:

(a) $x-y=2$ by elimination
$2x+3y=9$

(b) $4x+2y=10$ by substitution
$3x-5y=1$

3 Solve the following set of three equations in three unknowns:

$$x+y+z=6$$
$$2x-y+3z=9$$
$$x+2y-3z=-4$$

4 Simplify and solve the following set of simultaneous equations:

$$2(x+2y)-3(x+2z)-4(y-2z)=-1$$
$$3(2x-y)+2(3y-4z)-5(3x-2z)=-7$$
$$-(x-y)+(x-z)+(y+z)=-2$$

23

1 (a) $2(x-1)-4(x+2)=3(x+5)+(x-1)$.
Eliminate the brackets to give $2x-2-4x-8=3x+15+x-1$.
Simplify each side to give $-2x-10=4x+14$.
Add $2x-14$ to both sides of the equation to give $-24=6x$ so that $x=-4$.

(b) $\dfrac{x-1}{2}-\dfrac{x+1}{2}=5-\dfrac{x+2}{4}$.
Multiply throughout by 4 to give $2(x-1)-2(x+1)=20-x-2$.
Simplify to give $-4=18-x$, therefore $x=22$.

(c) $\dfrac{3}{x+2}-\dfrac{5}{x}=-\dfrac{2}{x-1}$. Multiply throughout by $x(x+2)(x-1)$ to give $3x(x-1)-5(x+2)(x-1)=-2x(x+2)$. Eliminate the brackets to give $3x^2-3x-5x^2-5x+10=-2x^2-4x$.
Simplify to give $-2x^2-8x+10=-2x^2-4x$. Add $2x^2$ to both sides to give $-8x+10=-4x$. Add $8x$ to both sides of the equation to give $10=4x$ so that $x=\dfrac{5}{2}$.

▶

2 (a) $x - y = 2$ [1]

$2x + 3y = 9$ [2] Multiply [1] by -2

$-2x + 2y = -4$ Add [2] to give $5y = 5$ so that $y = 1$. Substitution in [1] gives $x = 3$

(b) $4x + 2y = 10$ [1]

$3x - 5y = 1$ [2] Subtract $4x$ from both sides of [1] and then divide by 2 to give: $y = 5 - 2x$. Substitute this expression for y in [2] to give $3x - 5(5 - 2x) = 1$. Simplify to give $13x - 25 = 1$ so that $x = 2$ and $y = 5 - 4 = 1$.

3 $x + y + z = 6$ [1]

$2x - y + 3z = 9$ [2]

$x + 2y - 3z = -4$ [3] Add [1] and [3] and subtract [2] to give $(2x + 3y - 2z) - (2x - y + 3z) = (6 - 4) - 9$. That is

$4y - 5z = -7$ [4] Subtract [1] from [3] to give

$y - 4z = -10$ [5] Multiply this by 4 to give

$4y - 16z = -40$ [6] Subtract [6] from [4] to give $11z = 33$ so that $z = 3$. Substitute this value in [5] to give $y = -10 + 12 = 2$ so from [1], $x = 6 - 3 - 2 = 1$.

4 $2(x + 2y) - 3(x + 2z) - 4(y - 2z) = -1$ [1]

$3(2x - y) + 2(3y - 4z) - 5(3x - 2z) = -7$ [2]

$-(x - y) + (x - z) + (y + z) = -2$ [3] Eliminate the brackets to give

$-x + 2z = -1$ [4]

$-9x + 3y + 2z = -7$ [5]

$2y = -2$ [6] Thus $y = -1$. Substituting this value in [5] gives

$-9x + 2z = -4$ Subtracting [4] gives $-8x = -3$ so that $x = \frac{3}{8}$. Substitution in [4] gives $z = -\frac{5}{16}$.

24

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through them at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

Checklist F.5

Frames 25

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:

- Solve any linear equation? — Frames 1 to 6
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Solve simultaneous linear equations in two unknowns? — Frames 7 to 11
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Solve simultaneous linear equations in three unknowns? — Frames 12 to 20
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.5

Frames 26

1 Solve the following linear equations:

(a) $4(x+5)-6(2x+3)=3(x+14)-2(5-x)+9$

(b) $\dfrac{2x+1}{3}-\dfrac{2x+5}{5}=2+\dfrac{x-1}{6}$

(c) $\dfrac{2}{x-2}+\dfrac{3}{x}=\dfrac{5}{x-4}$

(d) $(4x-3)(3x-1)-(7x+2)(x+1)=(5x+1)(x-2)-10$ — 1 to 6

2 Solve the following pairs of simultaneous equations:

(a) $2x+3y=7$
$5x-2y=8$ by substitution

(b) $4x+2y=5$
$3x+y=9$ by elimination — 7 to 11

3 Solve the following set of three equations in three unknowns:

$2x+3y-z=-5$
$x-4y+2z=21$
$5x+2y-3z=-4$ — 12 to 14

4 Simplify and solve the following set of simultaneous equations:

$4(x+3y)-2(4x+3z)-3(x-2y-4z)=17$
$2(4x-3y)+5(x-4z)+4(x-3y+2z)=23$
$3(y+4z)+4(2x-y-z)+2(x+3y-2z)=5$ — 15 to 20

Further problems F.5

27

Solve the following equations, numbered **1** to **15**:

1 $(4x+5)(4x-3)+(5-4x)(5+4x)=(3x-2)(3x+2)-(9x-5)(x+1)$

2 $(3x-1)(x-3)-(4x-5)(3x-4)=6(x-7)-(3x-5)^2$

3 $\dfrac{4x+7}{3}-\dfrac{x-7}{5}=\dfrac{x+6}{9}$

4 $\dfrac{2x-5}{2}+\dfrac{5x+2}{6}=\dfrac{2x+3}{4}-\dfrac{x+5}{3}$

5 $(3x+1)(x-2)-(5x-2)(2x-1)=16-(7x+3)(x+2)$

6 $\dfrac{3x-4}{3}+\dfrac{2x-5}{8}=\dfrac{4x+5}{6}-\dfrac{x+2}{4}$

7 $8x-5y=10$
$6x-4y=11$

8 $\dfrac{2}{x+3}+\dfrac{5}{x}=\dfrac{7}{x-2}$

9 $(3-4x)(3+4x)=3(3x-2)(x+1)-(5x-3)(5x+3)$

10 $3x-2y+z=2$
$x-3y+2z=1$
$2x+y-3z=-5$

11 $2x-y+3z=20$
$x-6y-z=-41$
$3x+6y+2z=70$

12 $3x+2y+5z=2$
$5x+3y-2z=4$
$2x-5y-3z=14$

13 $5x-6y+3z=-9$
$2x-3y+2z=-5$
$3x-7y+5z=-16$

14 $\dfrac{3x+2}{4}-\dfrac{x+2y}{2}=\dfrac{x-3}{12}$
$\dfrac{2y+1}{5}+\dfrac{x-3y}{4}=\dfrac{3x+1}{10}$

15 $\dfrac{x-5}{x+5}+\dfrac{x-7}{x+7}=2$

▶

16 Simplify each side separately and hence determine the solution:

$$\frac{x-2}{x-4}-\frac{x-4}{x-6}=\frac{x-1}{x-3}-\frac{x-3}{x-5}$$

17 Solve the simultaneous equations:

$$\frac{3x-2y}{2}=\frac{2x+y}{7}+\frac{3}{2}$$

$$7-\frac{2x-y}{6}=x+\frac{y}{4}$$

18 Writing u for $\dfrac{1}{x+8y}$ and v for $\dfrac{1}{8x-y}$, solve the following equations for u and v, and hence determine the values of x and y:

$$\frac{2}{x+8y}-\frac{1}{8x-y}=4; \qquad \frac{1}{x+8y}+\frac{2}{8x-y}=7$$

19 Solve the pair of equations:

$$\frac{6}{x-2y}-\frac{15}{x+y}=0{\cdot}5$$

$$\frac{12}{x-2y}-\frac{9}{x+y}=-0{\cdot}4$$

20 Solve:

$$\frac{4}{x+1}+\frac{7}{4(2-y)}=9$$

$$\frac{3}{2(x+1)}+\frac{7}{2-y}=7$$

Programme F.6

Polynomial equations

Frames 1 to 41

Learning outcomes

When you have completed this Programme you will be able to:

- Solve quadratic equations by factors, completing the square and by formula
- Solve cubic equations with at least one linear factor
- Solve quartic equations with at least two linear factors

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.6

		Frames
1	Solve x by factorizing: $6x^2 - 13x + 6 = 0$	2 to 7
2	Solve for x (accurate to 3 dp) by completing the square: $3x^2 + 4x - 1 = 0$	8 to 13
3	Solve x (accurate to 3 dp) by using the formula: $x^2 - 3x + 1 = 0$	14 to 17
4	Solve for x: $x^3 - 8x^2 + 18x - 9 = 0$	18 to 26
5	Solve for x: $2x^4 - 9x^3 + 12x^2 - 3x - 2 = 0$	27 to 33

Polynomial equations

1

In Programme F.3 we looked at polynomial equations. In particular we selected a value for the variable x in a polynomial expression and found the resulting value of the polynomial expression. In other words, we *evaluated* the polynomial expression. Here we reverse the process by giving the polynomial expression a value of zero and finding those values of x which satisfy the resulting equation. We start with quadratic equations.

On now to the second frame

Quadratic equations, $ax^2 + bx + c = 0$

2

1 Solution by factors

We dealt at some length in Programmes F.2 and F.3 with the representation of a quadratic expression as a product of two simple linear factors, where such factors exist.

For example, $x^2 + 5x - 14$ can be factorized into $(x + 7)(x - 2)$. The equation $x^2 + 5x - 14 = 0$ can therefore be written in the form:

$(x + 7)(x - 2) = 0$

and this equation is satisfied if either factor has a zero value.

$\therefore\ x + 7 = 0$ or $x - 2 = 0$

i.e. $x = -7$ or $x = 2$

are the solutions of the given equation $x^2 + 5x - 14 = 0$.

By way of revision, then, you can solve the following equations with no trouble:

(a) $x^2 - 9x + 18 = 0$

(b) $x^2 + 11x + 28 = 0$

(c) $x^2 + 5x - 24 = 0$

(d) $x^2 - 4x - 21 = 0$

3

(a) $x = 3$ or $x = 6$
(b) $x = -4$ or $x = -7$
(c) $x = 3$ or $x = -8$
(d) $x = 7$ or $x = -3$

Not all quadratic expressions can be factorized as two simple linear factors. Remember that the test for the availability of factors with $ax^2 + bx + c$ is to calculate whether $(b^2 - 4ac)$ is ....

4

$\boxed{\text{a perfect square}}$

The test should always be applied at the begining to see whether, in fact, simple linear factors exist.

For example, with $x^2 + 8x + 15$, $(b^2 - 4ac) =$ ....

5

$\boxed{4, \text{ i.e. } 2^2}$

So $x^2 + 8x + 15$ can be written as the product of two simple linear factors.

But with $x^2 + 8x + 20$, $a = 1$, $b = 8$, $c = 20$

and $(b^2 - 4ac) = 8^2 - 4 \times 1 \times 20 = 64 - 80 = -16$, which is not a perfect square.

So $x^2 + 8x + 20$ cannot be written as the product of two simple linear factors.

On to the next frame

6

If the coefficient of the x^2 term is other than unity, the factorization process is a trifle more involved, but follows the routine already established in Programme F.2. You will remember the method.

Example 1

To solve $2x^2 - 3x - 5 = 0$

In this case, $ax^2 + bx + c = 0$, $a = 2$, $b = -3$, $c = -5$.

(a) Test for simple factors:

$$(b^2 - 4ac) = (-3)^2 - 4 \times 2 \times (-5)$$
$$= 9 + 40 = 49 = 7^2 \qquad \therefore \text{ simple factors exist.}$$

▶

(b) $|ac| = 10$. Possible factors of 10 are (1, 10) and (2, 5).

c is negative: $\therefore$ factors differ by $|b|$, i.e. 3

$\therefore$ Required factors are $(2, 5)$

c is negative: $\therefore$ factors are of different sign, the numerically larger having the sign of b, i.e. negative.

$$\therefore\ 2x^2 - 3x - 5 = 2x^2 + 2x - 5x - 5$$
$$= 2x(x+1) - 5(x+1)$$
$$= (x+1)(2x-5)$$

$\therefore$ The equation $2x^2 - 3x - 5 = 0$ becomes $(x+1)(2x-5) = 0$

$\therefore\ x + 1 = 0$ or $2x - 5 = 0 \quad \therefore\ x = -1$ or $x = 2{\cdot}5$

Example 2

In the same way, the equation:

$3x^2 + 14x + 8 = 0$

has solutions $x = \dots\dots\dots\dots$ or $x = \dots\dots\dots\dots$

7

$$\boxed{x = -4 \text{ or } x = -\frac{2}{3}}$$

Because $3x^2 + 14x + 8 = 0 \quad a = 3,\ b = 14,\ c = 8$

Test for simple factors: $(b^2 - 4ac) = 14^2 - 4 \times 3 \times 8 = 196 - 96 = 100 = 10^2$

$= \text{perfect square}, \therefore$ simple factors exist.

$|ac| = 24$. Possible factors of 24 are $(1, 24)$, $(2, 12)$, $(3, 8)$ and $(4, 6)$.
c is positive: $\therefore$ factors add up to $|b|$, i.e. 14 $\therefore$ $(2, 12)$
c is positive: $\therefore$ both factors have same sign as b, i.e. positive.

$$\therefore\ 3x^2 + 14x + 8 = 3x^2 + 2x + 12x + 8 = 0$$
$$= x(3x+2) + 4(3x+2) = 0$$
$$= (3x+2)(x+4) = 0$$

$\therefore\ x + 4 = 0$ or $3x + 2 = 0 \ \therefore\ x = -4$ or $x = -\dfrac{2}{3}$

They are all done in the same way – provided that simple linear factors exist, so always test to begin with by evaluating $(b^2 - 4ac)$.

Here is another: work right through it on your own. Solve the equation $4x^2 - 16x + 15 = 0$

$x = \dots\dots\dots\dots$ or $x = \dots\dots\dots\dots$

8

$$\boxed{x = \frac{3}{2} \text{ or } x = \frac{5}{2}}$$

2 Solution by completing the square

We have already seen that some quadratic equations are incapable of being factorized into two simple factors. In such cases, another method of solution must be employed. The following example will show the procedure.

Solve $x^2 - 6x - 4 = 0 \qquad a = 1, b = -6, c = -4.$

$(b^2 - 4ac) = 36 - 4 \times 1 \times (-4) = 36 + 16 = 52$. Not a perfect square.
$\therefore$ No simple factors.

So: $\qquad x^2 - 6x - 4 = 0$

Add 4 to both sides: $\qquad x^2 - 6x = 4$

Add to each side the square of half the coefficient of x:

$x^2 - 6x + (-3)^2 = 4 + (-3)^2$

$x^2 - 6x + 9 = 4 + 9 = 13$

$\therefore \ (x - 3)^2 = 13$

$\therefore \ x - 3 = \pm\sqrt{13}$. Remember to include the two signs.

$\therefore \ x = 3 \pm \sqrt{13} = 3 \pm 3{\cdot}6056 \ \therefore \ x = 6{\cdot}606$ or $x = -0{\cdot}606$

9

Now this one. Solve $x^2 + 8x + 5 = 0$ by the method of completing the square. First we take the constant term to the right-hand side:

$x^2 + 8x + 5 = 0$

$x^2 + 8x \qquad = -5$

Then we add to both sides ....

10

$$\boxed{\text{the square of half the coefficient of } x}$$

$x^2 + 8x + 4^2 = -5 + 4^2$

$\therefore \ x^2 + 8x + 16 = -5 + 16 = 11$

$\therefore \ (x + 4)^2 = 11$

And now we can finish it off, finally getting:

$x = \ldots\ldots\ldots\ldots$ or $x = \ldots\ldots\ldots\ldots$

11

$$\boxed{x = -0{\cdot}683 \text{ or } x = -7{\cdot}317}$$

If the coefficient of the squared term is not unity, the first step is to divide both sides of the equation by the existing coefficient.

For example: $2x^2 + 10x - 7 = 0$

Dividing throughout by 2: $x^2 + 5x - 3{\cdot}5 = 0$

We then proceed as before, which finally gives:

$x = \ldots\ldots\ldots\ldots$ or $x = \ldots\ldots\ldots\ldots$

12

$$\boxed{x = 0{\cdot}622 \text{ or } x = -5{\cdot}622}$$

Because

$$x^2 + 5x - 3{\cdot}5 = 0$$
$$\therefore\ x^2 + 5x = 3{\cdot}5$$
$$x^2 + 5x + 2{\cdot}5^2 = 3{\cdot}5 + 6{\cdot}25 = 9{\cdot}75$$
$$\therefore\ (x + 2{\cdot}5)^2 = 9{\cdot}75$$
$$\therefore\ x + 2{\cdot}5 = \pm\sqrt{9{\cdot}75} = \pm 3{\cdot}122$$
$$\therefore\ x = -2{\cdot}5 \pm 3{\cdot}122\ \therefore\ x = 0{\cdot}622 \text{ or } x = -5{\cdot}622$$

One more. Solve the equation $4x^2 - 16x + 3 = 0$ by completing the square.

$x = \ldots\ldots\ldots\ldots$ or $x = \ldots\ldots\ldots\ldots$

13

$$\boxed{x = 0{\cdot}197 \text{ or } x = 3{\cdot}803}$$

Because

$$4x^2 - 16x + 3 = 0 \qquad \therefore\ x^2 - 4x + 0{\cdot}75 = 0$$
$$\therefore\ x^2 - 4x = -0{\cdot}75$$
$$x^2 - 4x + (-2)^2 = -0{\cdot}75 + (-2)^2$$
$$x^2 - 4x + 4 = -0{\cdot}75 + 4 = 3{\cdot}25$$
$$\therefore\ (x - 2)^2 = 3{\cdot}25$$
$$\therefore\ x - 2 = \pm\sqrt{3{\cdot}25} = \pm 1{\cdot}8028$$
$$\therefore\ x = 2 \pm 1{\cdot}803\ \therefore\ x = 0{\cdot}197 \text{ or } x = 3{\cdot}803$$

On to the next topic

14

3 Solution by formula

We can establish a formula for the solution of the general quadratic equation $ax^2 + bx + c = 0$ which is based on the method of completing the square:

$$ax^2 + bx + c = 0$$

Dividing throughout by the coefficient of x, i.e. a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Subtracting $\frac{c}{a}$ from each side gives $x^2 + \frac{b}{a}x = -\frac{c}{a}$

We then add to each side the square of half the coefficient of x:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore\ x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \therefore\ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ \text{If } ax^2 + bx + c = 0,\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a, b and c for any particular quadratic equation gives the solutions of the equation.

Make a note of the formula: it is important

15

As an example, we shall solve the equation $2x^2 - 3x - 4 = 0$.

Here $a = 2$, $b = -3$, $c = -4$ and $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{3 \pm \sqrt{9 - 4 \times 2 \times (-4)}}{4} = \frac{3 \pm \sqrt{9 + 32}}{4} = \frac{3 \pm \sqrt{41}}{4}$$

$$= \frac{3 \pm 6{\cdot}403}{4} = \frac{-3{\cdot}403}{4} \text{ or } \frac{9{\cdot}403}{4}$$

$$\therefore\ x = -0{\cdot}851 \text{ or } x = 2{\cdot}351$$

It is just a case of careful substitution. You need, of course, to remember the formula. For

$$ax^2 + bx + c + 0 \quad x = \ldots\ldots\ldots\ldots$$

16

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As an exercise, use the formula method to solve the following:

(a) $5x^2 + 12x + 3 = 0$ (c) $x^2 + 15x - 7 = 0$
(b) $3x^2 - 10x + 4 = 0$ (d) $6x^2 - 8x - 9 = 0$

17

(a) $x = -2{\cdot}117$ or $x = -0{\cdot}283$ (c) $x = 0{\cdot}453$ or $x = -15{\cdot}453$
(b) $x = 0{\cdot}465$ or $x = 2{\cdot}869$ (d) $x = -0{\cdot}728$ or $x = 2{\cdot}061$

At this stage of our work, we can with advantage bring together a number of the items that we have studied earlier.

Solution of cubic equations having at least one linear factor in the algebraic expression

18

In Programme F.3, we dealt with the factorization of cubic polynomials, with application of the remainder theorem and factor theorem, and the evaluation of the polynomial functions by nesting. These we can now reapply to the solution of cubic equations.

So move on to the next frame for some examples

19

Example 1

To solve the cubic equation:

$$2x^3 - 11x^2 + 18x - 8 = 0$$

The first step is to find a linear factor of the cubic expression:

$$f(x) = 2x^3 - 11x^2 + 18x - 8$$

by application of the remainder theorem. To facilitate the calculation, we first write $f(x)$ in nested form:

$$f(x) = [(2x - 11)x + 18]x - 8$$

Now we seek a value for x, $(x = k)$ which gives a zero remainder on division by $(x - k)$. We therefore evaluate $f(1)$, $f(-1)$, $f(2)\ldots$ etc.

$f(1) = 1 \;\therefore\; (x - 1)$ is not a factor of $f(x)$
$f(-1) = -39 \therefore\; (x - 1)$ is not a factor of $f(x)$
$f(2) = 0 \therefore\; (x - 2)$ is a factor of $f(x)$

We therefore divide $f(x)$ by $(x - 2)$ to determine the remaining factor, which is …………

20

$$\boxed{2x^2 - 7x + 4}$$

Because

$$\begin{array}{r|llll}
 & & 2x^2 & -\;7x & +\;4 \\
\cline{2-5}
x-2 & 2x^3 & -\;11x^2 & +\;18x & -\;8 \\
 & 2x^3 & -\;4x^2 & & \\
\cline{2-3}
 & & -\;7x^2 & +\;18x & \\
 & & -\;7x^2 & +\;14x & \\
\cline{3-4}
 & & & 4x & -\;8 \\
 & & & 4x & -\;8 \\
\cline{4-5}
 & & & \cdot & \cdot \\
\cline{4-5}
\end{array}$$

$\therefore\; f(x) = (x - 2)(2x^2 - 7x + 4)$ and the cubic equation is now written:

$$(x - 2)(2x^2 - 7x + 4) = 0$$

which gives $x - 2 = 0$ or $2x^2 - 7x + 4 = 0$.

$\therefore\; x = 2$ and the quadratic equation can be solved in the usual way giving $x =$ ………… or $x =$ …………

21

$$\boxed{x = 0{\cdot}719 \text{ or } x = 2{\cdot}781}$$

$$2x^2 - 7x + 4 = 0 \therefore x = \frac{7 \pm \sqrt{49-32}}{4} = \frac{7 \pm \sqrt{17}}{4} = \frac{7 \pm 4{\cdot}1231}{4}$$

$$= \frac{2{\cdot}8769}{4} \text{ or } \frac{11{\cdot}1231}{4}$$

$$\therefore x = 0{\cdot}719 \text{ or } x = 2{\cdot}781$$

$\therefore$ $2x^3 - 11x^2 + 18x - 8 = 0$ has the solutions

$$x = 2, x = 0{\cdot}719, x = 2{\cdot}781$$

The whole method depends on the given expression in the equation having at least one linear factor.

Here is another.

Example 2

Solve the equation $3x^3 + 12x^2 + 13x + 4 = 0$

First, in nested form $f(x) = \dots\dots\dots\dots$

22

$$\boxed{f(x) = [(3x + 12)x + 13]x + 4}$$

Now evaluate $f(1), f(-1), f(2), \dots$

$f(1) = 32 \quad \therefore \ (x - 1)$ is not a factor of $f(x)$

$f(-1) = \dots\dots\dots\dots$

23

$$\boxed{f(-1) = 0}$$

$\therefore$ $(x + 1)$ is a factor of $f(x)$. Then, by long division, the remaining factor of $f(x)$ is $\dots\dots\dots\dots$

24

$$\boxed{3x^2 + 9x + 4}$$

$\therefore$ The equation $3x^3 + 12x^2 + 13x + 4 = 0$ can be written

$$(x + 1)(3x^2 + 9x + 4) = 0$$

so that $x + 1 = 0$ or $3x^2 + 9x + 4 = 0$

which gives $x = -1$ or $x = \dots\dots\dots\dots$ or $x = \dots\dots\dots\dots$

25

$$\boxed{x = -2{\cdot}457 \text{ or } x = -0{\cdot}543}$$

$$3x^2 + 9x + 4 = 0 \quad \therefore \; x = \frac{-9 \pm \sqrt{81-48}}{6} = \frac{-9 \pm \sqrt{33}}{6}$$
$$= \frac{-9 \pm 5{\cdot}7446}{6} = \frac{-14{\cdot}7446}{6} \text{ or } \frac{-3{\cdot}2554}{6}$$
$$= -2{\cdot}4574 \text{ or } -0{\cdot}5426$$

The complete solutions of $3x^3 + 12x^2 + 13x + 4 = 0$ are

$x = -1$, $x = -2{\cdot}457$, $x = -0{\cdot}543$

And now one more.

Example 3

Solve the equation $5x^3 + 2x^2 - 26x - 20 = 0$

Working through the method step by step, as before:

$x = \dots\dots\dots\dots$ or $x = \dots\dots\dots\dots$ or $x = \dots\dots\dots\dots$

26

$$\boxed{x = -2 \text{ or } x = -0{\cdot}825 \text{ or } x = 2{\cdot}425}$$

Here is the working, just as a check:

$f(x) = 5x^3 + 2x^2 - 26x - 20 = 0 \quad \therefore \; f(x) = [(5x+2)x - 26]x - 20$

$f(1) = -39 \quad \therefore \; (x-1)$ is not a factor of $f(x)$

$f(-1) = 3 \quad \therefore \; (x+1)$ is not a factor of $f(x)$

$f(2) = -24 \quad \therefore \; (x-2)$ is not a factor of $f(x)$

$f(-2) = 0 \quad \therefore \; (x+2)$ is a factor of $f(x)$:

$$\begin{array}{r|l}
 & 5x^2 - 8x - 10 \\
\hline
x+2 & 5x^3 + 2x^2 - 26x - 20 \\
 & 5x^3 + 10x^2 \\
\hline
 & -8x^2 - 26x \\
 & -8x^2 - 16x \\
\hline
 & -10x - 20 \\
 & -10x - 20 \\
\hline
 & \cdot \quad \cdot
\end{array}$$

$f(x) = (x+2)(5x^2 - 8x - 10) = 0 \quad \therefore \; x + 2 = 0$ or $5x^2 - 8x - 10 = 0$

$$\therefore \; x = -2 \text{ or } x = \frac{8 \pm \sqrt{64+200}}{10} = \frac{8 \pm \sqrt{264}}{10}$$
$$= \frac{8 \pm 16{\cdot}248}{10} = 0{\cdot}825 \text{ or } 2{\cdot}425$$

$\therefore \; x = -2$ or $x = -0{\cdot}825$ or $x = 2{\cdot}425$

Next frame

Solution of quartic equations, having at least two linear factors in the algebraic function

27

The method here is practically the same as that for solving cubic equations which we have just considered. The only difference is that we have to find two simple linear factors of $f(x)$. An example will reveal all.

Example 1

To solve the equation $4x^4 - 19x^3 + 24x^2 + x - 10 = 0$

(a) As before, express the polynomial in nested form:

$$f(x) = \{[(4x - 19)x + 24]x + 1\}x - 10$$

(b) Determine $f(1)$, $f(-1)$, $f(2)$ etc.

$f(1) = 0 \quad \therefore \; (x - 1)$ is a factor of $f(x)$:

$$\begin{array}{rrrrrr}
 & & 4x^3 & -15x^2 & +\;9x & +10 \\ \cline{2-6}
x-1 & \big) \; 4x^4 & -19x^3 & +24x^2 & +\;x & -10 \\
 & 4x^4 & -4x^3 & & & \\ \cline{2-3}
 & & -15x^3 & +24x^2 & & \\
 & & -15x^3 & +15x^2 & & \\ \cline{3-4}
 & & & 9x^2 & +\;x & \\
 & & & 9x^2 & -\;9x & \\ \cline{4-5}
 & & & & 10x & -10 \\
 & & & & 10x & -10 \\ \cline{5-6}
 & & & & \bullet & \bullet \\ \cline{5-6}
\end{array}$$

$\therefore \; f(x) = (x - 1)(4x^3 - 15x^2 + 9x + 10) = 0$

$\therefore \; x = 1$ or $4x^3 - 15x^2 + 9x + 10 = 0$, i.e. $F(x) = 0$

$F(x) = [(4x - 15)x + 9]x + 10$

$F(1) = 8 \quad \therefore \; (x - 1)$ is not a factor of $F(x)$

$F(-1) = -18 \quad \therefore \; (x + 1)$ is not a factor of $F(x)$

$F(2) = 0 \quad \therefore \; (x - 2)$ is a factor of $F(x)$

Division of $F(x)$ by $(x - 2)$ by long division gives $(4x^2 - 7x - 5)$:

$F(x) = (x - 2)(4x^2 - 7x - 5)$

$\therefore \; f(x) = (x - 1)(x - 2)(4x^2 - 7x - 5) = 0$

$\therefore \; x = 1, \; x = 2$, or $4x^2 - 7x - 5 = 0$

Solving the quadratic by formula gives solutions $x = -0{\cdot}545$ or $x = 2{\cdot}295$.

$\therefore \; x = 1, \;\; x = 2, \;\; x = -0{\cdot}545, \;\; x = 2{\cdot}295$

Now let us deal with another example

28

Example 2

Solve $2x^4 - 4x^3 - 23x^2 - 11x + 6 = 0$

$f(x) = \{[(2x - 4)x - 23]x - 11\}x + 6$

$f(1) = -30 \quad \therefore \ (x - 1)$ is not a factor of $f(x)$

$f(-1) = 0 \quad \therefore \ (x + 1)$ is a factor of $f(x)$:

$$
\begin{array}{r|llllll}
 & 2x^3 & -\ 6x^2 & -\ 17x & +\ 6 & & \\
\hline
x+1 & 2x^4 & -\ 4x^3 & -\ 23x^2 & -\ 11x & +\ 6 & \\
 & 2x^4 & +\ 2x^3 & & & & \\
\cline{2-3}
 & & -\ 6x^3 & -\ 23x^2 & & & \\
 & & -\ 6x^3 & -\ 6x^2 & & & \\
\cline{3-4}
 & & & -\ 17x^2 & -\ 11x & & \\
 & & & -\ 17x^2 & -\ 17x & & \\
\cline{4-5}
 & & & & 6x & +\ 6 & \\
 & & & & 6x & +\ 6 & \\
\cline{5-6}
 & & & & \cdot & \cdot & \\
\cline{5-6}
\end{array}
$$

$\therefore \ f(x) = (x + 1)(2x^3 - 6x^2 - 17x + 6) = 0$

$\therefore \ x = -1$ or $2x^3 - 6x^2 - 17x + 6 = 0$, i.e. $F(x) = 0$

$F(x) = [(2x - 6)x - 17]x + 6$

$F(1) = -15 \quad \therefore \ (x - 1)$ is not a factor of $F(x)$

Now you can continue and finish it off:

$x = -1, \ x =$ ...., $x =$ ...., $x =$ ....

29

$$\boxed{x = -2, \ x = 0{\cdot}321, \ x = 4{\cdot}679}$$

Because

$F(-1) = 15 \quad \therefore \ (x + 1)$ is not a factor of $F(x)$

$F(2) \ = -36 \quad \therefore \ (x - 2)$ is not a factor of $F(x)$

$F(-2) = 0 \quad \therefore \ (x + 2)$ is a factor of $F(x)$

Division of $F(x)$ by $(x + 2)$ by long division gives:

$F(x) = (x + 2)(2x^2 - 10x + 3)$

$\therefore \ f(x) = (x + 1)(x + 2)(2x^2 - 10x + 3) = 0$

$$\therefore \ x = -1, \ x = -2, \ x = \frac{10 \pm \sqrt{100 - 24}}{4} = \frac{10 \pm \sqrt{76}}{4}$$

$$= \frac{10 \pm 8{\cdot}7178}{4} = \frac{1{\cdot}2822}{4} \text{ or } \frac{18{\cdot}7178}{4}$$

$$= 0{\cdot}3206 \text{ or } 4{\cdot}6794$$

$\therefore \ x = 1, \ x = -2, \ x = 0{\cdot}321, \ x = 4{\cdot}679$

30

Example 3

Solve $f(x) = 0$ when $f(x) = 3x^4 + 2x^3 - 15x^2 + 12x - 2$

$$\begin{aligned} f(x) &= 3x^4 + 2x^3 - 15x^2 + 12x - 2 \\ &= \{[(3x+2)x - 15]x + 12\}x - 2 \end{aligned}$$

Now you can work through the solution, taking the same steps as before. There are no snags, so you will have no trouble:

$$x = \dots\dots\dots\dots$$

31

$$\boxed{x = 1,\ x = 1,\ x = -2{\cdot}897,\ x = 0{\cdot}230}$$

Because $f(1) = 0$, $\therefore$ $(x - 1)$ is a factor of $f(x)$:

$$\begin{array}{r|lllll}
 & & 3x^3 + & 5x^2 & -10x & +2 \\ \hline
x - 1 & 3x^4 & +2x^3 & -15x^2 & +12x & -2 \\
 & 3x^4 & -3x^3 & & & \\ \hline
 & & 5x^3 & -15x^2 & & \\
 & & 5x^3 & -\ \ 5x^2 & & \\ \hline
 & & & -10x^2 & +12x & \\
 & & & -10x^2 & +10x & \\ \hline
 & & & & 2x & -2 \\
 & & & & 2x & -2 \\ \hline
 & & & & \cdot & \cdot
\end{array}$$

$\therefore$ $f(x) = (x-1)(3x^3 + 5x^2 - 10x + 2) = (x-1) \times F(x) = 0$

$F(x) = [(3x+5)x - 10]x + 2$

$F(1) = 0$ $\therefore$ $(x-1)$ is a factor of $F(x)$:

$$\begin{array}{r|llll}
 & & 3x^2 + & 8x & -2 \\ \hline
x - 1 & 3x^3 & +5x^2 & -10x & +2 \\
 & 3x^3 & -3x^2 & & \\ \hline
 & & 8x^2 & -10x & \\
 & & 8x^2 & -\ \ 8x & \\ \hline
 & & & -\ \ 2x & +2 \\
 & & & -\ \ 2x & +2 \\ \hline
 & & & \cdot & \cdot
\end{array}$$

$\therefore$ $f(x) = (x-1)(x-1)(3x^2 + 8x - 2) = 0$

Solving the quadratic by formula gives $x = -2{\cdot}8968$ or $x = 0{\cdot}2301$

$\therefore$ $x = 1$, $x = 1$, $x = -2{\cdot}897$, $x = 0{\cdot}230$

All correct?

32

Finally, one more for good measure.

Example 4

Solve the equation $2x^4 + 3x^3 - 13x^2 - 6x + 8 = 0$

Work through it using the same method as before.

$x = \ldots\ldots\ldots\ldots$

33

$$\boxed{x = -1,\ x = 2,\ x = 0{\cdot}637,\ x = -3{\cdot}137}$$

Here is an outline of the solution:

$$f(x) = 2x^4 + 3x^3 - 13x^2 - 6x + 8$$
$$= \{[(2x + 3)x - 13]x - 6\}x + 8$$
$$f(-1) = 0 \quad \therefore\ (x + 1) \text{ is a factor of } f(x).$$

Division of $f(x)$ by $(x + 1)$ gives:

$$f(x) = (x + 1)(2x^3 + x^2 - 14x + 8) = (x + 1) \times F(x)$$
$$F(x) = 2x^3 + x^2 - 14x + 8 = [(2x + 1)x - 14]x + 8$$
$$F(2) = 0 \quad \therefore\ (x - 2) \text{ is a factor of } F(x).$$

Division of $F(x)$ by $(x - 2)$ gives $F(x) = (x - 2)(2x^2 + 5x - 4)$.

$$\therefore\ f(x) = (x + 1)(x - 2)(2x^2 + 5x - 4) = 0$$

Solution of the quadratic by formula shows $x = 0{\cdot}637$ or $x = -3{\cdot}137$.

$$\therefore\ x = 1,\ x = 2,\ x = 0{\cdot}637,\ x = -3{\cdot}137$$

34

The methods we have used for solving cubic and quartic equations have depended on the initial finding of one or more simple factors by application of the remainder theorem.

There are, however, many cubic and quartic equations which have no simple factors and other methods of solution are necessary. These are more advanced and will be dealt with later in the course.

35

Revision summary

1 *Quadratic equations* $ax^2 + bx + c = 0$

(a) *Solution by factors*
Test for availability of factors: evaluate $(b^2 - 4ac)$.
If the value of $(b^2 - 4ac)$ is a perfect square, simple linear factors exist.
If $(b^2 - 4ac)$ is not a perfect square, there are no simple linear factors.

(b) *Solution by completing the square*
(i) Remove the constant term to the RHS.
(ii) Add to both sides the square of half the coefficient of x. This completes the square on the LHS.
(iii) Take the square root of each side – including both signs.
(iv) Simplify the results to find the values of x.

(c) *Solution by formula*
Evaluate $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to obtain two values for x.

2 *Cubic equations* – with at least one linear factor

(a) Rewrite the polynomial function, $f(x)$, in nested form.

(b) Apply the remainder theorem by substituting values for x until a value, $x = k$, gives zero remainder. Then $(x - k)$ is a factor of $f(x)$.

(c) By long division, determine the remaining quadratic factor of $f(x)$, i.e. $(x - k)(ax^2 + bx + c) = 0$. Then $x = k$, and $ax^2 + bx + c = 0$ can be solved by the usual methods. $\therefore\ x = k,\ x = x_1,\ x = x_2$.

3 *Quartic equations* – with at least two linear factors.

(a) Find the first linear factor as in section **2** above.

(b) Divide by $(x - k)$ to obtain the remaining cubic expression. Then $f(x) = (x - k) \times F(x)$ where $F(x)$ is a cubic.

(c) $f(x) = 0 \quad \therefore\ (x - k) \times F(x) = 0 \quad \therefore\ x = k$ or $F(x) = 0$.

(d) The cubic $F(x)$ is now solved as in section **2** above, giving:
$(x - m)(ax^2 + bx + c) = 0$
$\therefore\ x = k,\ x = m$ and $ax^2 + bx + c = 0$
giving the four solutions $x = k$, $x = m$, $x = x_1$ and $x = x_2$.

36

Revision exercise

1 Solve for x by factorizing:

$$12x^2 - 5x - 2 = 0$$

2 Solve for x (accurate to 3 dp) by completing the square:

$$3x^2 - 4x - 2 = 0$$

3 Solve for x (accurate to 3 dp) by using the formula:

$$x^2 - 3x + 1 = 0$$

4 Solve for x:

$$x^3 - 5x^2 + 7x - 2 = 0$$

5 Solve for x:

$$x^4 + x^3 - 2x^2 - x + 1 = 0$$

37

1 $12x^2 - 5x - 2 = 0$

Here, $ax^2 + bx + c = 0$ where $a = 12$, $b = -5$, $c = -2$.

(a) Test for factors: $(b^2 - 4ac) = (-5)^2 - [4 \times 12 \times (-2)] = 25 + 96 = 121$ so that factors exist.

(b) $|ac| = 24$. Possible factors of 24 are (1, 24), (2, 12), (3, 8) and (4, 6).
c is negative $\therefore$ factors differ by $|b| = 5$ $\therefore$ Required factors are (3, 8)
c is negative $\therefore$ factors of different sign, the numerically larger having the same sign of b, that is negative, therefore:

$$\begin{aligned} 12x^2 - 5x - 2 &= 12x^2 - 8x + 3x - 2 \\ &= 4x(3x - 2) + (3x - 2) \\ &= (4x + 1)(3x - 2) = 0 \end{aligned}$$

So that $(4x + 1) = 0$ or $(3x - 2) = 0$. That is $x = -1/4$ or $x = 2/3$.

2 $3x^2 - 4x - 2 = 0$, that is $x^2 - (4/3)x - (2/3) = 0$.

Here $a = 1$, $b = -4/3$, $c = -2/3$. So that $(b^2 - 4ac) = \frac{16}{9} + 4 \times 1 \times \frac{2}{3} = \frac{40}{9}$

– not a perfect square so we proceed by completing the square:

$$\begin{aligned} x^2 - (4/3)x - 2/3 &= 0 \\ x^2 - (4/3)x &= 2/3 \\ x^2 - (4/3)x + (2/3)^2 &= 2/3 + (2/3)^2 \\ x^2 - (4/3)x + 4/9 &= 10/9 \\ (x - 2/3)^2 &= 10/9 \end{aligned}$$

So that $x - 2/3 = \pm\sqrt{10/9}$ giving $x = 2/3 \pm \sqrt{10/9} = 0{\cdot}6667 \pm 1{\cdot}0541$ to 4 dp. Therefore $x = 1{\cdot}721$ or $x = -0{\cdot}387$.

▶

3 $x^2 - 3x + 1 = 0$. Here $a = 1$, $b = -3$, $c = 1$ where:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that $x = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2} = 2{\cdot}618$ or $0{\cdot}382$

4 $x^3 - 5x^2 + 7x - 2 = 0$.

$$\begin{aligned} \text{Let } f(x) &= x^3 - 5x^2 + 7x - 2 \\ &= ((x-5)x + 7)x - 2 \text{ so} \\ f(2) &= ((2-5)2 + 7)2 - 2 = 0 \end{aligned}$$

Thus $x - 2$ is a factor of $f(x)$ and so $x = 2$ is a solution of the cubic. To find the other two we first need to divide the cubic by this factor:

$$\frac{x^3 - 5x^2 + 7x - 2}{x - 2} = x^2 - 3x + 1$$

That is $x^3 - 5x^2 + 7x - 2 = (x-2)(x^2 - 3x + 1) = 0$. So, from the previous question, the complete solution is $x = 2$, $x = 2{\cdot}6180$ or $x = 0{\cdot}3820$ to 4 dp.

5 $x^4 + x^3 - 2x^2 - x + 1 = 0$.

$$\begin{aligned} \text{Let } f(x) &= x^4 + x^3 - 2x^2 - x + 1 \\ &= (((x+1)x - 2)x - 1)x + 1 \text{ so} \\ f(1) &= (((1+1)1 - 2)1 - 1)1 + 1 = 0 \text{ and} \\ f(-1) &= ((([-1] + 1)[-1] - 2)[-1] - 1)[-1] + 1 = 0 \end{aligned}$$

Thus $x - 1$ and $x + 1$ are factors of $f(x)$. Now $(x-1)(x+1) = x^2 - 1$ and:

$$\frac{x^4 + x^3 - 2x^2 - x + 1}{x^2 - 1} = x^2 + x - 1$$

so that $x^4 + x^3 - 2x^2 - x + 1 = (x^2 - 1)(x^2 + x - 1)$ giving the solution of $x^4 + x^3 - 2x^2 - x + 1 = 0$ as $x = \pm 1$, $x = 0{\cdot}618$ or $x = -1{\cdot}618$.

38

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the begining of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through them at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

39

Checklist F.6

Check this list before you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Solve quadratic equations by factors, completing the square and by formula? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	2 to 17
• Solve cubic equations with at least one linear factor? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	18 to 26
• Solve quartic equations with at least two linear factors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	27 to 33

Test exercise F.6

40

	Frames
1 Solve the following equations by the method of factors: (a) $x^2 + 11x + 18 = 0$ (d) $2x^2 + 13x + 20 = 0$ (b) $x^2 - 13x + 42 = 0$ (e) $3x^2 + 5x - 12 = 0$ (c) $x^2 + 4x - 21 = 0$ (f) $5x^2 - 26x + 24 = 0$	2 to 7
2 Solve the following by completing the square: (a) $2x^2 - 4x - 3 = 0$ (c) $3x^2 + 12x - 18 = 0$ (b) $5x^2 + 10x + 2 = 0$	8 to 13
3 Solve by means of the formula: (a) $4x^2 + 11x + 2 = 0$ (c) $5x^2 - 9x - 4 = 0$ (b) $6x^2 - 3x - 5 = 0$	14 to 17
4 Solve the cubic equation: $2x^3 + 11x^2 + 6x - 3 = 0$	18 to 26
5 Solve the quartic equation: $4x^4 - 18x^3 + 23x^2 - 3x - 6 = 0$	27 to 33

Further problems F.6

41

1 Solve the following by the method of factors:

(a) $x^2 + 3x - 40 = 0$ (c) $x^2 + 10x + 24 = 0$

(b) $x^2 - 11x + 28 = 0$ (d) $x^2 - 4x - 45 = 0$

2 Solve the following:

(a) $4x^2 - 5x - 6 = 0$ (d) $7x^2 + 4x - 5 = 0$

(b) $3x^2 + 7x + 3 = 0$ (e) $6x^2 - 15x + 7 = 0$

(c) $5x^2 + 8x + 2 = 0$ (f) $8x^2 + 11x - 3 = 0$

3 Solve the following cubic equations:

(a) $5x^3 + 14x^2 + 7x - 2 = 0$ (d) $2x^3 + 4x^2 - 3x - 3 = 0$

(b) $4x^3 + 7x^2 - 6x - 5 = 0$ (e) $4x^3 + 2x^2 - 17x - 6 = 0$

(c) $3x^3 - 2x^2 - 21x - 10 = 0$ (f) $4x^3 - 7x^2 - 17x + 6 = 0$

4 Solve the following quartic equations:

(a) $2x^4 - 4x^3 - 23x^2 - 11x + 6 = 0$

(b) $5x^4 + 8x^3 - 8x^2 - 8x + 3 = 0$

(c) $4x^4 - 3x^3 - 30x^2 + 41x - 12 = 0$

(d) $2x^4 + 14x^3 + 33x^2 + 31x + 10 = 0$

(e) $3x^4 - 6x^3 - 25x^2 + 44x + 12 = 0$

(f) $5x^4 - 12x^3 - 6x^2 + 17x + 6 = 0$

(g) $2x^4 - 16x^3 + 37x^2 - 29x + 6 = 0$

Programme F.7

Partial fractions

Frames 1 to 49

Learning outcomes

When you have completed this Programme you will be able to:

- Factorize the denominator of an algebraic fraction into its prime factors
- Separate an algebraic fraction into its partial fractions
- Recognize the rules of partial fractions

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.7

Express each of the following in partial fraction form:	Frames
1 $\dfrac{3x+9}{x^2+8x+12}$	1 to 7
2 $\dfrac{x^2+x+1}{x^2+3x+2}$	8 to 21
3 $\dfrac{7x^2+6x+5}{(x+1)(x^2+x+1)}$	25 to 31
4 $\dfrac{5x+6}{(x-1)^2}$	32 to 34
5 $\dfrac{2x^3-5x+13}{(x+4)^2}$	8 to 34

Partial fractions

1

To simplify an arithmetical expression consisting of a number of fractions, we first convert the individual fractions to a new form having a common denominator which is the LCM of the individual denominators.

With $\frac{2}{5}-\frac{3}{4}+\frac{1}{2}$ the LCM of the denominators, 5, 4 and 2, is 20.

$$\therefore \frac{2}{5}-\frac{3}{4}+\frac{1}{2}=\frac{8-15+10}{20}=\frac{3}{20}$$

In just the same way, algebraic fractions can be combined by converting them to a new denominator which is the LCM of the individual denominators.

For example, with $\frac{2}{x-3}-\frac{4}{x-1}$, the LCM of the denominators is $(x-3)(x-1)$. Therefore:

$$\begin{aligned}\frac{2}{x-3}-\frac{4}{x-1}&=\frac{2(x-1)-4(x-3)}{(x-3)(x-1)}\\&=\frac{2x-2-4x+12}{(x-3)(x-1)}\\&=\frac{10-2x}{(x-3)(x-1)}\end{aligned}$$

In practice, the reverse process is often required. That is, presented with a somewhat cumbersome algebraic fraction there is a need to express this as a number of simpler component fractions.

From the previous example:

$$\frac{2}{x-3}-\frac{4}{x-1}=\frac{10-2x}{(x-3)(x-1)}$$

The two simple fractions on the left-hand side are called the *partial fractions* of the expression on the right-hand side. What follows describes how these partial fractions can be obtained from the original fraction.

So, on to the next frame

2

Let us consider a simple case and proceed step by step.

To separate:

$$\frac{8x-28}{x^2-6x+8}$$

into its partial fractions we must first factorize the denominator into its prime factors. You can do this:

$$x^2-6x+8=(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

3

$$\boxed{(x-2)(x-4)}$$

Therefore:

$$\frac{8x-28}{x^2-6x+8}=\frac{8x-28}{(x-2)(x-4)}$$

We now assume that each simple factor in the denominator gives rise to a single partial fraction. That is, we assume that we can write:

$$\frac{8x-28}{(x-2)(x-4)}=\frac{A}{x-2}+\frac{B}{x-4}$$

where A and B are constants. We shall now show that this assumption is valid by finding the values of A and B. First we add the two partial fractions on the RHS to give:

$$\frac{8x-28}{(x-2)(x-4)}=\dots\dots\dots\dots$$

The answer is in the next frame

4

$$\boxed{\frac{8x-28}{x^2-6x+8}=\frac{A(x-4)+B(x-2)}{(x-2)(x-4)}}$$

Because

$$\begin{aligned}\frac{A}{x-2}+\frac{B}{x-4}&=\frac{A(x-4)}{(x-2)(x-4)}+\frac{B(x-2)}{(x-2)(x-4)}\\&=\frac{A(x-4)+B(x-2)}{(x-2)(x-4)}\end{aligned}$$

The equation $\dfrac{8x-28}{x^2-6x+8}=\dfrac{A(x-4)+B(x-2)}{(x-2)(x-4)}$ is an identity because the RHS is just an alternative way of writing the LHS. Also, because the denominator on the RHS is an alternative way of writing the denominator on the LHS, the same must be true of the numerators. Consequently, equating the numerators we find that …………

Next frame

5

$$\boxed{8x-28\equiv A(x-4)+B(x-2)}$$

Because this is an identity it must be true for all values of x. It is convenient to choose a value of x that makes one of the brackets zero. For example:

Letting $x=4$ gives $B=\dots\dots\dots\dots$

6

$$\boxed{B = 2}$$

Because

$32 - 28 = A(0) + B(2)$ so that $4 = 2B$ giving $B = 2$.

Similarly if we let $x = 2$ then $A = \dots\dots\dots\dots$

7

$$\boxed{A = 6}$$

Because

$16 - 28 = A(-2) + B(0)$ so that $-12 = -2A$ giving $A = 6$.

Therefore:

$$\frac{8x - 28}{(x-2)(x-4)} = \frac{6}{x-2} + \frac{2}{x-4}$$

the required partial fraction breakdown.

This example has demonstrated the basic process whereby the partial fractions of a given rational expression can be obtained. There is, however, one important proviso that has not been mentioned:

To effect the partial fraction breakdown of a rational algebraic expression it is necessary for the degree of the numerator to be less than the degree of the denominator.

If, in the original algebraic rational expression, the degree of the numerator is not less than the degree of the denominator then we divide out by long division. This gives a polynomial with a rational remainder where the remainder has a numerator with degree less than the denominator. The remainder can then be broken down into its partial fractions. In the following frames we consider some examples of this type.

8

Example 1

Express $\dfrac{x^2 + 3x - 10}{x^2 - 2x - 3}$ in partial fractions.

The first consideration is $\dots\dots\dots\dots$

9

Is the numerator of lower degree than the denominator?

No, it is not, so we have to divide out by long division:

$$\begin{array}{r|l} & 1 \\ \hline x^2-2x-3 & x^2+3x-10 \\ & x^2-2x-3 \\ \hline & 5x-7 \end{array} \qquad \therefore \frac{x^2+3x-10}{x^2-2x-3}=1+\frac{5x-7}{x^2-2x-3}$$

Now we factorize the denominator into its prime factors, which gives ....

10

$$\boxed{(x+1)(x-3)}$$

$$\therefore \frac{x^2+3x-10}{x^2-2x-3}=1+\frac{5x-7}{(x+1)(x-3)}$$

The remaining fraction will give partial fractions of the form:

$$\frac{5x-7}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}$$

Multiplying both sides by the denominator $(x+1)(x-3)$:

$$5x-7= \ldots\ldots\ldots\ldots$$

11

$$\boxed{A(x-3)+B(x+1)}$$

$5x-7 \equiv A(x-3)+B(x+1)$ is an identity, since the RHS is the LHS merely written in a different form. Therefore the statement is true for any value of x we choose to substitute.

As was said previously, it is convenient to select a value for x that makes one of the brackets zero. So if we put $x=3$ in both sides of the identity, we get ....

12

$$\boxed{15-7=A(0)+B(4)}$$

i.e. $8=4B \quad \therefore B=2$

Similarly, if we substitute $x=-1$, we get ....

13

$$\boxed{-5-7=A(-4)+B(0)}$$

$\therefore \quad -12=-4A \qquad \therefore \; A=3$

$$\therefore \quad \frac{5x-7}{(x+1)(x-3)}=\frac{3}{x+1}+\frac{2}{x-3}$$

So, collecting our results together:

$$\frac{x^2+3x-10}{x^2-2x-3}=1+\frac{5x-7}{x^2-2x-3}=1+\frac{5x-7}{(x+1)(x-3)}$$
$$=1+\frac{3}{x+1}+\frac{2}{x-3}$$

Example 2

Express $\dfrac{2x^2+18x+31}{x^2+5x+6}$ in partial fractions.

The first step is ....

14

to divide the numerator by the denominator

since the numerator is not of lower degree than that of the denominator.
$\therefore \; \dfrac{2x^2+18x+31}{x^2+5x+6}=2+\dfrac{8x+19}{x^2+5x+6}$. Now we attend to $\dfrac{8x+19}{x^2+5x+6}$.
Factorizing the denominator, we have ....

15

$$\boxed{\frac{8x+19}{(x+2)(x+3)}}$$

so the form of the partial fractions will be ....

16

$$\boxed{\frac{A}{x+2}+\frac{B}{x+3}}$$

i.e. $\dfrac{8x+19}{(x+2)(x+3)}=\dfrac{A}{x+2}+\dfrac{B}{x+3}$

You can now multiply both sides by the denominator $(x+2)(x+3)$ and finish it off:

$$\frac{2x^2+18x+31}{x^2+5x+6}=\;............$$

17

$$\boxed{2+\frac{3}{x+2}+\frac{5}{x+3}}$$

So move on to the next frame

18

Example 3

Express $\dfrac{3x^3-x^2-13x-13}{x^2-x-6}$ in partial fractions.

Applying the rules, we first divide out:

$$\frac{3x^3-x^2-13x-13}{x^2-x-6}=\dots\dots\dots\dots$$

19

$$\boxed{3x+2+\frac{7x-1}{x^2-x-6}}$$

Now we attend to $\dfrac{7x-1}{x^2-x-6}$ in the normal way.

Finish it off

20

$$\boxed{3x+2+\frac{3}{x+2}+\frac{4}{x-3}}$$

Because

$$\frac{7x-1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}$$

$$\therefore\ 7x-1=A(x-3)+B(x+2)$$

$x=3 \qquad 20=A(0)+B(5) \qquad \therefore\ B=4$

$x=-2 \qquad -15=A(-5)+B(0) \qquad \therefore\ A=3$

Remembering to include the polynomial part:

$$\therefore\ \frac{3x^3-x^2-13x-13}{x^2-x-6}=3x+2+\frac{3}{x+2}+\frac{4}{x-3}$$

Now one more entirely on your own just like the last one.

Example 4

Express $\dfrac{2x^3+3x^2-54x+50}{x^2+2x-24}$ in partial fractions.

Work right through it: then check with the next frame.

$$\frac{2x^3+3x^2-54x+50}{x^2+2x-24}=\dots\dots\dots\dots$$

21

$$\boxed{2x - 1 + \frac{1}{x-4} - \frac{5}{x+6}}$$

Here it is:

$$\begin{array}{r|l}
 & \quad 2x \; - \; 1 \\
\cline{2-2}
x^2 + 2x - 24 & 2x^3 + 3x^2 - 54x + 50 \\
 & \underline{2x^3 + 4x^2 - 48x} \\
 & \quad\;\; - x^2 - \;\; 6x + 50 \\
 & \quad\;\; \underline{- x^2 - \;\; 2x + 24} \\
 & \qquad\qquad\;\; - 4x + 26
\end{array}$$

$$\therefore \; \frac{2x^3 + 3x^2 - 54x + 50}{x^2 + 2x - 24} = 2x - 1 - \frac{4x - 26}{x^2 + 2x - 24}$$

$$\frac{4x - 26}{(x-4)(x+6)} = \frac{A}{x-4} + \frac{B}{x+6} \qquad \therefore \; 4x - 26 = A(x+6) + B(x-4)$$

$$x = 4 \qquad -10 = A(10) + B(0) \qquad \therefore \; A = -1$$

$$x = -6 \qquad -50 = A(0) + B(-10) \qquad \therefore \; B = 5$$

$$\therefore \; \frac{2x^3 + 3x^2 - 54x + 50}{x^2 + 2x - 24} = 2x - 1 - \left\{ -\frac{1}{x-4} + \frac{5}{x+6} \right\}$$

$$= 2x - 1 + \frac{1}{x-4} - \frac{5}{x+6}$$

At this point let us pause and summarize the main facts so far on the breaking into partial fractions of rational algebraic expressions with denominators in the form of a product of two simple factors

Revision summary

22

1 To effect the partial fraction breakdown of a rational algebraic expression it is necessary for the degree of the numerator to be less than the degree of the denominator. In such an expression whose denominator can be expressed as a product of simple prime factors, each of the form $ax + b$:

(a) Write the rational expression with the denominator given as a product of its prime factors.

(b) Each factor then gives rise to a partial fraction of the form $\dfrac{A}{ax+b}$ where A is a constant whose value is to be determined.

(c) Add the partial fractions together to form a single algebraic fraction whose numerator contains the unknown constants and whose denominator is identical to that of the original expression.

(d) Equate the numerator so obtained with the numerator of the original algebraic fraction.

(e) By substituting appropriate values of x in this equation determine the values of the unknown constants.

▶

2 If, in the original algebraic rational expression, the degree of the numerator is not less than the degree of the denominator then we divide out by long division. This gives a polynomial with a rational remainder where the remainder has a numerator with degree less than the denominator. The remainder can then be broken down into its partial fractions.

23

Revision exercise

Express the following in partial fractions:

1 $\dfrac{x+7}{x^2-7x+10}$ **3** $\dfrac{3x^2-8x-63}{x^2-3x-10}$

2 $\dfrac{10x+37}{x^2+3x-28}$ **4** $\dfrac{2x^2+6x-35}{x^2-x-12}$

24

1 $$\frac{x+7}{x^2-7x+10}=\frac{x+7}{(x-2)(x-5)}=\frac{A}{x-2}+\frac{B}{x-5}$$

$$\therefore\ x+7=A(x-5)+B(x-2)$$

$x=5 \qquad 12=A(0)+B(3) \qquad \therefore\ B=4$

$x=2 \qquad 9=A(-3)+B(0) \qquad \therefore\ A=-3$

$$\frac{x+7}{x^2-7x+10}=\frac{4}{x-5}-\frac{3}{x-2}$$

2 $$\frac{10x+37}{x^2+3x-28}=\frac{10x+37}{(x-4)(x+7)}=\frac{A}{x-4}+\frac{B}{x+7}$$

$$\therefore\ 10x+37=A(x+7)+B(x-4)$$

$x=-7 \qquad -33=A(0)+B(-11) \qquad \therefore\ B=3$

$x=4 \qquad 77=A(11)+B(0) \qquad \therefore\ A=7$

$$\therefore\ \frac{10x+37}{x^2+3x-28}=\frac{7}{x-4}+\frac{3}{x+7}$$

3 $$\frac{3x^2-8x-63}{x^2-3x-10}=3+\frac{x-33}{(x+2)(x-5)}$$

$$\frac{x-33}{(x+2)(x-5)}=\frac{A}{x+2}+\frac{B}{x-5}$$

$$\therefore\ x-33=A(x-5)+B(x+2)$$

$x=5 \qquad -28=A(0)+B(7) \qquad \therefore\ B=-4$

$x=-2 \qquad -35=A(-7)+B(0) \qquad \therefore\ A=5$

$$\therefore\ \frac{3x^2-8x-63}{x^2-3x-10}=3+\frac{5}{x+2}-\frac{4}{x-5}$$

▶

4 $\dfrac{2x^2+6x-35}{x^2-x-12} = 2 + \dfrac{8x-11}{(x+3)(x-4)}$

$$\frac{8x-11}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}$$

$$8x - 11 = A(x-4) + B(x+3)$$

$x = 4 \qquad 21 = A(0) + B(7) \qquad \therefore\ B = 3$

$x = -3 \qquad -35 = A(-7) + B(0) \qquad \therefore\ A = 5$

$$\therefore\ \frac{2x^2+6x-35}{x^2-x-12} = 2 + \frac{5}{x+3} + \frac{3}{x-4}$$

Denominators with repeated and quadratic factors

25

Now let's look at a rational algebraic fraction where the denominator contains a quadratic factor that will not factorize into two simple factors.

Express $\dfrac{15x^2-x+2}{(x-5)(3x^2+4x-2)}$ in partial fractions.

Here the degree of the numerator is less than the degree of the denominator so no initial division is required. However, the denominator contains a quadratic factor that cannot be factorized further into simple factors. The usual test confirms this because $(b^2 - 4ac) = 16 - 4 \times 3 \times (-2) = 40$ which is not a perfect square. In this situation there is a rule that applies:

An irreducible quadratic factor in the denominator of the original rational expression of the form $(ax^2 + bx + c)$ gives rise to a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$

$$\therefore\ \frac{15x^2-x+2}{(x-5)(3x^2+4x-2)} = \frac{A}{x-5} + \frac{Bx+C}{3x^2+4x-2}$$

Multiplying throughout by the denominator:

$$\begin{aligned} 15x^2 - x + 2 &= A(3x^2+4x-2) + (Bx+C)(x-5) \\ &= 3Ax^2 + 4Ax - 2A + Bx^2 + Cx - 5Bx - 5C \end{aligned}$$

Collecting up like terms on the RHS gives:

$$15x^2 - x + 2 = \dots\dots\dots\dots$$

26

$$\boxed{15x^2 - x + 2 = (3A + B)x^2 + (4A - 5B + C)x - 2A - 5C}$$

This is an identity, so we can equate coefficients of like terms on each side:

$[x^2]$ $15 = 3A + B$ (1)

constant term [CT] $2 = -2A - 5C$ (2)

$[x]$ $-1 = 4A - 5B + C$ (3)

From (1): $B = 15 - 3A$

From (2): $2 = -2A - 5C \quad \therefore \; 5C = -2A - 2 \quad \therefore \; C = \dfrac{-(2A + 2)}{5}$

Substituting for B and C in (3), we have:

$$-1 = 4A - 5(15 - 3A) - \frac{2A + 2}{5}$$

$$\therefore \; -5 = 20A - 25(15 - 3A) - (2A + 2)$$

$$-5 = 20A - 375 + 75A - 2A - 2 = 93A - 377$$

$$\therefore \; 93A = 377 - 5 = 372 \quad \therefore \; A = 4$$

Sub. in (1): $15 = 12 + B \quad \therefore \; B = 3$

Sub. in (3): $-1 = 16 - 15 + C \quad \therefore \; C = -2$

$$\therefore \; \frac{15x^2 - x + 2}{(x - 5)(3x^2 + 4x - 2)} = \frac{4}{x - 5} + \frac{3x - 2}{3x^2 + 4x - 2}$$

Now on to another example

27

Express $\dfrac{7x^2 - 18x - 7}{(x - 4)(2x^2 - 6x + 3)}$ in partial fractions.

Here again, for the factor $(2x^2 - 6x + 3)$, $(b^2 - 4ac) = 36 - 24 = 12$ which is not a perfect square. $\therefore$ $(2x^2 - 6x + 3)$ is irreducible.

$\therefore$ The partial fractions of $\dfrac{7x^2 - 18x - 7}{(x - 4)(2x^2 - 6x + 3)}$ will be of the form ....

28

$$\boxed{\frac{A}{x - 4} + \frac{Bx + C}{2x^2 - 6x + 3}}$$

Multiplying throughout by the complete denominator:

$$7x^2 - 18x - 7 = \ldots\ldots\ldots\ldots$$

29

$$\boxed{A(2x^2 - 6x + 3) + (Bx + C)(x - 4)}$$

Then multiply out and collect up like terms, and that gives:

$$7x^2 - 18x - 7 = \ldots\ldots\ldots\ldots$$

30

$$\boxed{7x^2 - 18x - 7 = (2A + B)x^2 - (6A + 4B - C)x + 3A - 4C}$$

Now you can equate coefficients of like terms on each side and finish it. The required partial fractions for

$$\frac{7x^2 - 18x - 7}{(x-4)(2x^2 - 6x + 3)} = \ldots\ldots\ldots\ldots$$

31

$$\boxed{\frac{3}{x-4} + \frac{x+4}{2x^2 - 6x + 3}}$$

Because

$$[x^2] \qquad 7 = 2A + B \quad \therefore\ B = 7 - 2A \qquad (1)$$

$$[\mathrm{CT}] \qquad -7 = 3A - 4C \quad \therefore\ C = \frac{3A + 7}{4} \qquad (2)$$

$$[x] \quad -18 = -\left(6A + 28 - 8A - \frac{3A + 7}{4}\right)$$

$$\therefore\ 72 = 24A + 112 - 32A - 3A - 7$$

$$= -11A + 105 \quad \therefore\ 11A = 33 \quad \therefore\ A = 3$$

Substitution in (1) and (2) gives $B = 1$ and $C = 4$.

$$\therefore\ \frac{7x^2 - 18x - 7}{(x-4)(2x^2 - 6x + 3)} = \frac{3}{x-4} + \frac{x+4}{2x^2 - 6x + 3}$$

Next frame

32

Now let's look at a rational algebraic fraction where the denominator contains a repeated simple factor.

Express $\dfrac{35x - 14}{(7x-2)^2}$ in partial fractions.

Again, there is a rule that applies:

Repeated factors in the denominator of the algebraic expression of the form $(ax + b)^2$ *give partial fractions of the form* $\dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2}$. *Similarly* $(ax + b)^3$ *gives rise to partial fractions of the form:*

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$$

▶

Consequently, we write:

$$\frac{35x-14}{(7x-2)^2}=\frac{A}{7x-2}+\frac{B}{(7x-2)^2}$$

Then we multiply throughout as usual by the original denominator:

$$\begin{aligned}35x-14&=A(7x-2)+B\\&=7Ax-2A+B\end{aligned}$$

Now we simply equate coefficients and A and B are found:

$$\frac{35x-14}{(7x-2)^2}=\dots\dots\dots\dots$$

33

$$\boxed{\frac{5}{7x-2}-\frac{4}{(7x-2)^2}}$$

Similarly:

$$\frac{42x+44}{(6x+5)^2}\text{ in partial fractions }=\dots\dots\dots\dots$$

Complete it

34

$$\boxed{\frac{7}{6x+5}+\frac{9}{(6x+5)^2}}$$

$$\frac{42x+44}{(6x+5)^2}=\frac{A}{6x+5}+\frac{B}{(6x+5)^2}$$

$\therefore\ 42x+44=A(6x+5)+B=6Ax+5A+B$

$[x]\quad 42=6A\quad\therefore\ A=7$

$[\mathrm{CT}]\quad 44=5A+B=35+B\quad\therefore\ B=9$

$$\therefore\ \frac{42x+44}{(6x+5)^2}=\frac{7}{6x+5}+\frac{9}{(6x+5)^2}$$

And now this one:

Express $\dfrac{18x^2+3x+6}{(3x+1)^3}$ in partial fractions.

Complete the work with that one and then check with the next frame

35

$$\boxed{\frac{2}{3x+1}-\frac{3}{(3x+1)^2}+\frac{7}{(3x+1)^3}}$$

Here $\dfrac{18x^2+3x+6}{(3x+1)^3}=\dfrac{A}{3x+1}+\dfrac{B}{(3x+1)^2}+\dfrac{C}{(3x+1)^3}$

$$\begin{aligned}\therefore\ 18x^2+3x+6 &= A(3x+1)^2+B(3x+1)+C\\ &= A(9x^2+6x+1)+B(3x+1)+C\\ &= 9Ax^2+6Ax+A+3Bx+B+C\\ &= 9Ax^2+(6A+3B)x+(A+B+C)\end{aligned}$$

Equating coefficients:

$[x^2]$ $\quad 18=9A \qquad \therefore\ A=2$

$[x]$ $\quad 3=6A+3B \qquad 3=12+3B \quad 3B=-9 \quad \therefore\ B=-3$

[CT] $\quad 6=A+B+C \qquad 6=2-3+C \quad \therefore\ C=7$

$$\therefore\ \frac{18x^2+3x+6}{(3x+1)^3}=\frac{2}{3x+1}-\frac{3}{(3x+1)^2}+\frac{7}{(3x+1)^3}$$

Now determine the partial fractions of $\dfrac{20x^2-54x+35}{(2x-3)^3}$

The working is just the same as with the previous example:

$$\frac{20x^2-54x+35}{(2x-3)^3}=\ldots\ldots\ldots\ldots$$

36

$$\boxed{\frac{5}{2x-3}+\frac{3}{(2x-3)^2}-\frac{1}{(2x-3)^3}}$$

Because

$$\frac{20x^2-54x+35}{(2x-3)^3}=\frac{A}{2x-3}+\frac{B}{(2x-3)^2}+\frac{C}{(2x-3)^3}$$

$$\therefore\ 20x^2-54x+35=A(2x-3)^2+B(2x-3)+C$$

Multiplying out and collecting up like terms:

$$20x^2-54x+35=4Ax^2-(12A-2B)x+(9A-3B+C)$$

Then, equating coefficients in the usual way:

$A=5$; $B=3$; $C=-1$

$$\therefore\ \frac{20x^2-54x+35}{(2x-3)^3}=\frac{5}{2x-3}+\frac{3}{(2x-3)^2}-\frac{1}{(2x-3)^3}$$

Next frame

37

Let us now consider the case where the denominator is a cubic expression with different linear factors.

Express $\dfrac{10x^2+7x-42}{(x-2)(x+4)(x-1)}$ in partial fractions.

Here the factors are all different, so the partial fractions will be ....

38

$$\boxed{\frac{A}{x-2}+\frac{B}{x+4}+\frac{C}{x-1}}$$

$$\frac{10x^2+7x-42}{(x-2)(x+4)(x-1)}=\frac{A}{x-2}+\frac{B}{x+4}+\frac{C}{x-1}$$

$$\begin{aligned}\therefore\ 10x^2+7x-42&=A(x+4)(x-1)+B(x-2)(x-1)+C(x+4)(x-2)\\&=A(x^2+3x-4)+B(x^2-3x+2)+C(x^2+2x-8)\end{aligned}$$

Multiplying out and collecting up like terms:

$$10x^2+7x-42=\text{............}$$

39

$$\boxed{(A+B+C)x^2+(3A-3B+2C)x-(4A-2B+8C)}$$

Equating coefficients:

$$\begin{array}{rll}[x^2] & 10=A+B+C & (1)\\ [x] & 7=3A-3B+2C & (2)\\ [\mathrm{CT}] & -42=-4A+2B-8C & (3)\end{array}$$

Solving these three simultaneous equations provides the values of A, B and C:

$$\begin{array}{lll}(1)\times 2 & 2A+2B+2C=20 & \\ (2) & 3A-3B+2C=\ \ 7 & \\ \hline & -A+5B \qquad =13 & (4)\end{array}$$

$$\begin{array}{lll}(2)\times 4 & 12A-12B+8C=\ \ 28 & \\ (3) & -4A+\ \ 2B-8C=-42 & \\ \hline & 8A-10B \qquad =-14 & (5)\end{array}$$

Now we can solve (4) and (5) to find A and B, and then substitute in (1) to find C. So finally:

$$\frac{10x^2+7x-42}{(x-2)(x+4)(x-1)}=\text{............}$$

40

$$\boxed{\frac{2}{x-2}+\frac{3}{x+4}+\frac{5}{x-1}}$$

In this latest example, the denominator has conveniently been given as the product of three linear factors. It may well be that this could be given as a cubic expression, in which case factorization would have to be carried out using the remainder theorem before further progress could be made. Here then is an example which brings us to the peak of this programme.

Determine the partial fractions of $\dfrac{8x^2-14x-10}{x^3-4x^2+x+6}$.

First we see that no initial division is necessary. Then we have to factorize the denominator into its prime factors, as we did in Programme F.3.

So, putting $f(x)=x^3-4x^2+x+6$, we determine the three simple factors of $f(x)$, if they exist. These are ....

41

$$\boxed{(x+1)(x-2)(x-3)}$$

Because

$f(x)=x^3-4x^2+x+6=[(x-4)x+1]x+6$ in nested form.

$f(1)=4 \quad \therefore \ (x-1)$ is not a factor

$f(-1)=0 \quad \therefore \ (x+1)$ is a factor of $f(x)$:

$$\begin{array}{r|l}
 & x^2 \quad -5x+6 \\
\hline
x+1 & x^3 \quad -4x^2+\ x+6 \\
 & x^3 \quad +\ x^2 \\
\hline
 & \quad -5x^2+\ x \\
 & \quad -5x^2-5x \\
\hline
 & \qquad\qquad 6x+6 \\
 & \qquad\qquad 6x+6 \\
\hline
 & \qquad\qquad \cdot \quad \cdot
\end{array}$$

$\therefore \ f(x)=(x+1)(x^2-5x+6)=(x+1)(x-2)(x-3)$

$\therefore \ \dfrac{8x^2-14x-10}{x^3-4x^2+x+6}=\dfrac{8x^2-14x-10}{(x+1)(x-2)(x-3)}$

and now we can proceed as in the previous example. Work right through it and then check the results with the following frame.

$$\frac{8x^2-14x-10}{(x+1)(x-2)(x-3)}=\text{............}$$

42

$$\boxed{\frac{1}{x+1}+\frac{2}{x-2}+\frac{5}{x-3}}$$

Because

$$\frac{8x^2-14x-10}{(x+1)(x-2)(x-3)}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$\begin{aligned}\therefore\ 8x^2-14x-10 &= A(x-2)(x-3)+B(x+1)(x-3)+C(x+1)(x-2)\\ &= A(x^2-5x+6)+B(x^2-2x-3)+C(x^2-x-2)\\ &= (A+B+C)x^2-(5A+2B+C)x+(6A-3B-2C)\end{aligned}$$

$[x^2]$	$A+B+C=8$	(1)
$[x]$	$5A+2B+C=14$	(2)
[CT]	$6A-3B-2C=-10$	(3)
$(1)\times 2$	$2A+2B+2C=\ 16$	
(3)	$6A-3B-2C=-10$	
	$8A-B=6$	(4)
(2)	$5A+2B+C=14$	
(1)	$A+B+C=8$	
	$4A+B=6$	(5)
(5)	$4A+B=6$	
(6)	$8A-B=6$	
	$12A=12$ $\quad\therefore A=1$	
(5)	$4A+B=6 \quad 4+B=6 \quad \therefore B=2$	
(1)	$A+B+C=8 \quad 1+2+C=8 \quad \therefore C=5$	

$$\begin{aligned}\therefore\ \frac{8x^2-14x-10}{x^3-4x^2+x+6} &= \frac{8x^2-14x-10}{(x+1)(x-2)(x-3)}\\ &= \frac{1}{x+1}+\frac{2}{x-2}+\frac{5}{x-3}\end{aligned}$$

At this point let us pause and summarize the main facts on the breaking into partial fractions of rational algebraic expressions with denominators containing irreducible quadratic factors or repeated simple factors

43

Revision summary

To effect the partial fraction breakdown of a rational algebraic expression it is necessary for the degree of the numerator to be less than the degree of the denominator. In such an expression whose denominator contains:

1 A quadratic factor of the form $ax^2 + bx + c$ which cannot be expressed as a product of simple factors, the partial fraction breakdown gives rise to a partial fraction of the form $\dfrac{Ax + B}{ax^2 + bx + c}$

2 Repeated factors of the form $(ax + b)^2$, the partial fraction breakdown gives rise to a partial fraction of the form $\dfrac{A}{ax + b} + \dfrac{B}{(ax + b)^2}$. Similarly $(ax + b)^3$ gives rise to partial fractions of the form

$$\frac{A}{ax + b} + \frac{B}{(ax + b)^2} + \frac{C}{(ax + b)^3}$$

44

Revision exercise

Express in partial fractions:

1 $\dfrac{32x^2 - 28x - 5}{(4x - 3)^3}$

2 $\dfrac{9x^2 + 48x + 18}{(2x + 1)(x^2 + 8x + 3)}$

3 $\dfrac{12x^2 + 36x + 6}{x^3 + 6x^2 + 3x - 10}$

45

1 $\dfrac{2}{4x - 3} + \dfrac{5}{(4x - 3)^2} - \dfrac{8}{(4x - 3)^3}$

2 $\dfrac{5}{2x + 1} + \dfrac{2x + 3}{x^2 + 8x + 3}$

3 $\dfrac{3}{x - 1} + \dfrac{2}{x + 2} + \dfrac{7}{x + 5}$

46

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through them at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

47

Checklist F.7

Check this list before you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Factorize the denominator of an algebraic fraction into its prime factors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 3
• Separate an algebraic fraction into its partial fractions? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	3 to 7
• Recognize the rules of partial fractions? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	8 to 42

Test exercise F.7

48

Express each of the following in partial fractions:	Frames
1 $\dfrac{x-14}{x^2-10x+24}$	1 to 7
2 $\dfrac{13x-7}{10x^2-11x+3}$	1 to 7
3 $\dfrac{4x^2+9x-73}{x^2+x-20}$	8 to 21
4 $\dfrac{6x^2+19x-11}{(x+1)(x^2+5x-2)}$	25 to 31
5 $\dfrac{10x-13}{(2x-3)^2}$	32 to 34
6 $\dfrac{3x^2-34x+97}{(x-5)^3}$	35 to 36
7 $\dfrac{9x^2-16x+34}{(x+4)(2x-3)(3x+1)}$	37 to 40
8 $\dfrac{8x^2+27x+13}{x^3+4x^2+x-6}$	41 to 42

Further problems F.7

49

Express each of the following in partial fractions:

1 $\dfrac{7x+36}{x^2+12x+32}$

2 $\dfrac{5x-2}{x^2-3x-28}$

3 $\dfrac{x+7}{x^2-7x+10}$

4 $\dfrac{3x-9}{x^2-3x-18}$

5 $\dfrac{7x-9}{2x^2-7x-15}$

6 $\dfrac{14x}{6x^2-x-2}$

7 $\dfrac{13x-7}{10x^2-11x+3}$

8 $\dfrac{7x-7}{6x^2+11x+3}$

9 $\dfrac{18x+20}{(3x+4)^2}$

10 $\dfrac{35x+17}{(5x+2)^2}$

11 $\dfrac{12x-16}{(4x-5)^2}$

12 $\dfrac{5x^2-13x+5}{(x-2)^3}$

13 $\dfrac{75x^2+35x-4}{(5x+2)^3}$

14 $\dfrac{64x^2-148x+78}{(4x-5)^3}$

15 $\dfrac{8x^2+x-3}{(x+2)(x-1)^2}$

16 $\dfrac{4x^2-24x+11}{(x+2)(x-3)^2}$

17 $\dfrac{14x^2+31x+5}{(x-1)(2x+3)^2}$

18 $\dfrac{4x^2-47x+141}{x^2-13x+40}$

19 $\dfrac{5x^2-77}{x^2-2x-15}$

20 $\dfrac{8x^2-19x-24}{(x-2)(x^2-2x-5)}$

21 $\dfrac{5x^2+9x-1}{(2x+3)(x^2+5x+2)}$

22 $\dfrac{7x^2-18x-7}{(3x-1)(2x^2-4x-5)}$

23 $\dfrac{11x+23}{(x+1)(x+2)(x+3)}$

24 $\dfrac{5x^2+28x+47}{(x-1)(x+3)(x+4)}$

25 $\dfrac{8x^2-60x-43}{(4x+1)(2x+3)(3x-2)}$

26 $\dfrac{74x^2-39x-6}{(2x-1)(3x+2)(4x-3)}$

27 $\dfrac{16x^2-x+3}{6x^3-5x^2-2x+1}$

28 $\dfrac{11x^2+11x-32}{6x^3+5x^2-16x-15}$

29 $\dfrac{4x^2-3x-4}{6x^3-29x^2+46x-24}$

30 $\dfrac{2x^2+85x+36}{20x^3+47x^2+11x-6}$

Programme F.8

Trigonometry

Frames 1 to 47

Learning outcomes

When you have completed this Programme you will be able to:

- Convert angles measured in degrees, minutes and seconds into decimal degrees
- Convert degrees into radians and vice versa
- Use a calculator to determine the values of trigonometric ratios for any acute angle
- Verify trigonometric identities

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.8

Frames

1 Convert the angle $253°18'42''$ to decimal degree format. — 1 to 2

2 Convert the angle $73{\cdot}415°$ to degrees, minutes and seconds. — 2 to 3

3 Convert the following to radians to 2 dp:
(a) $47°$ (b) $12{\cdot}61°$ (c) $135°$ (as a multiple of π) — 4 to 6

4 Convert the following to degrees to 2 dp:
(a) $4{\cdot}621$ rad (b) $9\pi/4$ rad (c) $13\pi/5$ rad — 6 to 8

5 Find the value of each of the following to 4 dp:
(a) $\cos 24°$ (b) $\sin 5\pi/12$ (c) $\cot \pi/3$
(d) $\operatorname{cosec} 17{\cdot}9°$ (e) $\sec 5{\cdot}42°$ (f) $\tan 3{\cdot}24$ rad — 9 to 19

6 Given one side and the hypotenuse of a right-angled triangle as $5{\cdot}6$ and $12{\cdot}3$, find the length of the other side. — 20 to 23

7 Show that the triangle with sides 7, 24 and 25 is a right-angled triangle. — 20 to 23

8 A ship sails 12 km due north of a port and then sails 14 km due east. How far is the ship from the port? How much further east will it have sailed when it is 30 km from the port? — 24 to 27

9 Verify each of the following trigonometric identities:
(a) $(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2 \equiv 2$
(b) $(1 - \cos\theta)^{\frac{1}{2}}(1 + \cos\theta)^{\frac{1}{2}} \equiv \sin\theta$
(c) $\tan\theta + \sec\theta \equiv \dfrac{1}{\sec\theta - \tan\theta}$
(d) $\cos\theta - \cos\phi \equiv -2\sin\dfrac{\theta + \phi}{2}\sin\dfrac{\theta - \phi}{2}$ — 31 to 40

10 Show that:
(a) $\tan 75° = \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1}$
(b) $\sin 15° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ — 38 to 39

Angles

Rotation

1

When a straight line is rotated about a point it sweeps out an angle that can be measured either in *degrees* or in *radians*. By convention a straight line rotating through a *full angle* and returning to its starting position is said to have rotated through 360 degrees – 360° – where each degree is subdivided into 60 minutes – $60'$ – and each minute further subdivided into 60 seconds – $60''$. A *straight angle* is half of this, namely 180° and a *right angle* is half of this again, namely 90°. Any angle less than 90° is called an *acute* angle and any angle greater than 90° is called an *obtuse* angle.

An angle that is measured in degrees, minutes and seconds can be converted to a decimal degree as follows:

$$\begin{aligned} 45^\circ 36'18'' &= 45^\circ + \left(\frac{36}{60}\right)^\circ + \left(\frac{18}{60\times 60}\right)^\circ \\ &= (45 + 0{\cdot}6 + 0{\cdot}005)^\circ \\ &= 45{\cdot}605^\circ \end{aligned}$$

That was easy, so the decimal form of $53^\circ 29'7''$ to 3 dp is ....

The answer is in the next frame

2

$$\boxed{53{\cdot}485^\circ}$$

Because

$$\begin{aligned} 53^\circ 29'7'' &= 53^\circ + \left(\frac{29}{60}\right)^\circ + \left(\frac{7}{60\times 60}\right)^\circ \\ &= (53 + 0{\cdot}48\dot{3} + 0{\cdot}0019\dot{4})^\circ \\ &= 53{\cdot}485^\circ \text{ to 3 dp} \end{aligned}$$

How about the other way? For example, to convert $18{\cdot}478^\circ$ to degrees, minutes and seconds we proceed as follows:

$18{\cdot}478^\circ = 18^\circ + (0{\cdot}478 \times 60)'$ — Multiply the fractional part of the degree by 60

$= 18^\circ + 28{\cdot}68'$

$= 18^\circ + 28' + (0{\cdot}68 \times 60)''$ — Multiply the fractional part of the minute by 60

$= 18^\circ + 28' + 40{\cdot}8''$

$= 18^\circ 28'41''$ to the nearest second

So that $236{\cdot}986^\circ =$ .... (in degrees, minutes and seconds)

Next frame

3

$\boxed{236°59'10''}$

Because

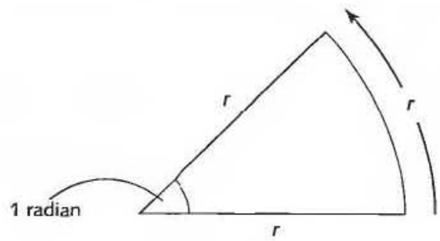
$$\begin{aligned}236{\cdot}986° &= 236° + (0{\cdot}986 \times 60)' \\ &= 236° + 59{\cdot}16' \\ &= 236° + 59' + (0{\cdot}16 \times 60)'' \\ &= 236° + 59' + 9{\cdot}6'' \\ &= 236°59'10'' \text{ to the nearest second}\end{aligned}$$

Move now to the next frame

4

Radians

An alternative unit of measure of an angle is the radian. If a straight line of length r rotates about one end so that the other end describes an arc of length r, the line is said to have rotated through 1 radian – 1 rad.



Because the arc described when the line rotates through a full angle is the circumference of a circle which measures $2\pi r$, the number of radians in a full angle is 2π rad. Consequently, relating degrees to radians we see that:

$$\begin{aligned}360° &= 2\pi \text{ rad} \\ &= 6{\cdot}2831\ldots \text{ rad}\end{aligned}$$

So that $1° = \ldots\ldots\ldots\ldots$ rad (to 3 sig fig)

The answer is in the next frame

5

$\boxed{0{\cdot}0175 \text{ rad}}$

Because

$$360° = 2\pi \text{ rad, so } 1° = \frac{2\pi}{360} = \frac{\pi}{180} = 0{\cdot}0175 \text{ rad to 3 sig fig}$$

Often, when degrees are transformed to radians they are given as multiples of π. For example:

$360° = 2\pi$ rad, so that $180° = \pi$ rad, $90° = \pi/2$ rad, $45° = \pi/4$ rad and so on

So, $30°$, $120°$ and $270°$ are given in multiples of π as ……, ……, …….

Answers in the next frame

6

$$\boxed{\pi/6 \text{ rad}, 2\pi/3 \text{ rad}, 3\pi/2 \text{ rad}}$$

Because

$30^\circ = 180^\circ/6 = \pi/6$ rad
$120^\circ = 2 \times 60^\circ = 2 \times (180^\circ/3) = 2\pi/3$ rad
$270^\circ = 3 \times 90^\circ = 3 \times (180^\circ/2) = 3\pi/2$ rad

Also, 1 rad = ... degrees (to 3 dp)

Check your answer in the next frame

7

$$\boxed{57{\cdot}296^\circ}$$

Because

$$2\pi \text{ rad} = 360^\circ, \text{ so } 1 \text{ rad} = \frac{360}{2\pi} = \frac{180}{\pi} = 57{\cdot}296^\circ$$

So, the degree equivalents of 2·34 rad, $\pi/3$ rad, $5\pi/6$ rad and $7\pi/4$ rad are ....

Check with the next frame

8

$$\boxed{134{\cdot}1^\circ \text{ to 1 dp}, 60^\circ, 150^\circ \text{ and } 315^\circ}$$

Because

$$2{\cdot}34 \text{ rad} = \left(2{\cdot}34 \times \frac{180}{\pi}\right)^\circ = 134{\cdot}1^\circ \text{ to 1 dp}$$

$$\pi/3 \text{ rad} = \left(\frac{\pi}{3} \times \frac{180}{\pi}\right)^\circ = 60^\circ$$

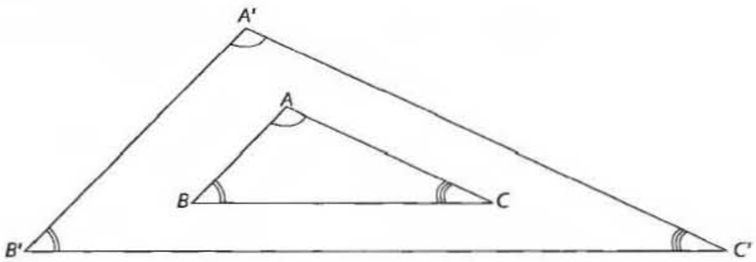
$$5\pi/6 \text{ rad} = \left(\frac{5\pi}{6} \times \frac{180}{\pi}\right)^\circ = 150^\circ$$

$$7\pi/4 \text{ rad} = \left(\frac{7\pi}{4} \times \frac{180}{\pi}\right)^\circ = 315^\circ$$

Move to the next frame

9 Triangles

All triangles possess shape and size. The shape of a triangle is governed by the three angles and the size by the lengths of the three sides. Two triangles can possess the same shape – possess the same angles – but be of different sizes. We say that two such triangles are *similar*. It is the similarity of figures of different sizes that permits an artist to draw a picture of a scene that looks like the real thing – the lengths of the corresponding lines in the picture and the scene are obviously different but the corresponding angles in the picture and the scene are the same.



A significant feature of similar figures is that lengths of corresponding sides are all in the same ratio so that, for example, in the similar triangles ABC and $A'B'C'$ in the figure:

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

So from a knowledge of the ratios of the sides of a given triangle we can make deductions about any triangle that is similar to it. For example, if in triangle ABC of the above figure:

$AB = 2$ cm, $AC = 5$ cm and $BC = 4$ cm

and in triangle $A'B'C'$, $A'B' = 3$ cm, the length of $A'C'$ can be found as follows:

Since $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$ and $\dfrac{AB}{A'B'} = \dfrac{2}{3}$ then $\dfrac{AC}{A'C'} = \dfrac{5}{A'C'} = \dfrac{2}{3}$ giving

$$A'C' = \frac{5 \times 3}{2} = 7{\cdot}5 \text{ cm}$$

This means that the length of $B'C' = \ldots\ldots\ldots\ldots$

Check your answer in the next frame

10

$\boxed{6 \text{ cm}}$

Because

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{2}{3} \text{ then } \frac{4}{B'C'} = \frac{2}{3} \text{ so that } B'C' = \frac{4 \times 3}{2} = 6 \text{ cm}$$

Ratios of side lengths of a given triangle are also equal to the corresponding ratios for a similar triangle. For example, since in the figure on page 246

$\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$ then multiplying both sides of this equation by $\dfrac{A'B'}{AC}$ we find that:

$$\frac{AB}{A'B'} \times \frac{A'B'}{AC} = \frac{AC}{A'C'} \times \frac{A'B'}{AC}, \text{ that is } \frac{AB}{AC} = \frac{A'B'}{A'C'}$$

So that $\dfrac{AB}{BC} = \ldots\ldots\ldots\ldots$

Answer in the next frame

11

$$\boxed{\frac{A'B'}{B'C'}}$$

Because

$\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$ then multiplying both sides of this equation by $\dfrac{A'B'}{BC}$ we find that:

$$\frac{AB}{A'B'} \times \frac{A'B'}{BC} = \frac{BC}{B'C'} \times \frac{A'B'}{BC}, \text{ that is } \frac{AB}{BC} = \frac{A'B'}{B'C'}$$

Similarly, $\dfrac{AC}{BC} = \ldots\ldots\ldots\ldots$

Next frame

12

$$\boxed{\frac{A'C'}{B'C'}}$$

Because

$\frac{AC}{A'C'} = \frac{BC}{B'C'}$ then multiplying both sides of this equation by $\frac{A'C'}{BC}$ gives:

$\frac{AC}{A'C'} \times \frac{A'C'}{BC} = \frac{BC}{B'C'} \times \frac{A'C'}{BC}$, that is $\frac{AC}{BC} = \frac{A'C'}{B'C'}$

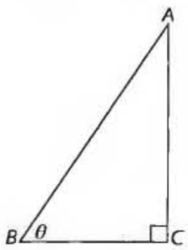
All triangles whose corresponding ratios of side lengths are equal have the same shape – they are similar triangles because corresponding angles are equal. Consequently, while the lengths of the sides of a triangle dictate the size of the triangle, the *ratios* of the side lengths dictate the angles of the triangle.

Because we need to know the properties of similar triangles we shall now link these ratios of side lengths to specific angles by using a right-angled triangle; the ratios are then called the *trigonometric ratios*.

On now to the next frame

13

Trigonometric ratios



Given the right-angled triangle ABC of the figure with angle θ at vertex B where side AC is *opposite* θ, side BC is *adjacent* to θ and side AB is called the *hypotenuse*, we define the trigonometric ratios as:

sine of angle θ as $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{AC}{AB}$ – this ratio is denoted by $\sin\theta$

cosine of angle θ as $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{BC}{AB}$ – this ratio is denoted by $\cos\theta$

tangent of angle θ as $\frac{\text{opposite}}{\text{adjacent}} = \frac{AC}{BC}$ – this ratio is denoted by $\tan\theta$

Every angle possesses its respective set of values for the trigonometric ratios and these are most easily found by using a calculator. For example, with the calculator in degree mode, enter 58 and press the *sin* key to display 0·84804... which is the value of $\sin 58°$ (that is the ratio of the opposite side over the hypotenuse of all right-angled triangles with an angle of 58°).

▶

Now, with your calculator in radian mode enter 2 and press the *sin* key to display 0·90929... which is the value of sin 2 rad – ordinarily we shall omit the rad and just write sin 2. Similar results are obtained using the *cos* key to find the cosine of an angle and the *tan* key to find the tangent of an angle.

Use a calculator in degree mode to find to 4 dp the values of:

(a) $\sin 27^{\circ}$
(b) $\cos 84^{\circ}$
(c) $\tan 43^{\circ}$

The answers are in the next frame

14

(a) 0·4340
(b) 0·1045
(c) 0·9325

That was easy enough. Now use a calculator in radian mode to find to 4 dp the values of the following where the angles are measured in radians:

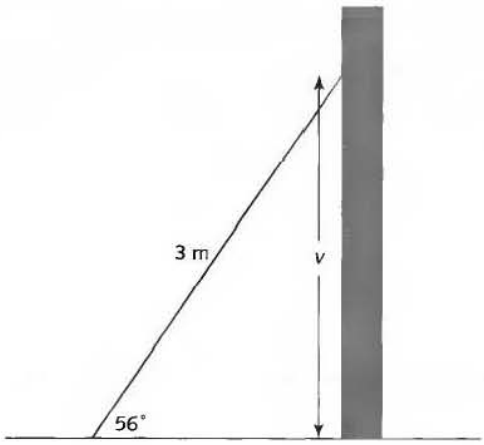
(a) $\cos 1{\cdot}321$
(b) $\tan 0{\cdot}013$
(c) $\sin \pi/6$

Check with the next frame

15

(a) 0·2472
(b) 0·0130
(c) 0·5000

We can now use these ratios to find unknowns. For example (see figure), a ladder of length 3 m leans against a wall at an angle of 56° to the horizontal.



▶

The vertical height of the ladder can now be found as follows. Dividing the vertical height v (the opposite) by the length of the ladder (the hypotenuse) gives the sine of the angle of inclination 56°. That is:

$\dfrac{\text{vertical height}}{\text{length of ladder}} = \sin 56°$. That is $\dfrac{v}{3} = 0{\cdot}82903\ldots$ giving the vertical height v as

$3 \times 0{\cdot}82903\ldots = 2{\cdot}49$ m (to 3 sig fig)

So if a ladder of length L leans against a wall at an angle of 60° to the horizontal with the top of the ladder 4·5 m above the ground, the length of the ladder is:

$L = \ldots\ldots$

The answer is in the next frame

16

$\boxed{5{\cdot}20 \text{ m}}$

Because

$$\frac{\text{vertical height}}{L} = \frac{4{\cdot}5}{L} = \sin 60° = 0{\cdot}8660\ldots$$

$$\text{so that } L = \frac{4{\cdot}5}{0{\cdot}8660} = 5{\cdot}20 \text{ m (to 2 dp)}$$

Next frame

17

Reciprocal ratios

In addition to the three trigonometrical ratios there are three *reciprocal ratios*, namely:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta} \equiv \frac{\cos\theta}{\sin\theta}$$

The values of these for a given angle can also be found using a calculator by finding the appropriate trigonometric ratio and then pressing the *reciprocal* key – the $\dfrac{1}{x}$ key.

So that, to 4 dp:

(a) $\cot 12° = \ldots\ldots\ldots\ldots$
(b) $\sec 37° = \ldots\ldots\ldots\ldots$
(c) $\operatorname{cosec} 71° = \ldots\ldots\ldots\ldots$

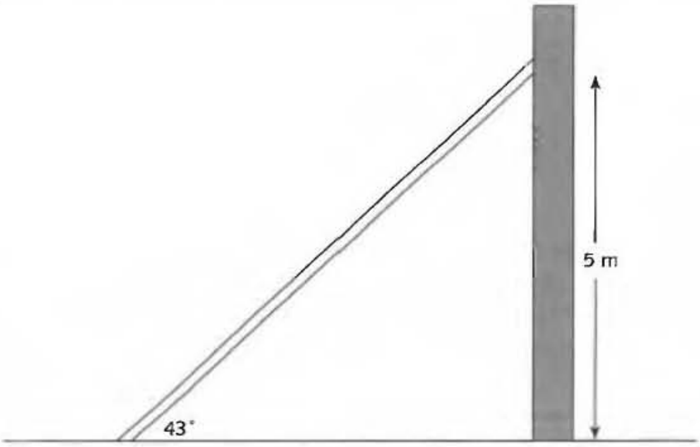
Next frame

18

(a) 4·7046
(b) 1·2521
(c) 1·0576

Because

(a) $\tan 12^\circ = 0{\cdot}21255\ldots$ and the reciprocal of that is 4·7046 to 4 dp
(b) $\cos 37^\circ = 0{\cdot}79863\ldots$ and the reciprocal of that is 1·2521 to 4 dp
(c) $\sin 71^\circ = 0{\cdot}94551\ldots$ and the reciprocal of that is 1·0576 to 4 dp



To strengthen a vertical wall a strut has to be placed 5 m up the wall and inclined at an angle of 43° to the ground. To do this the length of the strut must be ....

Check the next frame

19

$\boxed{7{\cdot}33\text{ m}}$

Because

$\dfrac{\text{length of strut}}{5} = \dfrac{1}{\sin 43^\circ} = \operatorname{cosec} 43^\circ$ that is $\dfrac{L}{5} = 1{\cdot}4662\ldots$
giving $L = 7{\cdot}33$ to 2 dp

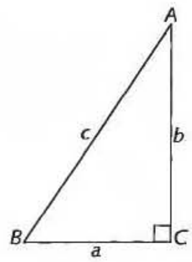
Now go to the next frame

20

Pythagoras' theorem

All right-angled triangles have a property in common that is expressed in Pythagoras' theorem:

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides



So in the figure:

$a^2 + b^2 = c^2$

Notice how the letter for each side length corresponds to the opposite angle (a is opposite angle A etc.); this is the common convention.
So, if a right-angled triangle has a hypotenuse of length 8 and one other side of length 3, the length of the third side to 3 dp is ....

Check your answer in the next frame

21

7·416

Because

If a represents the length of the third side then:

$$a^2 + 3^2 = 8^2 \text{ so } a^2 = 64 - 9 = 55 \text{ giving } a = 7{\cdot}416 \text{ to } 3 \text{ dp}$$

Here's another. Is the triangle with sides 7, 24 and 25 a right-angled triangle?

Answer in the next frame

22

$\boxed{\text{Yes}}$

Because

Squaring the lengths of the sides gives:

$7^2 = 49$, $24^2 = 576$ and $25^2 = 625$.

Now, $49 + 576 = 625$ so that $7^2 + 24^2 = 25^2$

The sum of the squares of the lengths of the two smaller sides is equal to the square on the longest side. Because the lengths satisfy Pythagoras' theorem, the triangle is right-angled.

How about the triangle with sides 5, 11 and 12? Is this a right-angled triangle?

Check in the next frame

23

$\boxed{\text{No}}$

Because

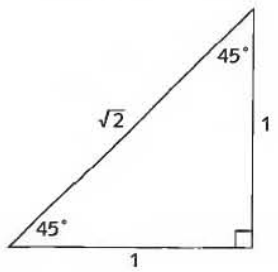
$5^2 = 25$ and $11^2 = 121$ so $5^2 + 11^2 = 146 \neq 12^2$. The squares of the smaller sides do not add up to the square of the longest side so the triangle does not satisfy Pythagoras' theorem and so is not a right-angled triangle.

Next frame

Special triangles

24

Two right-angled triangles are of special interest because the trigonometric ratios of their angles can be given in surd or fractional form. The first is the right-angled *isosceles* triangle (an isosceles triangle is any triangle with two sides of equal length) whose angles are 90°, 45° and 45° with side lengths, therefore, in the ratio $1 : 1 : \sqrt{2}$ (by Pythagoras' theorem).



Here we see that:

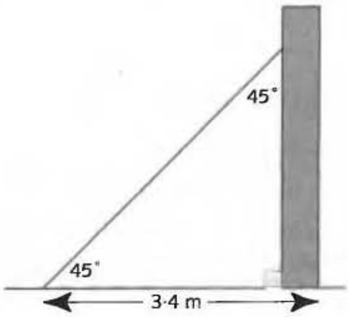
$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} \text{ and } \tan 45° = 1$$

Or, measuring the angles in radians:

$$\sin \pi/4 = \cos \pi/4 = \frac{1}{\sqrt{2}} \text{ and } \tan \pi/4 = 1$$

▶

Now, a problem using these ratios:



A prop in the form of an isosceles triangle constructed out of timber is placed against a vertical wall. If the length of the side along the horizontal ground is 3·4 m the length of the hypotenuse to 2 dp is obtained as follows:

$$\frac{\text{ground length}}{\text{hypotenuse}} = \frac{3{\cdot}4}{\text{hypotenuse}} = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

so that:

$$\text{hypotenuse} = \sqrt{2} \times 3{\cdot}4 = 4{\cdot}81 \text{ m}$$

Now one for you to try.

A bicycle frame is in the form of an isosceles triangle with the horizontal crossbar forming the hypotenuse. If the crossbar is 53 cm long, the length of each of the other two sides to the nearest mm is ....

The answer is in the next frame

25

$\boxed{37{\cdot}5 \text{ cm}}$

Because

$$\frac{\text{side length}}{\text{hypotenuse}} = \frac{\text{side length}}{53} = \cos 45^\circ = \frac{1}{\sqrt{2}} = 0{\cdot}7071\ldots$$

so that:

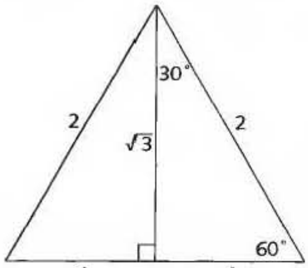
$$\text{side length} = 53 \times 0{\cdot}7071 = 37{\cdot}5 \text{ cm}$$

Next frame for some more surd forms

Half equilateral

26

The second right-angled triangle of interest is the *half equilateral* triangle (an equilateral triangle is a triangle whose sides are all the same length) with side lengths (again, by Pythagoras) in the ratio $1 : \sqrt{3} : 2$.



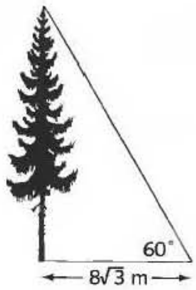
Here we see that:

$$\sin 30° = \cos 60° = \frac{1}{2}, \ \sin 60° = \cos 30° = \frac{\sqrt{3}}{2} \text{ and } \tan 60° = \frac{1}{\tan 30°} = \sqrt{3}$$

Again, if we measure the angles in radians:

$$\sin \pi/6 = \cos \pi/3 = \frac{1}{2}, \ \sin \pi/3 = \cos \pi/6 = \frac{\sqrt{3}}{2} \text{ and } \tan \pi/3 = \frac{1}{\tan \pi/6} = \sqrt{3}$$

Here's an example using these new ratios.



A tree casts a horizontal shadow $8\sqrt{3}$ m long. If a line were to be drawn from the end of the shadow to the top of the tree it would be inclined to the horizontal at 60°. The height of the tree is obtained as follows:

$$\frac{\text{height of tree}}{\text{length of shadow}} = \tan 60° = \sqrt{3}$$

so that

$$\text{height of tree} = \sqrt{3} \times \text{ length of shadow } = \sqrt{3} \times 8\sqrt{3} = 8 \times 3 = 24 \text{ m}$$

Now try this one.

When a small tent is erected the front forms an equilateral triangle. If the tent pole is $\sqrt{3}$ m long, the lengths of the sides of the tent are both ....

Check your answer in the next frame

27

$$\boxed{2\ \text{m}}$$

Because

$$\frac{\text{length of tent pole}}{\text{length of tent side}} = \frac{\sqrt{3}}{L} = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

so that $L = 2$ m.

28

Revision summary

1 Angles can be measured in degrees, minutes and seconds or radians where a full angle has a magnitude of 360° or 2π radians.

2 Similar triangles have the same shape – the same angles – but different sizes.

3 Ratios of the sides of one triangle have the same values as ratios of the corresponding sides in a similar triangle.

4 The trigonometric ratios are defined within a right-angled triangle. They are:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}}$$

and their reciprocal ratios are:

$$\operatorname{cosec}\theta = 1/\sin\theta$$
$$\sec\theta = 1/\cos\theta$$
$$\cot\theta = 1/\tan\theta$$

5 Pythagoras' theorem states that:

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides

$$a^2 + b^2 = c^2$$

where a and b are the lengths of the two smaller sides and c is the length of the hypotenuse.

6 The right-angled isosceles triangle has angles $\pi/2$, $\pi/4$ and $\pi/4$, and sides in the ratio $1 : 1 : \sqrt{2}$.

7 The right-angled half equilateral triangle has angles $\pi/2$, $\pi/3$ and $\pi/6$, and sides in the ratio $1 : \sqrt{3} : 2$.

29

Revision exercise

1 Convert the angle $164°49'13''$ to decimal degree format.

2 Convert the angle $87{\cdot}375°$ to degrees, minutes and seconds.

3 Convert the following to radians to 2 dp:

(a) $73°$ (b) $18{\cdot}34°$ (c) $240°$

4 Convert the following to degrees to 2 dp:

(a) 3·721 rad (b) $7\pi/6$ rad (c) $11\pi/12$ rad

5 Find the value of each of the following to 4 dp:

(a) $\sin 32°$ (b) $\cos \pi/12$ (c) $\tan 2\pi/5$
(d) $\sec 57{\cdot}8°$ (e) $\operatorname{cosec} 13{\cdot}33°$ (f) $\cot 0{\cdot}99$ rad

6 Given one side and the hypotenuse of a right-angled triangle as 5·6 cm and 12·3 cm respectively, find the length of the other side.

7 Show that the triangle with sides 9 m, 40 m and 41 m is a right-angled triangle.

8 A rod of length $7\sqrt{2}$ cm is inclined to the horizontal at an angle of $\pi/4$ radians. A shadow is cast immediately below it from a lamp directly overhead. What is the length of the shadow? What is the new length of the shadow if the rod's inclination is changed to $\pi/3$ to the vertical?

30

1 $164{\cdot}8203°$ to 4 dp.

2 $87°50'15''$

3 (a) 1·27 rad (b) 0·32 (c) $4\pi/3$ rad $= 4{\cdot}19$ rad

4 (a) $213{\cdot}20°$ (b) $210°$ (c) $165°$

5 (a) 0·5299 (b) 0·9659 (c) 3·0777
(d) 1·8766 (e) 4·3373 (f) 0·6563

6 If the sides are a, b and c where c is the hypotenuse then $a^2 + b^2 = c^2$. That is, $(5{\cdot}6)^2 + b^2 = (12{\cdot}3)^2$ so that $b = \sqrt{(12{\cdot}3)^2 - (5{\cdot}6)^2} = 11{\cdot}0$ to 1 dp.

7 $40^2 + 9^2 = 1681 = 41^2$ thereby satisfying Pythagoras' theorem.

8 If l is the length of the shadow then $\dfrac{l}{7\sqrt{2}} = \cos \pi/4 = \dfrac{1}{\sqrt{2}}$ so that $l = 7$ cm.

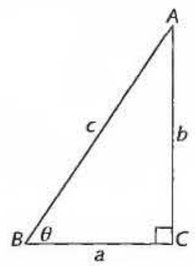
If the angle is $\pi/3$ to the vertical then $\dfrac{l}{7\sqrt{2}} = \sin \pi/3 = \dfrac{\sqrt{3}}{2}$ so that

$l = \dfrac{7\sqrt{3}\sqrt{2}}{2} = 7\sqrt{\dfrac{3}{2}} = 8{\cdot}6$ cm.

On now to the next topic

Trigonometric identities

31 The fundamental identity



Given the right-angled triangle of the above figure with vertices A, B and C, sides opposite the vertices of a, b and hypotenuse c and angle θ at B then:

$$a^2 + b^2 = c^2$$

Dividing both sides by c^2 gives:

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Because $\frac{a}{c} = \cos\theta$ and $\frac{b}{c} = \sin\theta$ this equation can be written as:

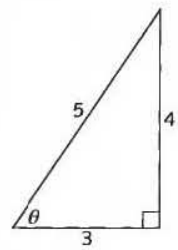
$$\cos^2\theta + \sin^2\theta = 1$$

where the notation $\cos^2\theta \triangleq (\cos\theta)^2$ and $\sin^2\theta \triangleq (\sin\theta)^2$. Since this equation is true for any angle θ the equation is in fact an identity:

$$\cos^2\theta + \sin^2\theta \equiv 1$$

and is called the *fundamental trigonometrical identity*.

For example, to show that the triangle with sides 3 cm, 4 cm and 5 cm is a right-angled triangle it is sufficient to show that it satisfies the fundamental trigonometrical identity. That is, taking the side of length 3 cm to be adjacent to θ (the side with length 5 cm is obviously the hypotenuse as it is the longest side) then:



$\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$ and so

$$\cos^2\theta + \sin^2\theta = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25}$$
$$= \frac{25}{25} = 1$$

Is the triangle with sides of length 8 cm, 12 cm and 10 cm a right-angled triangle?

The answer is in the next frame

32

$$\boxed{\text{No}}$$

Because

Letting $\cos\theta = \dfrac{8}{12}$ and $\sin\theta = \dfrac{10}{12}$, then

$$\cos^2\theta + \sin^2\theta = \left(\frac{8}{12}\right)^2 + \left(\frac{10}{12}\right)^2$$

$$= \frac{64}{144} + \frac{100}{144} = \frac{164}{144} \neq 1$$

Since the fundamental trigonometric identity is not satisfied this is not a right-angled triangle.

Move to the next frame

Two more identities

33

Two more identities can be derived directly from the fundamental identity; dividing both sides of the fundamental identity by $\cos^2\theta$ gives the identity …………

Check your answer in the next frame

34

$$\boxed{1 + \tan^2\theta \equiv \sec^2\theta}$$

Because

$\dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} \equiv \dfrac{1}{\cos^2\theta}$ that is $1 + \tan^2\theta \equiv \sec^2\theta$

Dividing the fundamental identity by $\sin^2\theta$ gives a third identity …………

Next frame

35

$$\boxed{\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta}$$

Because

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta} \text{ that is } \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

Using these three identities and the definitions of the trigonometric ratios it is possible to demonstrate the validity of other identities. For example, to demonstrate the validity of the identity:

$$\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \equiv 2\operatorname{cosec}^2\theta$$

we start with the left-hand side of this identity and demonstrate that it is equivalent to the right-hand side:

$$\begin{aligned}
\text{LHS} &= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \\
&\equiv \frac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)} && \text{Adding the two fractions together} \\
&\equiv \frac{2}{1-\cos^2\theta} \\
&\equiv \frac{2}{\sin^2\theta} && \text{From the fundamental identity} \\
&\equiv 2\operatorname{cosec}^2\theta \\
&= \text{RHS}
\end{aligned}$$

Try this one. Show that:

$\tan\theta + \cot\theta \equiv \sec\theta\operatorname{cosec}\theta$

Next frame

36

We proceed as follows:

$$\begin{aligned}
\text{LHS} &= \tan\theta + \cot\theta \\
&\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} && \text{Writing explicitly in terms of sines and cosines} \\
&\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} && \text{Adding the two fractions together} \\
&\equiv \frac{1}{\cos\theta\sin\theta} && \text{Since } \sin^2\theta + \cos^2\theta = 1 \text{ (the fundamental identity)} \\
&\equiv \sec\theta\operatorname{cosec}\theta \\
&= \text{RHS}
\end{aligned}$$

So demonstrate the validity of each of the following identities:

(a) $\tan^2\theta - \sin^2\theta \equiv \sin^4\theta\sec^2\theta$ (b) $\dfrac{1+\sin\theta}{\cos\theta} \equiv \dfrac{\cos\theta}{1-\sin\theta}$

Take care with the second one – it is done by performing an operation on both sides first.

The answers are in the next frame.

37

(a) $\text{LHS} = \tan^2\theta - \sin^2\theta$

$\equiv \dfrac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta$ Writing explicitly in terms of sines and cosines

$\equiv \sin^2\theta\sec^2\theta - \sin^2\theta$

$\equiv \sin^2\theta(\sec^2\theta - 1)$ Factorizing out the $\sin^2\theta$

$\equiv \sin^2\theta\tan^2\theta$ Using the identity $1 + \tan^2\theta \equiv \sec^2\theta$

$\equiv \sin^2\theta\dfrac{\sin^2\theta}{\cos^2\theta}$

$\equiv \sin^4\theta\sec^2\theta$

$= \text{RHS}$

(b) $\dfrac{1+\sin\theta}{\cos\theta} \equiv \dfrac{\cos\theta}{1-\sin\theta}$

Multiplying both sides by $\cos\theta(1 - \sin\theta)$ transforms the identity into:

$(1 - \sin\theta)(1 + \sin\theta) \equiv \cos^2\theta$. From this we find that:

$\text{LHS} = (1 - \sin\theta)(1 + \sin\theta)$

$\equiv 1 - \sin^2\theta$

$\equiv \cos^2\theta$ since $\cos^2\theta + \sin^2\theta \equiv 1$

$= \text{RHS}$

Move on now to the next frame

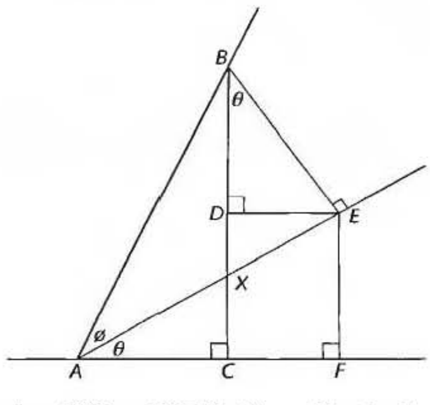
Identities for compound angles

38

The trigonometric ratios of the sum or difference of two angles can be given in terms of the ratios of the individual angles. For example, the cosine of a sum of angles is given by:

$\cos(\theta + \phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi$

To demonstrate the validity of this, consider the following figure:



(Notice that in triangles AXC and BXE, $\angle C = \angle E$ as both are right angles, and $\angle AXC = \angle BXE$ as they are equal and opposite. Consequently, the third angles must also be equal so that $\angle EBX = \angle CAX = \theta$.)

Hence we see that:

$$\begin{aligned}\cos(\theta+\phi) &= \frac{AC}{AB} && \text{Adjacent over hypotenuse}\\ &= \frac{AF-CF}{AB} \\ &= \frac{AF-DE}{AB} && \text{Because } DE = CF\\ &= \frac{AF}{AB}-\frac{DE}{AB} && \text{Separating out the fraction}\end{aligned}$$

Now, $\cos\theta = \dfrac{AF}{AE}$ so that $AF = AE\cos\theta$. Similarly, $\sin\theta = \dfrac{DE}{BE}$ so that $DE = BE\sin\theta$. This means that:

$$\begin{aligned}\cos(\theta+\phi) &= \frac{AF}{AB}-\frac{DE}{AB}\\ &= \frac{AE\cos\theta}{AB}-\frac{BE\sin\theta}{AB}. \text{ Now, } \frac{AE}{AB} = \cos\phi \text{ and } \frac{BE}{AB} = \sin\phi, \text{ therefore}\end{aligned}$$

$$\cos(\theta+\phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi$$

A similar identity can be demonstrated for the difference of two angles, namely:

$$\cos(\theta-\phi) \equiv \cos\theta\cos\phi + \sin\theta\sin\phi$$

Using these identities it is possible to obtain the cosine of angles other than 30°, 60° and 45° in surd form. For example:

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ+30^\circ) && \text{Expressing } 75^\circ \text{ in angles where we know the surd form for the trigonometric ratios}\\ &= \cos 45^\circ\cos 30^\circ - \sin 45^\circ \sin 30^\circ && \text{Using the new formula}\\ &= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

So the value of $\cos 15^\circ$ in surd form is ....

The answer is in the next frame

39

$$\boxed{\dfrac{1+\sqrt{3}}{2\sqrt{2}}}$$

Because

$$\begin{aligned}\cos 15^\circ &= \cos(60^\circ - 45^\circ)\\ &= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ\\ &= \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

Just as it is possible to derive the cosine of a sum of angles, it is also possible to derive other trigonometric ratios of sums and differences of angles. In the next frame a list of such identities is given for future reference.

Trigonometric formulas

Sums and differences of angles

40

$$\cos(\theta+\phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi \qquad \sin(\theta+\phi) \equiv \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta-\phi) \equiv \cos\theta\cos\phi + \sin\theta\sin\phi \qquad \sin(\theta-\phi) \equiv \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\tan(\theta+\phi) \equiv \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \equiv \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$ Now divide numerator and denominator by $\cos\theta\cos\phi$

$$\equiv \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$\tan(\theta-\phi) \equiv \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

Double angles

Double angle formulas come from the above formulas for sums when $\theta = \phi$:

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$$

$$\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$$

For future reference we now list identities for sums, differences and products of the trigonometric ratios. Each of these can be proved by using the earlier identities and showing that RHS $\equiv$ LHS (rather than showing LHS $\equiv$ RHS as we have done hitherto).

▶

Sums and differences of ratios

$$\sin\theta + \sin\phi \equiv 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$
$$\sin\theta - \sin\phi \equiv 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$
$$\cos\theta + \cos\phi \equiv 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$
$$\cos\theta - \cos\phi \equiv -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

Products of ratios

$$2\sin\theta\cos\phi \equiv \sin(\theta+\phi) + \sin(\theta-\phi)$$
$$2\cos\theta\cos\phi \equiv \cos(\theta+\phi) + \cos(\theta-\phi)$$
$$2\sin\theta\sin\phi \equiv \cos(\theta-\phi) - \cos(\theta+\phi)$$

41

Revision summary

1 The fundamental trigonometric identity is $\cos^2\theta + \sin^2\theta \equiv 1$ and is derived from *Pythagoras'* theorem.

2 Trigonometric identities can be verified using both the fundamental identity and the definitions of the trigonometric ratios.

42

Revision exercise

1 Use the fundamental trigonometric identity to show that:

(a) the triangle with sides 5 cm, 12 cm and 13 cm is a right-angled triangle.

(b) the triangle with sides 7 cm, 15 cm and 16 cm is not a right-angled triangle.

2 Verify each of the following identities:

(a) $1 - \dfrac{\sin\theta\tan\theta}{1+\sec\theta} \equiv \cos\theta$

(b) $\sin\theta + \sin\phi \equiv 2\sin\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2}$

43

1 (a) $5^2 + 12^2 = 25 + 144 = 169 = 13^3$

(b) $7^2 + 15^2 = 49 + 225 = 274 \neq 16^2$

2 (a)
$$\begin{aligned}
\text{LHS} &= 1 - \frac{\sin\theta\tan\theta}{1+\sec\theta}\\
&\equiv \frac{1+\sec\theta - \sin\theta\tan\theta}{1+\sec\theta}\\
&\equiv \frac{\cos\theta + 1 - \sin^2\theta}{\cos\theta + 1} \quad \text{multiplying top and bottom by } \cos\theta\\
&\equiv \frac{\cos\theta + \cos^2\theta}{\cos\theta + 1}\\
&\equiv \frac{\cos\theta(1+\cos\theta)}{\cos\theta + 1}\\
&\equiv \cos\theta\\
&= \text{RHS}
\end{aligned}$$

(b)
$$\begin{aligned}
\text{RHS} &= 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\\
&\equiv 2\left(\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\right)\\
&\equiv 2\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos^2\frac{\phi}{2} + \sin\frac{\phi}{2}\cos\frac{\phi}{2}\cos^2\frac{\theta}{2}\right.\\
&\qquad \left. + \sin\frac{\phi}{2}\cos\frac{\phi}{2}\sin^2\frac{\theta}{2} + \sin^2\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\\
&\equiv \sin\theta\cos^2\frac{\phi}{2} + \sin\phi\cos^2\frac{\theta}{2} + \sin\phi\sin^2\frac{\theta}{2} + \sin^2\frac{\phi}{2}\sin\theta\\
&\equiv \sin\theta\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\sin\theta + \sin\phi\cos^2\frac{\theta}{2} + \sin\phi\sin^2\frac{\theta}{2}\\
&\equiv \sin\theta\left(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\right) + \sin\phi\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)\\
&\equiv \sin\theta + \sin\phi\\
&= \text{LHS}
\end{aligned}$$

44

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through these at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

45

Checklist F.8

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Convert angles measured in degrees, minutes and seconds into decimal degrees? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 3
• Convert degrees into radians and vice versa? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	4 to 8
• Use a calculator to determine the values of trigonometric ratios for any angle? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	9 to 27
• Verify trigonometric identities? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	31 to 43

Test exercise F.8

46

	Frames
1 Convert the angle $39°57'2''$ to decimal degree format.	1 to 2
2 Convert the angle $52{\cdot}505°$ to degrees, minutes and seconds.	2 to 3
3 Convert the following to radians to 2 dp: (a) $84°$ (b) $69{\cdot}12°$ (c) $240°$ (as a multiple of π)	4 to 6
4 Convert the following to degrees to 2 dp: (a) $2{\cdot}139$ rad (b) $5\pi/3$ rad (c) $9\pi/10$ rad	6 to 8
5 Find the value of each of the following to 4 dp: (a) $\cos 18°$ (b) $\sin \pi/11$ (c) $\cos 2\pi/7$ (d) $\cot 48{\cdot}7°$ (e) $\operatorname{cosec} 1{\cdot}04$ rad (f) $\sec 0{\cdot}85$ rad	9 to 19
6 Given one side and the hypotenuse of a right-angled triangle as $4{\cdot}3$ and $11{\cdot}2$, find the length of the other side.	20 to 23

▶

Frames

7 Show that the triangle with sides 9, 12 and 15 is a right-angled triangle. 20 to 23

8 A triangle has its three sides in the ratio 1 : 0·6 : 0·8. Is it a right-angled triangle? 24 to 27

9 Verify each of the following trigonometric identities:

(a) $\dfrac{(\cos\theta - \sin\theta)^2}{\cos\theta} \equiv \sec\theta - 2\sin\theta$

(b) $\dfrac{\operatorname{cosec}\theta\sec\theta}{\cot\theta} \equiv 1 + \tan^2\theta$

(c) $2\sin\theta\cos\phi \equiv \sin(\theta+\phi) + \sin(\theta-\phi)$ 31 to 40

10 Show that:

(a) $\sin 75^\circ = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$

(b) $\tan 15^\circ = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ 38 to 39

Further problems F.8

47

1 Convert the angle 81°18′23″ to decimal degree format.

2 Convert the angle 63·216° to degrees, minutes and seconds.

3 Convert the following to radians to 2 dp:
(a) 31° (b) 48·15° (c) 225° (as a multiple of π)

4 Convert the following to degrees to 2 dp:
(a) 1·784 rad (b) $3\pi/4$ rad (c) $4\pi/5$ rad

5 Find the value of each of the following to 4 dp:
(a) $\tan 27^\circ$ (b) $\sin \pi/5$ (c) $\tan 4\pi/9$
(d) $\sec 89{\cdot}2^\circ$ (e) $\operatorname{cosec} 0{\cdot}04^\circ$ (f) $\cot 1{\cdot}18$ rad

6 Given one side and the hypotenuse of a right-angled triangle as 6·4 and 9·1, find the length of the other side.

7 Show that the triangle with sides 5, 11 and 12 is not a right-angled triangle.

8 What is the length of the diagonal of a square of side length $\sqrt{2}$?

▶

9 Verify each of the following trigonometric identities:

(a) $\dfrac{\cos\theta - 1}{\sec\theta + \tan\theta} + \dfrac{\cos\theta + 1}{\sec\theta - \tan\theta} \equiv 2(1 + \tan\theta)$

(b) $\sin^3\theta - \cos^3\theta \equiv (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

(c) $\operatorname{cosec}^2\theta - \operatorname{cosec}\theta \equiv \dfrac{\cot^2\theta}{1 + \sin\theta}$

(d) $\cot\theta\cos\theta + \tan\theta\sin\theta \equiv (\operatorname{cosec}\theta + \sec\theta)(1 - \sin\theta\cos\theta)$

(e) $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \equiv 1 + \dfrac{2\tan\theta}{1 - \tan\theta}$

(f) $(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2 \equiv 2$

(g) $\sqrt{\dfrac{1 + \tan^2\theta}{1 + \cot^2\theta}} \equiv \tan\theta$

Binomial series

Frames 1 to 55

Learning outcomes

When you have completed this Programme you will be able to:

- Define $n!$ and recognize that there are $n!$ different combinations of n different items
- Evaluate $n!$ for moderately sized n using a calculator
- Manipulate expressions containing factorials
- Recognize that there are $\dfrac{n!}{(n-r)!r!}$ different combinations of r identical items in n locations
- Recognize simple properties of combinatorial coefficients
- Construct Pascal's triangle
- Write down the binomial expansion for natural number powers
- Obtain specific terms in the binomial expansion using the general term
- Use the sigma notation
- Recognize and reproduce the expansion for e^x where e is the exponential number

If you already feel confident about these why not try the short quiz over the page? You can check your answers at the end of the book.

Quiz F.9

		Frames
1	In how many different ways can 7 identical bottles of wine be arranged in a wine rack with spaces for 12 bottles?	1 to 3
2	Find the value of: (a) $6!$ (b) $11!$ (c) $\frac{15!}{12!}$ (d) $(13-7)!$ (e) $\frac{0!}{3!}$	4 to 9
3	Evaluate each of the following: (a) ${}^{4}C_{2}$ (b) ${}^{10}C_{6}$ (c) ${}^{89}C_{0}$ (d) ${}^{36}C_{35}$	10 to 16
4	Expand $(3a+4b)^4$ as a binomial series.	20 to 29
5	In the binomial expansion of $(1-x/2)^9$ written in terms of ascending powers of x, find: (a) the 4th term (b) the coefficient of x^5.	30 to 31
6	Evaluate: (a) $\sum_{r=1}^{20} r$ (b) $\sum_{r=1}^{n}(2r+3)$	35 to 45
7	Determine the 6th term and the sum of the first 10 terms of the series: $2+4+6+8+\ldots\ldots\ldots\ldots$	35 to 45
8	Using the series expansion of e^x find $e^{-0{\cdot}45}$ accurate to 3 decimal places.	46 to 48

Factorials and combinations

Factorials

1

Answer this question. How many different three-digit numbers can you construct using the three numerals 5, 7 and 8 once each?

The answer is in the next frame

2

$\boxed{6}$

They are: 578 587
758 785
857 875

Instead of listing them like this you can work it out. There are 3 choices for the first numeral and for each choice there are a further 2 choices for the second numeral. That is, there are:

$3 \times 2 = 6$ choices of first and second numeral combined

The third numeral is then the one that is left.

So, how many four-digit numbers can be constructed using the numerals 1, 2, 3 and 4 once each?

Answer in the next frame

3

$\boxed{4 \times 3 \times 2 = 24}$

Because

The first numeral can be selected in one of 4 ways, each selection leaving 3 ways to select the second numeral. So there are:

$4 \times 3 = 12$ ways of selecting the first two numerals

Each combination of the first two numerals leaves 2 ways to select the third numeral. So there are:

$4 \times 3 \times 2 = 24$ ways of selecting the first three numerals

The last numeral is the one that is left.

Can you see the pattern here? If you have n different items then you can form

$n \times (n-1) \times (n-2) \times \ldots \times 2$

different arrangements, or *combinations*, using each item just once.

This type of product of decreasing natural numbers occurs quite often in mathematics so a general notation has been devised. For example, the product:

$3 \times 2 \times 1$

is called *3-factorial* and is written as 3!

So the value of 5! is ....

Next frame

4

$\boxed{120}$

Because

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

The factorial expression can be generalized to the factorial of an arbitrary natural number n as:

$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

To save time a calculator can be used to evaluate $n!$ for moderately sized n. For example, on your calculator:

Enter the number 9
Press the **!** key
The display changes to 362880. that is:

$9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362\,880$

So the value of 11! is ....

5

$\boxed{39\,916\,800}$

For reasons that will soon become clear the value of 0! is defined to be 1. Try it on your calculator.

Try a few examples. Evaluate each of the following:

(a) $6!$ (b) $\frac{8!}{3!}$ (c) $(7-2)!$ (d) $\frac{3!}{0!}$ (e) $\frac{5!}{(5-2)!2!}$

6

(a) 720	(b) 6720	(c) 120	(d) 6	(e) 10

Because

(a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(b) $\frac{8!}{3!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 8 \times 7 \times 6 \times 5 \times 4 = 6720$

(c) $(7-2)! = 5! = 120$

(d) $\frac{3!}{0!} = \frac{6}{1} = 6$ Notice that we have used the fact that $0! = 1$

(e) $\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (2 \times 1)} = \frac{5 \times 4}{(2 \times 1)} = 10$

▶

Now try some for general n. Because

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \\ &= n \times (n-1)! \end{aligned}$$

then:

$$\frac{n!}{(n-1)!} = \frac{n \times (n-1)!}{(n-1)!} = n$$

Also note that while

$$2n! = 2 \times n!, (2n)! = (2n) \times (2n-1) \times (2n-2) \times \ldots \times 2 \times 1$$

so that, for example:

$$\begin{aligned} \frac{(2n+1)!}{(2n-1)!} &= \frac{(2n+1) \times (2n+1-1) \times (2n+1-2) \times \ldots \times 2 \times 1}{(2n-1) \times (2n-1-1) \times (2n-1-2) \times \ldots \times 2 \times 1} \\ &= \frac{(2n+1) \times (2n) \times (2n-1) \times \ldots \times 2 \times 1}{(2n-1) \times (2n-2) \times (2n-3) \times \ldots \times 2 \times 1} \\ &= (2n+1) \times (2n) \end{aligned}$$

So simplify each of these:

(a) $\dfrac{n!}{(n+1)!}$ (b) $\dfrac{(n+1)!}{(n-1)!}$ (c) $\dfrac{(2n)!}{(2n+2)!}$

Answers in the next frame

7

(a) $\dfrac{1}{n+1}$

(b) $n(n+1)$

(c) $\dfrac{1}{(2n+2) \times (2n+1)}$

Because

(a) $\dfrac{n!}{(n+1)!} = \dfrac{n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1}{(n+1) \times n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1} = \dfrac{1}{(n+1)}$

(b) $\dfrac{(n+1)!}{(n-1)!} = \dfrac{(n+1) \times (n) \times (n-1)!}{(n-1)!} = (n+1) \times (n) = n(n+1)$

(c) $\dfrac{(2n)!}{(2n+2)!} = \dfrac{(2n)!}{(2n+2) \times (2n+1) \times (2n)!} = \dfrac{1}{(2n+2) \times (2n+1)}$

8

Try some more. Write each of the following in factorial form:

(a) $4 \times 3 \times 2 \times 1$

(b) $6 \times 5 \times 4$

(c) $\dfrac{(7 \times 6) \times (3 \times 2 \times 1)}{2}$

Next frame for the answers

9

$$\text{(a) } 4! \quad \text{(b) } \frac{6!}{3!} \quad \text{(c) } \frac{7! \times 3!}{5! \times 2!}$$

Because

(a) $4 \times 3 \times 2 \times 1 = 4!$ by the definition of the factorial

(b) $6 \times 5 \times 4 = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \dfrac{6!}{3!}$

(c) $$\frac{(7 \times 6) \times (3 \times 2 \times 1)}{2} = \frac{(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}{(5 \times 4 \times 3 \times 2 \times 1) \times (2 \times 1)}$$
$$= \frac{7! \times 3!}{5! \times 2!}$$

Now let's use these ideas. Next frame

10

Combinations

Let's assume that you have a part-time job in the weekday evenings where you have to be at work just two evenings out of the five. Let's also assume that your employer is very flexible and allows you to choose which evenings you work provided you ring him up on Sunday and tell him. One possible selection could be:

Mon	*Tue*	*Wed*	*Thu*	*Fri*
W	W	–	–	–

another selection could be:

Mon	*Tue*	*Wed*	*Thu*	*Fri*
–	–	W	–	W

How many arrangements are there of two working evenings among the five days?

The answer is in the next frame

11

$$\boxed{5 \times 4 = 20}$$

Because

There are 5 weekdays from which to make a first selection and for each such selection there are 4 days left from which to make the second selection. This gives a total of $5 \times 4 = 20$ possible arrangements.

However, not all arrangements are *different*. For example, if, on the Sunday, you made

your first choice as Friday and your second choice as Wednesday,

this would be the same arrangement as making

your first choice as Wednesday and your second choice as Friday.

So every arrangement is duplicated.

How many *different* arrangements are there?

12

$$\boxed{\frac{5 \times 4}{2} = 10}$$

Because each arrangement is duplicated. List them:

Mon, Tue	Mon, Wed	Mon, Thu	Mon, Fri
	Tue, Wed	Tue, Thu	Tue, Fri
		Wed, Thu	Wed, Fri
			Thu, Fri

There are 10 different ways of combining two identical items in five different places.

The expression $\frac{5 \times 4}{2}$ can be written in factorial form as follows:

$$\frac{5 \times 4}{2} = \frac{5 \times 4}{2 \times 1} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = \frac{5!}{3!2!}$$ or better still:

$$\frac{5 \times 4}{2} = \frac{5!}{3!2!} = \frac{5!}{(5-2)!2!}$$ because this just contains the numbers 5 and 2 and will link to a general notation to be introduced in Frame 14.

There are $\frac{5!}{(5-2)!2!}$ *different combinations of* ***two*** *identical items in* ***five*** *different places.*

So, if your employer asked you to work 3 evenings out of the 5, how many different arrangements could you select? (Give your answer in terms of factorials.)

Next frame

13

$$\boxed{\frac{5!}{(5-3)!3!}}$$

Because

There are 5 weekdays from which to make a first selection and for each such selection there are 4 days left from which to make the second selection and then 3 days from which to make the third selection. This gives a total of $5 \times 4 \times 3 = 60$ possible arrangements.

However, not all arrangements are different. Any one arrangement can be rearranged within itself 3! times because the first selection in the arrangement can be placed against three different days and having placed the first selection the second can be placed against a choice of two days. The third selection is then placed against the remaining day. So any arrangement is repeated $3 \times 2 \times 1 = 3!$ times.

The total number of *different* arrangements is then:

▶

$$\frac{5 \times 4 \times 3}{3!} = \frac{5!}{2!3!} = \frac{5!}{(5-3)!3!}$$

Can you see the pattern emerging here?

There are $\dfrac{5!}{(5-2)!2!}$ different combinations of *two* identical items in *five* different places.

There are $\dfrac{5!}{(5-3)!3!}$ different combinations of *three* identical items in *five* different places.

So if you have r identical items to be located in n different places where $n \geq r$, the number of different combinations is ....

The answer is in the next frame

14

$$\boxed{\frac{n!}{(n-r)!r!}}$$

Because

The first item can be located in any one of n places.
The second item can be located in any one of the remaining $n - 1$ places.
The third item can be located in any one of the remaining $n - 2$ places.
⋮
The rth item can be located in any one of the remaining $n - (r - 1)$ places.

This means that there are $n - (r - 1) = \dfrac{n!}{(n-r)!}$ arrangements. However, every arrangement is repeated $r!$ times. (Any one arrangement can be rearranged within itself $r!$ times.) So the total number of *different* combinations is given as:

$$\frac{n!}{(n-r)!r}$$

This particular ratio of factorials is called a *combinatorial coefficient* and is denoted by ${}^nC_r$:

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

So, evaluate each of the following:

(a) ${}^6C_3$ (b) ${}^7C_2$ (c) ${}^4C_4$ (d) ${}^3C_0$ (e) ${}^5C_1$

15

(a) 20	(b) 21	(c) 1	(d) 1	(e) 5

Because

(a) $^6C_3 = \dfrac{6!}{(6-3)!3!} = \dfrac{6!}{3!3!} = \dfrac{720}{36} = 20$

(b) $^7C_2 = \dfrac{7!}{(7-2)!2!} = \dfrac{7!}{5!2!} = \dfrac{5040}{120 \times 2} = 21$

(c) $^4C_4 = \dfrac{4!}{(4-4)!4!} = \dfrac{4!}{0!4!} = \dfrac{4!}{4!} = 1$ Remember $0! = 1$

(d) $^3C_0 = \dfrac{3!}{(3-0)!0!} = \dfrac{3!}{3!0!} = 1$

(e) $^5C_1 = \dfrac{5!}{(5-1)!1!} = \dfrac{5!}{4!1!} = \dfrac{5!}{4!} = 5$

Three properties of combinatorial coefficients

16

1 $^nC_n = {}^nC_0 = 1$

This is quite straightforward to prove:

$$^nC_n = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = \frac{n!}{n!} = 1 \text{ and } {}^nC_0 = \frac{n!}{(n-0)!0!} = \frac{n!}{n!0!} = \frac{n!}{n!} = 1$$

2 $^nC_{n-r} = {}^nC_r$. This is a little more involved:

$$^nC_{n-r} = \frac{n!}{(n-[n-r])!(n-r)!} = \frac{n!}{(n-n+r)!(n-r)!} = \frac{n!}{r!(n-r)!} = {}^nC_r$$

3 $^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

This requires some care to prove:

$$\begin{aligned}
{}^nC_r + {}^nC_{r+1} &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-[r+1])!(r+1)!} \\
&= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r-1)!(r+1)!} \\
&= \frac{n!}{(n-r-1)!r!}\left(\frac{1}{n-r} + \frac{1}{r+1}\right) && \text{Taking out common factors} \\
&= \frac{n!}{(n-r-1)!r!}\left(\frac{r+1+n-r}{(n-r)(r+1)}\right) && \text{Adding the two fractions together} \\
&= \frac{n!}{(n-r)!(r+1)!}(n+1) && \text{Because } (n-r-1)!(n-r) = (n-r)! \text{ and } r!(r+1) = (r+1)! \\
&= \frac{(n+1)!}{(n-r)!(r+1)!} && \text{Because } n!(n+1) = (n+1)! \\
&= \frac{(n+1)!}{([n+1]-[r+1])!(r+1)!} && \text{Because } [n+1]-[r+1] = n-r \\
&= {}^{n+1}C_{r+1}
\end{aligned}$$

At this point let us pause and summarize the main facts on factorials and combinations

17

Revision summary

1 The product $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ is called n-factorial and is denoted by $n!$

2 The number of different combinations of n different items is $n!$

3 The number of different combinations of r identical items among n locations is given by the combinatorial coefficient ${}^nC_r = \dfrac{n!}{(n-r)!r!}$

4 ${}^nC_n = {}^nC_0 = 1$

5 ${}^nC_{n-r} = {}^nC_r$

6 ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

18

Revision exercise

1 Find the value of:

(a) $5!$ (b) $10!$ (c) $\dfrac{13!}{8!}$ (d) $(9-3)!$ (e) $0!$

2 How many different arrangements are there of 9 identical umbrellas on a rack of 15 coat hooks?

3 Evaluate each of the following:

(a) ${}^3C_1$ (b) ${}^9C_4$ (c) ${}^{100}C_0$ (d) ${}^{50}C_{50}$

4 Without evaluating, explain why each of the following equations are true:

(a) ${}^{15}C_5 = {}^{15}C_{10}$ (b) ${}^5C_3 + {}^5C_4 = {}^6C_4$

19

1 (a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

(b) $10! = 3\,628\,800$ (use a calculator)

(c) $\dfrac{13!}{8!} = 13 \times 12 \times 11 \times 10 \times 9 = 154\,440$

(d) $(9-3)! = 6! = 720$

(e) $0! = 1$

2 There are ${}^{15}C_9$ different arrangements of 9 identical umbrellas on a rack of 15 coat hooks, where:

$${}^{15}C_9 = \frac{15!}{(15-9)!9!} = \frac{15!}{6!9!} = 5005$$

3 (a) ${}^3C_1 = \dfrac{3!}{(3-1)!1!} = \dfrac{3!}{2!1!} = 3$

(b) ${}^9C_4 = \dfrac{9!}{(9-4)!4!} = \dfrac{9!}{5!4!} = 126$

(c) ${}^{100}C_0 = 1$

4 (a) ${}^{15}C_5 = {}^{15}C_{10}$ because ${}^nC_{n-r} = {}^nC_r$ where $n = 15$ and $r = 10$

(b) ${}^5C_3 + {}^5C_4 = {}^6C_4$ because ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ where $n = 5$ and $r = 3$

The next topic is related to this last one

Binomial series

Pascal's triangle 20

The following triangular array of combinatorial coefficients can be constructed where the *superscript* to the left of each coefficient indicates the row number and the *subscript* to the right indicates the column number:

Row	*Column*				
	0	*1*	*2*	*3*	*4*
0	$^0C_0$				
1	$^1C_0$	$^1C_1$			
2	$^2C_0$	$^2C_1$	$^2C_2$		
3	$^3C_0$	$^3C_1$	$^3C_2$	$^3C_3$	
4	......	......	......	......	......

Follow the pattern and fill in the next row.

The answer is in the following frame

21

$$\boxed{^4C_0,\ ^4C_1,\ ^4C_2,\ ^4C_3 \text{ and } ^4C_4}$$

Because

The superscript indicates row 4 and the subscripts indicate columns 0 to 4.

The pattern devised in this array can be used to demonstrate the third property of combinatorial coefficients that you considered in Frame 15, namely that:

$^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

In the following array, arrows have been inserted to indicate that any coefficient is equal to the coefficient immediately above it plus the one above it and to the left.

Row	*Column*					
	0	*1*	*2*	*3*	*4*	
0	$^0C_0$					
1	$^1C_0$	$^1C_1$				
2	$^2C_0$	$^2C_1$	$^2C_2$			$^1C_0 + {}^1C_1 = {}^{1+1}C_{0+1} = {}^2C_1$
3	$^3C_0$	$^3C_1$	$^3C_2$	$^3C_3$		$^2C_0 + {}^2C_1 = {}^{2+1}C_{0+1} = {}^3C_1$ etc.
4	$^4C_0$	$^4C_1$	$^4C_2$	$^4C_3$	$^4C_4$	$^3C_0 + {}^3C_1 = {}^{3+1}C_{0+1} = {}^4C_1$ etc.

▶

Now you have already seen from the first property of the combinatorial coefficients in Frame 16 that ${}^nC_n = {}^nC_0 = 1$ so the values of some of these coefficients can be filled in immediately:

Row			*Column*		
	0	*1*	*2*	*3*	*4*
0	1				
1	1	1			
2	1	${}^2C_1$	1		
3	1	${}^3C_1$	${}^3C_2$	1	
4	1	${}^4C_1$	${}^4C_2$	${}^4C_3$	1

Fill in the numerical values of the remaining combinatorial coefficients using the fact that ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$: that is, any coefficient is equal to the sum of the number immediately above it and the one above it and to the left.

The answer is in Frame 22

22

Row			*Column*			
	0	*1*	*2*	*3*	*4*	
0	1					
1	1	1				
2	1	2	1			$1+1=2$
3	1	3	3	1		$1+2=3$, $2+1=3$
4	1	4	6	4	1	$1+3=4$, $3+3=6$, $3+1=4$

This is called *Pascal's triangle.*

The numbers on row 5, reading from left to right are ....

Answers are in the next frame

23

$$\boxed{1, 5, 10, 10, 5, 1}$$

Because

The first number is ${}^5C_0 = 1$ and then $1+4=5$, $4+6=10$, $6+4=10$, $4+1=5$. Finally, ${}^5C_5 = 1$.

Now let's move on to a related topic in the next frame

Binomial expansions

24

A *binomial* is a pair of numbers raised to a power. In this Programme we shall only consider natural number powers, namely, binomials of the form:

$(a+b)^n$

where n is a natural number. In particular, look at the following expansions:

$(a+b)^1 = a+b$	Note the coefficients 1, 1
$(a+b)^2 = a^2+2ab+b^2$	Note the coefficients 1, 2, 1
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$	Note the coefficients 1, 3, 3, 1

So what is the expansion of $(a+b)^4$ and what are the coefficients?

Next frame for the answer

25

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$
Coefficients 1, 4, 6, 4, 1

Because

$$\begin{aligned}(a+b)^4 &= (a+b)^3(a+b)\\ &= (a^3+3a^2b+3ab^2+b^3)(a+b)\\ &= a^4+3a^3b+3a^2b^2+ab^3+a^3b+3a^2b^2+3ab^3+b^4\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

Can you see the connection with Pascal's triangle?

Row			*Column*			
	0	*1*	*2*	*3*	*4*	
0	1					
1	1	1				Coefficients of $(a+b)^1$
2	1	2	1			Coefficients of $(a+b)^2$
3	1	3	3	1		Coefficients of $(a+b)^3$
4	1	4	6	4	1	Coefficients of $(a+b)^4$

So what are the coefficients of $(a+b)^5$?

Next frame for the answer

26

1, 5, 10, 10, 5, 1

Because

The values in the next row of Pascal's triangle are 1, 5, 10, 10, 5, 1 and the numbers in row 5 are also the values of the coefficients of the binomial expansion:

$$\begin{aligned}(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\ &= 1a^5b^0+5a^4b^1+10a^3b^2+10a^2b^3+5a^1b^4+1a^0b^5\end{aligned}$$

▶

Notice how in this expansion, as the power of a decreases, the power of b increases so that in each term the two powers add up to 5.

Because the numbers in row 5 are also the values of the appropriate combinatorial coefficients this expansion can be written as:

$$(a+b)^5 = {}^5C_0a^5b^0 + {}^5C_1a^4b^1 + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4a^1b^4 + {}^5C_5a^0b^5$$

The general binomial expansion for natural number n is then given as:

$$\begin{aligned}(a+b)^n &= {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_na^{n-n}b^n \\ &= a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + b^n\end{aligned}$$

Now ${}^nC_1 = \dfrac{n!}{(n-1)!1!} = n$ and ${}^nC_2 = \dfrac{n!}{(n-2)!2!} = \dfrac{n(n-1)}{2!}$ so ${}^nC_3 = \ldots\ldots\ldots\ldots$

Next frame for the answer

27

$$\boxed{\dfrac{n(n-1)(n-2)}{3!}}$$

Because

$${}^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!}$$

From this, the binomial expansion can be written as:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$$

So use this form of the binomial expansion to expand $(a+b)^6$.

Next frame for the answer

28

$$\boxed{(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6}$$

Because

$$\begin{aligned}(a+b)^6 &= a^6 + 6a^5b + \frac{6\times5}{2!}a^4b^2 + \frac{6\times5\times4}{3!}a^3b^3 \\ &\quad + \frac{6\times5\times4\times3}{4!}a^2b^4 + \frac{6\times5\times4\times3\times2}{5!}ab^5 \\ &\quad + \frac{6\times5\times4\times3\times2\times1}{6!}b^6 \\ &= a^6 + 6a^5b + \frac{6\times5}{2}a^4b^2 + \frac{6\times5\times4}{6}a^3b^3 \\ &\quad + \frac{6\times5\times4\times3}{4\times3\times2\times1}a^2b^4 + \frac{6\times5\times4\times3\times2}{5\times4\times3\times2\times1}ab^5 + \frac{6!}{6!}b^6 \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

Notice that 1, 6, 15, 20, 15, 6 and 1 are the numbers in row 6 of Pascal's triangle.

There are now two ways of obtaining the binomial expansion of $(a+b)^n$:

1 Use Pascal's triangle. This is appropriate when n is small
2 Use the combinatorial coefficients. This is appropriate when n is large

So, expand each of the following binomials:

(a) $(1+x)^7$ using Pascal's triangle
(b) $(3-2x)^4$ using the combinatorial coefficients

Next frame for the answer

29

(a) $(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$
(b) $(3-2x)^4 = 81 - 216x + 216x^2 - 96x^3 + 16x^4$

Because

(a) Using Pascal's triangle:

$$\begin{aligned}(1+x)^7 &= 1 + (1+6)x + (6+15)x^2 + (15+20)x^3 \\ &\quad + (20+15)x^4 + (15+6)x^5 + (6+1)x^6 + x^7 \\ &= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7\end{aligned}$$

(b) Using the general form of the binomial expansion:

$$\begin{aligned}(3-2x)^4 &= (3+[-2x])^4 \\ &= 3^4 + 4 \times 3^3 \times (-2x) + \frac{4 \times 3}{2!} \times 3^2 \times (-2x)^2 \\ &\quad + \frac{4 \times 3 \times 2}{3!} \times 3 \times (-2x)^3 + \frac{4 \times 3 \times 2 \times 1}{4!} \times (-2x)^4 \\ &= 3^4 + 3^3 \times (-8x) + 6 \times 3^2 \times (4x^2) + 4 \times 3 \times (-8x^3) + (16x^4) \\ &= 81 - 216x + 216x^2 - 96x^3 + 16x^4\end{aligned}$$

The general term of the binomial expansion

30

In Frame 26 we found that the binomial expansion of $(a+b)^n$ is given as:

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^{n-n} b^n$$

Each term of this expansion looks like ${}^nC_r a^{n-r} b^r$ where the value of r ranges progressively from $r = 0$ to $r = n$ (there are $n+1$ terms in the expansion). Because the expression ${}^nC_r a^{n-r} b^r$ is typical of each and every term in the expansion we call it the *general term* of the expansion.

Any *specific* term can be derived from the general term. For example, consider the case when $r = 2$. The general term then becomes:

$${}^nC_2 a^{n-2} b^2 = \frac{n!}{(n-2)!2!} a^{n-2} b^2 = \frac{n(n-1)}{2!} a^{n-2} b^2$$

and this is the *third* term of the expansion:

The 3rd term is obtained from the general term by letting $r = 2$.

Consequently, we can say that:

${}^nC_r a^{n-r} b^r$ represents the $(r+1)$th term in the expansion for $0 \le r \le n$

▶

Let's look at an example. To find the 10th term in the binomial expansion of $(1+x)^{15}$ written in ascending powers of x, we note that $a=1$, $b=x$, $n=15$ and $r+1=10$ so $r=9$. This gives the 10th term as:

$$\begin{aligned} {}^{15}C_9 1^{15-9}x^9 &= \frac{15!}{(15-9)!9!}1^{15-9}x^9 \\ &= \frac{15!}{6!9!}x^9 \\ &= 5005x^9 \text{ obtained using a calculator} \end{aligned}$$

Try this one yourself. The 8th term in the binomial expansion of

$$\left(2-\frac{x}{3}\right)^{12} \text{ is} \ldots\ldots\ldots\ldots$$

The answer is in the next frame

31

$$\boxed{-\frac{2816}{243}x^7}$$

Because

Here $a=2, b=-x/3, n=12$ and $r+1=8$ so $r=7$. The 8th term is:

$$\begin{aligned} {}^{12}C_7 2^{12-7}(-x/3)^7 &= \frac{12!}{(12-7)!7!}2^5(-x/3)^7 \\ &= \frac{12!}{5!7!}32\times\frac{x^7}{(-3)^7} \\ &= \frac{792\times 32}{-2187}x^7 \\ &= -\frac{25\,344}{2187}x^7 \\ &= -\frac{2816}{243}x^7 \end{aligned}$$

At this point let us pause and summarize the main facts on binomial expansions

---

32

Revision summary

1. Row n in Pascal's triangle contains the coefficients of the binomial expansion of $(a+b)^n$.
2. An alternative form of the binomial expansion of $(a+b)^n$ is given in terms of combinatorial coefficients as:

$$\begin{aligned} (a+b)^n &= {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_na^{n-n}b^n \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \ldots + b^n \end{aligned}$$

33

Revision exercise

1 Using Pascal's triangle write down the binomial expansion of:

(a) $(a+b)^6$ (b) $(a+b)^7$

2 Write down the binomial expansion of each of the following:

(a) $(1+x)^5$ (b) $(2+3x)^4$ (c) $(2-x/2)^3$

3 In the binomial expansion of $\left(2-\dfrac{3}{x}\right)^8$ written in terms of descending powers of x, find:

(a) the 4th term (b) the coefficient of x^{-4}

34

1 From Pascal's triangle:

(a) $$(a+b)^6 = a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + a^0b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

(b) $$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

2 (a) $$(1+x)^5 = 1^5 + 5\times 1^4x^1 + 10\times 1^3x^2 + 10\times 1^2x^3 + 5\times 1^1x^4 + x^5$$
$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

(b) $$(2+3x)^4 = 2^4 + 4\times 2^3(3x)^1 + 6\times 2^2(3x)^2 + 4\times 2^1(3x)^3 + (3x)^4$$
$$= 16 + 96x + 216x^2 + 216x^3 + 81x^4$$

(c) $$(2-x/2)^3 = 2^3 + 3\times 2^2(-x/2)^1 + 3\times 2^1(-x/2)^2 + (-x/2)^3$$
$$= 8 - 6x + 3x^2/2 - x^3/8$$

3 In the binomial expansion of $\left(2-\dfrac{3}{x}\right)^8$:

(a) The 4th term is derived from the general term ${}^nC_ra^{n-r}b^r$ where $a=2$, $b=-3/x$, $n=8$ and $r+1=4$ so $r=3$. That is, the 4th term is:

$$^8C_32^{8-3}(-3/x)^3 = \frac{8!}{(8-3)!3!}2^5(-3/x)^3$$
$$= \frac{8!}{5!3!}\times 32\times(-27/x^3)$$
$$= -\frac{56}{x^3}\times 864 = -\frac{48\,384}{x^3}$$

(b) The coefficient of x^{-4} is derived from the general term when $r=4$. That is: $^8C_42^{8-4}(-3/x)^4$, giving the coefficient as:

$$^8C_42^{8-4}(-3)^4 = \frac{8!}{(8-4)!4!}2^4(-3)^4$$
$$= 70\times 16\times 81 = 90\,720$$

35 The $\sum$ (sigma) notation

The binomial expansion of $(a+b)^n$ is given as a sum of terms:

$$(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_na^{n-n}b^n$$

Instead of writing down each term in the sum in this way a shorthand notation has been devised. We write down the general term and then use the Greek letter $\sum$ (sigma) to denote the sum. That is:

$$\begin{aligned}(a+b)^n &= {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_na^{n-n}b^n\\ &= \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r\end{aligned}$$

Here the Greek letter $\sum$ denotes a *sum of terms* where the typical term is ${}^nC_ra^{n-r}b^r$ and where the value of r ranges in integer steps from $r=0$, as indicated at the bottom of the sigma, to $r=n$, as indicated on the top of the sigma.

One immediate benefit of this notation is that it permits further properties of the combinatorial coefficients to be proved. For example:

$\sum_{r=0}^{n} {}^nC_r = 2^n$ The sum of the numbers in any row of Pascal's triangle is equal to 2 raised to the power of the row number

This is easily proved using the fact that:

$(a+b)^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$. When $a=1$ and $b=1$, then:

$$(1+1)^n = 2^n = \sum_{r=0}^{n} {}^nC_r1^{n-r}1^r = \sum_{r=0}^{n} {}^nC_r$$

36 General terms

It is necessary for you to acquire the ability to form the general term from a sum of specific terms and so write the sum of specific terms using the sigma notation. To begin, consider the sum of the first n even numbers:

$2+4+6+8+\ldots\ldots\ldots\ldots$

Every even integer is divisible by 2 so every even integer can be written in the form $2r$ where r is some integer. For example:

$8 = 2\times 4$ so here $8 = 2r$ where $r=4$

Every odd integer can be written in the form $2r-1$ or as $2r+1$ – as an even integer minus or plus 1. For example:

$13 = 14-1 = 2\times 7-1$ so that $13 = 2r-1$ where $r=7$

▶

Alternatively:

$13 = 12 + 1 = 2 \times 6 + 1$ so that $13 = 2r + 1$ where $r = 6$

Writing (a) 16, 248, −32 each in the form $2r$, give the value of r in each case.
(b) 21, 197, −23 each in the form $2r - 1$, give the value of r in each case.

The answer is in the next frame

37

(a) $r = 8$	(b) $r = 11$
$r = 124$	$r = 99$
$r = -16$	$r = -11$

Because

(a) $16 = 2 \times 8 = 2r$ where $r = 8$
$248 = 2 \times 124 = 2r$ where $r = 124$
$-32 = 2 \times (-16)$ where $r = -16$

(b) $21 = 22 - 1 = 2 \times 11 - 1 = 2r - 1$ where $r = 11$
$197 = 198 - 1 = 2 \times 99 - 1 = 2r - 1$ where $r = 99$
$-23 = -22 - 1 = 2 \times (-11) - 1 = 2r - 1$ where $r = -11$

Next frame

38

We saw in Frame 35 that we can use the sigma notation to denote sums of general terms. We shall now use the notation to denote sums of terms involving integers. For example, in the sum of the odd natural numbers:

$1 + 3 + 5 + 7 + 9 + \ldots\ldots\ldots\ldots$

the general term can now be denoted by $2r - 1$ where $r \geq 1$. The symbol $\sum$ can then be used to denote a sum of terms of which the general term is typical:

$1 + 3 + 5 + 7 + 9 + \ldots\ldots\ldots\ldots = \sum(2r - 1)$

We can now also denote the range of terms over which we wish to extend the sum by inserting the appropriate values of the *counting number* r below and above the sigma sign. For example:

$\sum_{r=1}^{7}(2r - 1)$ indicates the sum of 7 terms where r ranges from $r = 1$ to $r = 7$.

That is:

$$\sum_{r=1}^{7}(2r - 1) = 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$$

▶

Now you try some. In each of the following write down the general term and then write down the sum of the first 10 terms using the sigma notation:

(a) $2+4+6+8+\dots$ (b) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

(c) $1+8+27+64+\dots$ (d) $-1+2-3+4-5+\dots$

(e) $\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$

Answers in the next frame

39

(a) $2r,\ \sum_{r=1}^{10} 2r$	(b) $\frac{1}{r},\ \sum_{r=1}^{10} \frac{1}{r}$
(c) $r^3,\ \sum_{r=1}^{10} r^3$	(d) $(-1)^r r,\ \sum_{r=1}^{10} (-1)^r r$
(e) $\frac{1}{r!},\ \sum_{r=0}^{9} \frac{1}{r!}$	

Because

(a) This is the sum of the first 10 even numbers and every number is divisible by 2 so the general even number can be denoted by $2r$, giving the sum as $\sum_{r=1}^{10} 2r$.

(b) This is the sum of the first 10 reciprocals and the general reciprocal can be denoted by $\frac{1}{r}$ where $r \neq 0$. This gives the sum as $\sum_{r=1}^{10} \frac{1}{r}$.

(c) This is the sum of the first 10 numbers cubed and every number can be denoted by r^3, giving the sum as $\sum_{r=1}^{10} r^3$.

(d) Here, every odd number is preceded by a minus sign. This can be denoted by $(-1)^r$ because when r is even $(-1)^r = 1$ and when r is odd $(-1)^r = -1$. This permits the general term to be written as $(-1)^r r$ and the sum is $\sum_{r=1}^{10} (-1)^r r$.

(e) This is the sum of the first 10 reciprocal factorials and the general reciprocal factorial can be denoted by $\frac{1}{r!}$, giving the sum as $\sum_{r=0}^{9} \frac{1}{r!}$. Notice that the sum of the first 10 terms starts with $r = 0$ and ends with $r = 9$.

40

If the sum of terms is required up to some final but unspecified value of the counting variable r, say $r = n$, then the symbol n is placed on the top of the sigma. For example, the sum of the first n terms of the series with general term r^2 is given by:

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Try some examples. In each of the following write down the general term and then write down the sum of the first n terms using the sigma notation:

(a) $\frac{5}{2} + \frac{5}{4} + \frac{5}{6} + \frac{5}{8} + \ldots$ (b) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

(c) $2 - 4 + 6 - 8 + \ldots$ (d) $1 - 3 + 9 - 27 + \ldots$

(e) $1 - 1 + 1 - 1 + \ldots$

Answers in the next frame

41

(a) $\frac{5}{2r}, \sum_{r=1}^{n} \frac{5}{2r}$ (b) $\frac{1}{(2r-1)^2}, \sum_{r=1}^{n} \frac{1}{(2r-1)^2}$

(c) $(-1)^{r+1}2r, \sum_{r=1}^{n} (-1)^{r+1}2r$ (d) $(-1)^r 3^r, \sum_{r=0}^{n-1} (-1)^r 3^r$

(e) $(-1)^r, \sum_{r=0}^{n-1} (-1)^r$

Because

(a) Each term is of the form of 5 divided by an even number.

(b) Each term is of the form of the reciprocal of an odd number squared. Notice that the first term could be written as $\frac{1}{1^2}$ to maintain the pattern.

(c) Here the alternating sign is positive for every odd term (r odd) and negative for every even term (r even). Consequently, to force a positive sign for r odd we must raise -1 to an even power – hence $r + 1$ which is even when r is odd and odd when r is even.

(d) Here the counting starts at $r = 0$ for the first term so while odd terms are preceded by a minus sign the value of r is even. Also the nth term corresponds to $r = n - 1$.

(e) Again, the nth term corresponds to $r = n - 1$ as the first term corresponds to $r = 0$.

42

The sum of the first *n* natural numbers

Consider the sum of the first n non-zero natural numbers:

$\sum_{r=1}^{n} r = 1+2+3+\ldots+n$. This can equally well be written as:

$\sum_{r=1}^{n} r = n+(n-1)+(n-2)+\ldots+1$ starting with n and working backwards.

If these two are added together term by term then:

$$2\sum_{r=1}^{n} r = (1+n)+(2+n-1)+(3+n-2)+\ldots+(n+1)$$

That is:

$$2\sum_{r=1}^{n} r = (n+1)+(n+1)+(n+1)+\ldots+(n+1) \quad (n+1) \text{ added } n \text{ times}$$

That is:

$2\sum_{r=1}^{n} r = n(n+1)$ so that $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ the sum of the first n non-zero natural numbers.

This is a useful formula to remember so make a note of it in your workbook.

Now for two rules to be used when manipulating sums. Next frame

43

Rules for manipulating sums

Rule 1

If $f(r)$ is some general term and k is a constant then:

$$\begin{aligned}\sum_{r=1}^{n} kf(r) &= kf(1)+kf(2)+kf(3)+\ldots+kf(n)\\ &= k(f(1)+f(2)+f(3)+\ldots+f(n))\\ &= k\sum_{r=1}^{n} f(r)\end{aligned}$$

Common constants can be factored out of the sigma.

In particular, when $f(r)=1$ for all values of r:

$$\begin{aligned}\sum_{r=1}^{n} k &= k\sum_{r=1}^{n} 1\\ &= k(1+1+\ldots+1) \quad k \text{ multiplied by 1 added to itself } n \text{ times}\\ &= kn\end{aligned}$$

Rule 2

If $f(r)$ and $g(r)$ are two general terms then:

$$\begin{aligned}\sum_{r=1}^{n}(f(r)+g(r)) &= \{f(1)+g(1)+f(2)+g(2)+\ldots\}\\ &= \{f(1)+f(2)+\ldots\}+\{g(1)+g(2)+\ldots\}\\ &= \sum_{r=1}^{n}f(r)+\sum_{r=1}^{n}g(r)\end{aligned}$$

Now for two worked examples

44

Example 1

Find the value of the first 100 natural numbers (excluding zero).

Solution

$$\begin{aligned}1+2+3+\ldots+100 &= \sum_{r=1}^{100} r\\ &= \frac{100(100+1)}{2} \quad \text{using the formula in Frame 42}\\ &= 5050\end{aligned}$$

Example 2

Find the value of $\sum_{r=1}^{n}(6r+5)$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(6r+5) &= \sum_{r=1}^{n}6r+\sum_{r=1}^{n}5 \quad \text{by Rule 2}\\ &= 6\sum_{r=1}^{n}r+\sum_{r=1}^{n}5 \quad \text{by Rule 1}\\ &= 6\frac{n(n+1)}{2}+5n \quad \text{using the formulas in Frames 42 and 43}\\ &= 3(n^2+n)+5n\\ &= 3n^2+8n\\ &= n(3n+8)\end{aligned}$$

So the values of:

(a) $\sum_{r=1}^{50} r$

(b) $\sum_{r=1}^{n}(8r-7)$ are ....

Answers in next frame

45

(a) 1275
(b) $n(4n-3)$

Because

(a) $\sum_{r=1}^{50} r = \dfrac{50 \times 51}{2} = 1275$

(b) $\sum_{r=1}^{n}(8r-7) = 8\sum_{r=1}^{n} r - \sum_{r=1}^{n} 7 = 4n^2 - 3n = n(4n-3)$

Now for the last topic

The exponential number *e*

46

The binomial expansion of $\left(1+\dfrac{1}{n}\right)^n$ is given as:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \ldots + \left(\frac{1}{n}\right)^n \\ &= 1 + \frac{n}{n} + \frac{n(n-1)}{2!n^2} + \frac{n(n-1)(n-2)}{3!n^3} + \ldots + \frac{1}{n^n} \\ &= 1 + 1 + \frac{(1-1/n)}{2!} + \frac{(1-1/n)(1-2/n)}{3!} + \ldots + \frac{1}{n^n}\end{aligned}$$

This expansion is true for any natural number value of n, large or small, but when n is a large natural number then $\dfrac{1}{n}$ is a small number. If we now let the value of n increase then, as it does so, the value of $\dfrac{1}{n}$ decreases. Indeed, the larger the value of n becomes, the closer $\dfrac{1}{n}$ becomes to zero. We have a notation for this, we write:

$\underset{n\to\infty}{Lim}\left(\dfrac{1}{n}\right) = 0$: the *limit* of $\dfrac{1}{n}$ as $n \to \infty$ is 0

Also, as $n \to \infty$, the closer the expansion above becomes to the expansion:

$$\begin{aligned}1 + 1 + \frac{(1-0)}{2!} + \frac{(1-0)(1-0)}{3!} + \ldots &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \\ &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\end{aligned}$$

Here the *ellipsis* (...) at the end of the expansion means that the expansion never ends – we say that it has an *infinite number of terms*. Indeed, we can use the sigma notation here and write:

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}$$

▶

Notice the symbol for infinity (∞) at the top of the sigma; this denotes the fact that the sum is a sum of an infinite number of terms.

It can be shown that this sum is a finite number which is denoted by e, the exponential number, whose value is 2·7182818...

That is:

$$\sum_{r=0}^{\infty} \frac{1}{r!} = e$$

You will find this number on your calculator:

Enter the number 1

Press the e^x key and the value of e is displayed.

In Part II we shall show that:

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Next frame

47

Use the series expansion to find the value of $e^{0\cdot1}$ accurate to 3 sig fig.

Since:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \text{ then:}$$

$$e^{0\cdot1} = 1 + 0\cdot1 + \frac{(0\cdot1)^2}{2!} + \frac{(0\cdot1)^3}{3!} + \frac{(0\cdot1)^4}{4!} + \dots$$

$$= 1 + 0\cdot1 + 0\cdot005 + 0\cdot0001\dot{6} + \dots$$ subsequent terms will not affect the value to 3 sig fig

$$= 1\cdot11$$

So, using the series expansion of e^x, it is found that the value of $e^{-0\cdot25}$ to 3 dp is ....

Next frame for the answer

48

$$\boxed{1\cdot779}$$

Because

$$e^{-0\cdot25} = 1 + (-0\cdot25) + \frac{(-0\cdot25)^2}{2!} + \frac{(-0\cdot25)^3}{3!} + \dots$$

$$= 1 - 0\cdot25 + 0\cdot03125 - 0\cdot0026 + \dots$$ subsequent terms will not affect the value to 3 dp

$$= 1\cdot03125 - 0\cdot2526$$

$$= 1\cdot779 \text{ to 3 dp}$$

Check this answer using your calculator.

At this point let us pause and summarize the main facts on the sigma notation and the series expansion of e^x.

49

Revision summary

1 The sigma notation is used as a shorthand notation for the sum of a number of terms, each term typified by a general term:

$$\sum_{r=1}^{n} f(r) = f(1) + f(2) + f(3) + \ldots + f(n)$$

2 The binomial expansion can be written using the sigma notation as:

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$$

3 The exponential expression e^x is given as a sum of an infinite number of terms:

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

50

Revision exercise

1 Evaluate:

(a) $\sum_{r=1}^{36} r$ (b) $\sum_{r=1}^{n} (4r+2)$

2 Determine the 5th term and the sum of the first 20 terms of the series: $3 + 6 + 9 + 12 + \ldots$

3 Using the series expansion of e^x find $e^{0.23}$ accurate to 3 decimal places.

51

1 (a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ so that $\sum_{r=1}^{36} r = \frac{36(37)}{2} = 666$

(b) $\sum_{r=1}^{n} (4r+2) = \sum_{r=1}^{n} 4r + \sum_{r=1}^{n} 2$

$= 4\sum_{r=1}^{n} r + (2 + 2 + \ldots + 2)$ 2 added to itself n times

$= 4\frac{n(n+1)}{2} + 2n$

$= 2n^2 + 4n$

$= 2n(n+2)$

2 The 5th term of $3 + 6 + 9 + 12 + \ldots$ is 15 because the general term is $3r$. The sum of the first 20 terms of the series is then:

$$\sum_{r=1}^{20} 3r = 3\sum_{r=1}^{20} r = 3\frac{20(21)}{2} = 630$$

▶

3 Given that:

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ then:}$$

$$e^{0\cdot23} = \sum_{r=1}^{\infty} \frac{(0\cdot23)^r}{r!} = 1 + 0\cdot23 + \frac{(0\cdot23)^2}{2!} + \frac{(0\cdot23)^3}{3!} + \frac{(0\cdot23)^4}{4!} + \dots$$

$$= 1 + 0\cdot23 + 0\cdot02645 + 0\cdot002028 + 0\cdot000117 + \dots$$

$$= 1\cdot259 \text{ to 3 dp}$$

52

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through these at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

Checklist F.9

53

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Define $n!$ and recognize that there are $n!$ different combinations of n different items? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate $n!$ for moderately sized n using a calculator? 4 to 5

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Manipulate expressions containing factorials? 6 to 9

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognize that there are $\dfrac{n!}{(n-r)!r!}$ different combinations of r identical items in n locations? 10 to 15

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognize simple properties of combinatorial coefficients? 16

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Construct Pascal's triangle? 20 to 23

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

	Frames
• Write down the binomial expansion for natural number powers? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	24 to 29
• Obtain specific terms in the binomial expansion using the general term? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	30 to 31
• Use the sigma notation? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	35 to 45
• Recognize and reproduce the expansion for e^x where e is the exponential number? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	46 to 48

Test exercise F.9

54

	Frames
1 In how many different ways can 6 different numbers be selected from the numbers 1 to 49 if the order in which the selection is made does not matter?	1 to 3
2 Find the value of: (a) $8!$ (b) $10!$ (c) $\dfrac{17!}{14!}$ (d) $(15-11)!$ (e) $\dfrac{4!}{0!}$	4 to 9
3 Evaluate each of the following: (a) ${}^{8}C_3$ (b) ${}^{15}C_{12}$ (c) ${}^{159}C_{158}$ (d) ${}^{204}C_0$	10 to 16
4 Expand $(2a-5b)^7$ as a binomial series.	20 to 29
5 In the binomial expansion of $(1+10/x)^{10}$ written in terms of descending powers of x, find: (a) the 8th term (b) the coefficient of x^{-8}	30 to 31
6 Evaluate: (a) $\sum_{r=1}^{45} r$ (b) $\sum_{r=1}^{n}(9-3r)$	35 to 45
7 Determine the 5th term and the sum of the first 20 terms of the series: $1+3+5+7+\dots$	35 to 45
8 Using the series expansion of e^x find e^{-2} accurate to 3 decimal places.	46 to 48

Further problems F.9

55

1 Given a row of 12 hat pegs, in how many different ways can:

(a) 5 identical red hardhats be hung?

(b) 5 identical red and 4 identical blue hardhats be hung?

(c) 5 identical red, 4 identical blue and 2 identical white hardhats be hung?

2 Show that:

(a) ${}^{n+1}C_1 - {}^{n}C_1 = 1$

(b) $\sum_{r=0}^{n}(-1)^r\,{}^{n}C_r = 0$

(c) $\sum_{r=0}^{n}{}^{n}C_r\,2^r = 3^n$

3 Write out the binomial expansions of:

(a) $(1-3x)^4$ (b) $(2+x/2)^5$ (c) $\left(1-\dfrac{1}{x}\right)^5$ (d) $(x+1/x)^6$

4 Evaluate:

(a) $\sum_{r=1}^{16}(5r-7)$ (b) $\sum_{r=1}^{n}(4-5r)$

5 Using the first 6 terms of the series for e^x, determine approximate values of e^2 and $\sqrt{e}$ to 4 sig fig.

6 Expand e^{3x} and e^{-2x} as a series of ascending powers of x using the first 5 terms in each case. Hence determine a series for $e^{3x} - e^{-2x}$.

7 Using the fact that $a^x = e^{x\ln a}$ and the series expansion of e^{kx}, evaluate $2^{-3\cdot 4}$ accurate to 3 dp.

Programme F.10

Differentiation

Frames 1 to 52

Learning outcomes

When you have completed this Programme you will be able to:

- Determine the gradient of a straight-line graph
- Evaluate from first principles the gradient at a point on a quadratic curve
- Differentiate powers of x and polynomials
- Evaluate second derivatives and use tables of standard derivatives
- Differentiate products and quotients of expressions
- Differentiate using the chain rule for a 'function of a function'

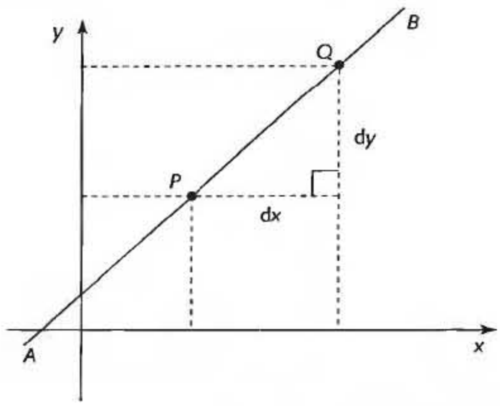
If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.10

	Frames
1 Calculate the gradient of the straight line joining: (a) $P(1, 3)$ and $Q(4, 9)$ (b) $P(-2, 6)$ and $Q(7, -5)$ (c) $P(0, 0)$ and $Q(4, 8)$	1 to 2
2 Determine, algebraically, from first principles, the gradient of the graph of $y = 3x^2 - 2$ at the point P where $x = 2{\cdot}4$.	3 to 10
3 If $y = 3x^4 - 5x^3 + x^2 + 2x - 4$, obtain an expression for $\dfrac{dy}{dx}$ and hence calculate the value of $\dfrac{dy}{dx}$ at $x = 4$.	11 to 27
4 (a) If $y = 4x^3 + x^2 - 6x - 3$, determine: (i) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ (ii) the values of x at which $\dfrac{dy}{dx} = 0$ (b) If $y = 2\cos(9x - 1) - \sin(3x + 7)$, obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.	11 to 27
5 Determine $\dfrac{dy}{dx}$ in each of the following cases: (a) $y = x^4 \cos x$ (b) $y = x^2 \ln x$ (c) $y = \dfrac{\sin x}{x}$ (d) $y = \dfrac{3\ln x}{x^4}$	28 to 38
6 Differentiate the following with respect to x: (a) $y = (3x - 2)^5$ (b) $y = 2\sin(3x - 1)$ (c) $y = -e^{(2x+3)}$ (d) $y = \ln(3 - 2x)$ (e) $y = e^{2x}\cos 3x$ (f) $y = (\ln x)\sin(2 - 5x)$	39 to 45

The gradient of a straight-line graph

1



The *gradient* of the sloping straight line shown in the figure is defined as:

$$\frac{\text{the vertical distance the line rises or falls between two points } P \text{ and } Q}{\text{the horizontal distance between } P \text{ and } Q}$$

where P is a point to the left of point Q on the straight line AB which slopes upwards from left to right. The changes in the x- and y-values of the points P and Q are denoted by dx and dy respectively. So the gradient of this line is given as:

$$\frac{dy}{dx}$$

We could have chosen any pair of points on the straight line for P and Q and by similar triangles this ratio would have worked out to the same value:

the gradient of a straight line is constant throughout its length

Its value is denoted by the symbol m.

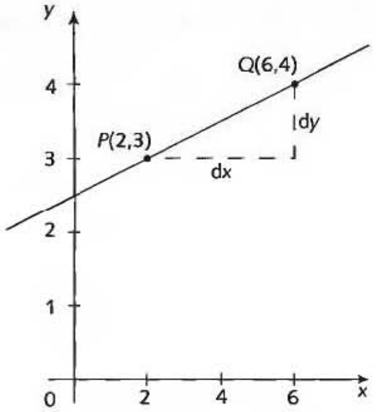
Therefore $m = \dfrac{dy}{dx}$.

For example, if, for some line (see the figure over the page), P is the point (2, 3) and Q is the point (6, 4), then P is to the left and *below* the point Q. In this case:

dy = the change in the y-values $= 4 - 3 = 1$ and
dx = the change in the x-values $= 6 - 2 = 4$ so that:

$m = \dfrac{dy}{dx} = \dfrac{1}{4} = 0{\cdot}25$. The sloping line *rises vertically* from left to right by 0·25 unit for every 1 unit horizontally.

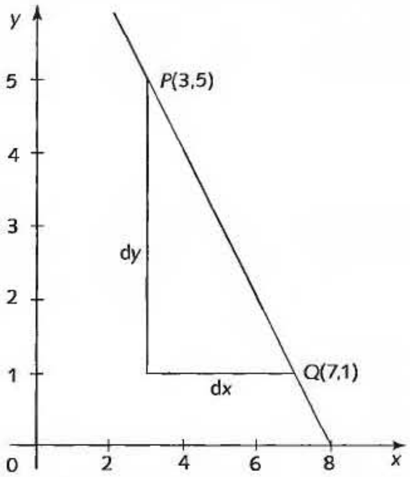
▶



If, for some other line (see the figure below), P is the point (3, 5) and Q is the point (7, 1), then P is to the left and *above* the point Q. In this case:

dy = the change in the y-values = $1 - 5 = -4$ and
dx = the change in the x-values = $7 - 3 = 4$ so that:

$m = \dfrac{dy}{dx} = \dfrac{-4}{4} = -1.$ The sloping line *rises vertically* from left to right by 0·25 unit for every 1 unit horizontally.



So, lines going *up* to the right have a *positive* gradient, lines going *down* to the right have a *negative* gradient.

Try the following exercises. Determine the gradients of the straight lines joining:

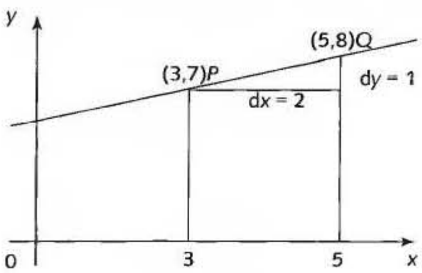
1 P (3, 7) and Q (5, 8)
2 P (2, 4) and Q (6, 9)
3 P (1, 6) and Q (4, 4)
4 P (−3, 6) and Q (5, 2)
5 P (−2, 4) and Q (3, −2)

Draw a diagram in each case

2

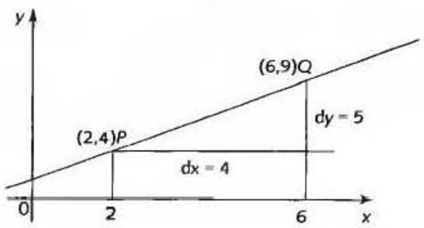
Here they are:

1



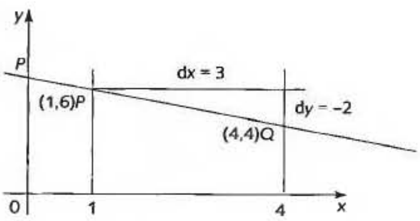
$$m = \frac{1}{2} = 0{\cdot}5$$

2



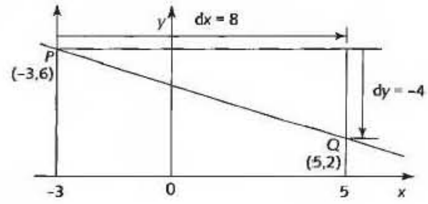
$$m = \frac{5}{4} = 1{\cdot}25$$

3



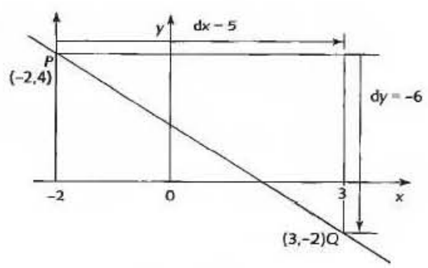
$$m = -\frac{2}{3} = -0{\cdot}667$$

4



$$m = -\frac{1}{2} = -0{\cdot}5$$

5



$$m = -\frac{6}{5} = -1{\cdot}2$$

Now let us extend these ideas to graphs that are not straight lines.

On then to the next frame

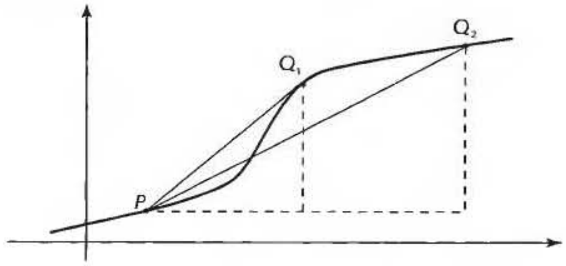
The gradient of a curve at a given point

3

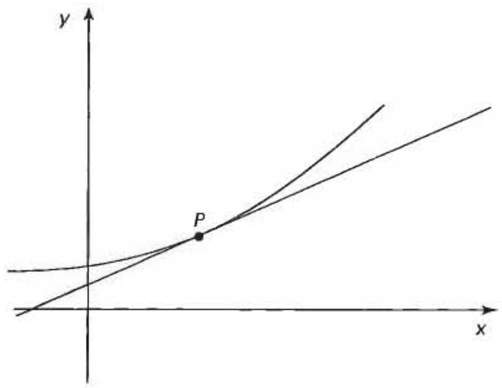
If we take two points P and Q on a curve and calculate, as we did for the straight line:

$$\frac{\text{the vertical distance the curve rises or falls between two points } P \text{ and } Q}{\text{the horizontal distance between } P \text{ and } Q}$$

the result will depend upon the choice of points P and Q as can be seen from the figure below:



This is because the gradient of the curve varies along its length, as anyone who has climbed a hill will appreciate. Because of this the gradient of a curve is not defined *between two points* as in the case of a straight line but *at a single point*. The gradient of a curve at a given point is defined to be the gradient of the straight line that touches the curve at that point – the *gradient of the tangent*.



The gradient of the curve at P is equal to the gradient of the tangent to the curve at P.

This is a straightforward but very important definition because all of the differential calculus depends upon it. Make a note of it in your workbook.

For example, find the gradient of the curve of $y = 2x^2 + 5$ at the point P at which $x = 1{\cdot}5$.

First we must compile a table giving the y-values of $y = 2x^2 + 5$ at 0·5 intervals of x between $x = 0$ and $x = 3$.

Complete the table and then move on to the next frame

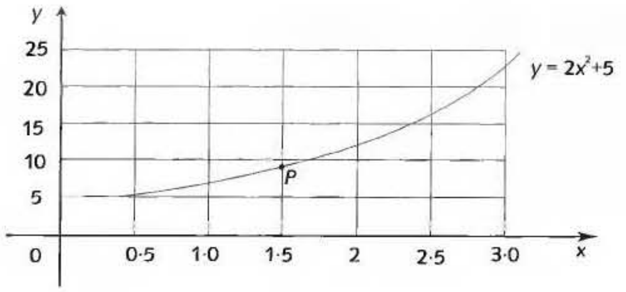
4

x	0	0·5	1·0	1·5	2·0	2·5	3·0
x^2	0	0·25	1·0	2·25	4·0	6·25	9·0
$2x^2$	0	0·5	2·0	4·5	8·0	12·5	18·0
$y = 2x^2 + 5$	5	5·5	7·0	9·5	13·0	17·5	23·0

Then we can plot the graph accurately, using these results, and mark the point P on the graph at which $x = 1{\cdot}5$.

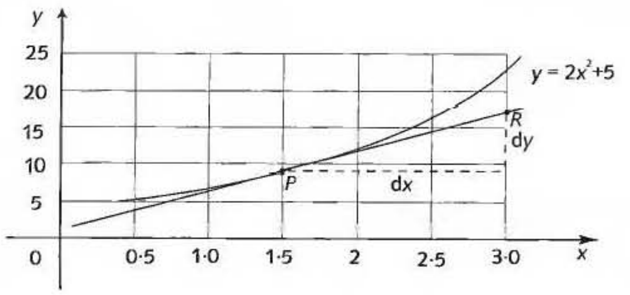
So do that

5



Now we bring a ruler up to the point P and adjust the angle of the ruler by eye until it takes on the position of the tangent to the curve. Then we carefully draw the tangent.

6



Now select a second point on the tangent at R, for example R at $x = 3{\cdot}0$ where at P, $x = 1{\cdot}5$. Determine dx and dy between them and hence calculate the gradient of the tangent, i.e. the gradient of the curve at P.

This gives $m = \dots\dots\dots\dots$

7

$$\boxed{m = 5{\cdot}9}$$

Because, at $x = 1{\cdot}5$, $y = 9{\cdot}3$ (gauged from the graph)
$x = 3{\cdot}0$, $y = 18{\cdot}1$ (again, gauged from the graph) therefore $dx = 1{\cdot}5$ and $dy = 8{\cdot}8$.

Therefore for the tangent $m = \dfrac{dy}{dx} = \dfrac{8{\cdot}8}{1{\cdot}5} = 5{\cdot}9$ to 2 sig fig.

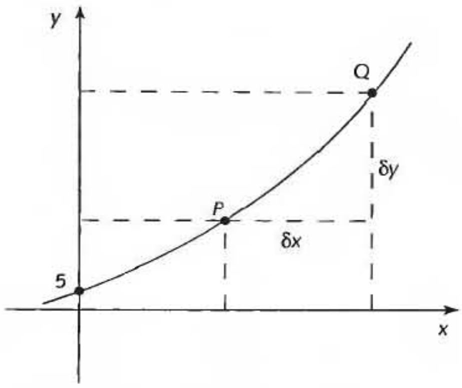
Therefore the gradient of the curve at P is approximately 5·9.

Note: Your results may differ slightly from those given here because the value obtained is the result of practical plotting and construction and may, therefore, contain minor differences. However, the description given is designed to clarify the method.

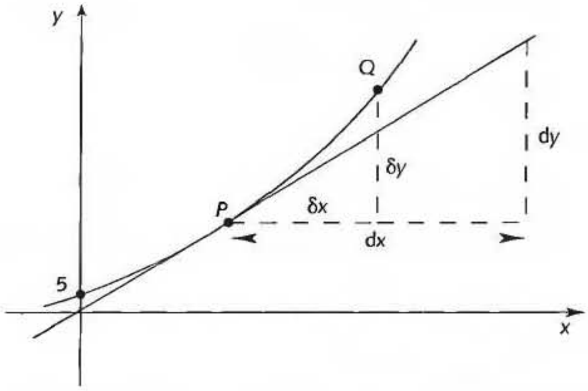
Algebraic determination of the gradient of a curve

8

The limitations of the practical construction in the previous method call for a more accurate method of finding the gradient, so let us start afresh from a different viewpoint and prove a general rule.



Let P be a fixed point (x, y) on the curve $y = 2x^2 + 5$, and Q be a neighbouring point, with coordinates $(x + \delta x, y + \delta y)$.



Notice that we use δx and δy to denote the respective differences in the x- and y-values of points P and Q on the curve. We reserve the notation dx and dy for the differences in the x- and y-values of two points on a *straight line* and particularly for a tangent. The quantities dx and dy are called *differentials*.

Next frame

9

At Q: $y + \delta y = 2(x + \delta x)^2 + 5$

$= 2(x^2 + 2x.\delta x + [\delta x]^2) + 5$ Expanding the bracket

$= 2x^2 + 4x.\delta x + 2[\delta x]^2 + 5$ Multiplying through by 2

Subtracting y from both sides:

$$y + \delta y - y = 2x^2 + 4x.\delta x + 2[\delta x]^2 + 5 - (2x^2 + 5)$$
$$= 4x.\delta x + 2[\delta x]^2$$

Therefore: $\delta y = 4x.\delta x + 2[\delta x]^2$

δy is the *vertical* distance between point P and point Q and this has now been given in terms of x and δx where δx is the *horizontal* distance between point P and point Q.

Now, if we divide both sides by δx:

$$\frac{\delta y}{\delta x} = 4x + 2.\delta x$$

This expression, giving the change in vertical distance per unit change in horizontal distance, is the *gradient* of the straight line through P and Q.

If the line through P and Q now rotates clockwise about P, the point Q moves down the curve and approaches P. Also, both δx and δy approach 0 ($\delta x \to 0$ and $\delta y \to 0$). However, *their ratio*, which is the gradient of PQ, approaches the gradient of the tangent at P:

$$\frac{\delta y}{\delta x} \to \text{the gradient of the tangent at } P = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots$

10

$$\boxed{\frac{dy}{dx} = 4x}$$

Because

$\frac{\delta y}{\delta x} = 4x + 2.\delta x$ and as $\delta x \to 0$ so $\frac{\delta y}{\delta x} \to 4x + 2 \times 0 = 4x$

This is a general result giving the slope of the curve at any point on the curve $y = 2x^2 + 5$.

$\therefore$ At $x = 1{\cdot}5$ $\frac{dy}{dx} = 4(1{\cdot}5) = 6$

$\therefore$ The real slope of the curve $y = 2x^2 + 5$ at $x = 1{\cdot}5$ is 6.

The graphical solution previously obtained, i.e. 5·9, is an approximation.

The expression $\frac{dy}{dx}$ is called *the derivative of y with respect to x* because it is *derived* from the expression for y. The process of finding the derivative is called *differentiation* because it involves manipulating differences in coordinate values.

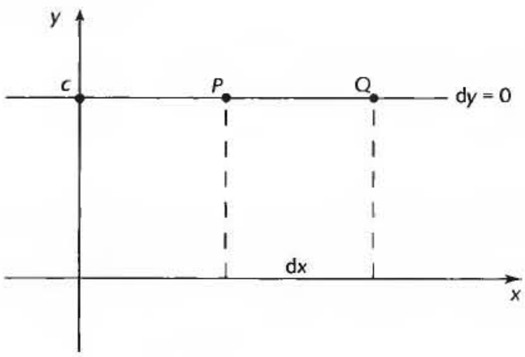
Derivatives of powers of *x*

11

Two straight lines

(a) $y = c$ (constant)

The graph of $y = c$ is a straight line parallel to the x-axis. Therefore its gradient is zero.



$y = c$, $dy = 0$ therefore $\frac{dy}{dx} = \frac{0}{dx} = 0$

$$\text{If } y = c,\ \frac{dy}{dx} = 0$$

Move to the next frame

12

(b) $y = ax$

So for a point further up the line

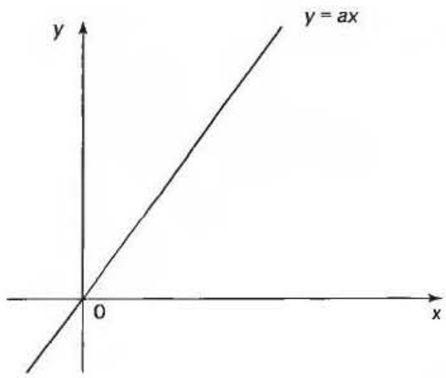
$$y + \text{d}y = a(x + \text{d}x)$$
$$= ax + a.\text{d}x$$

Subtract y:

$$\text{d}y = ax + a.\text{d}x - ax \text{ where } y = ax$$
$$= a.\text{d}x$$

Therefore:

$$\frac{\text{d}y}{\text{d}x} = a$$



Meaning that the gradient of a straight line is constant along its length.

$$\text{If } y = ax,\ \frac{\text{d}y}{\text{d}x} = a$$

In particular, if $a = 1$ then $y = x$ and $\dfrac{\text{d}y}{\text{d}x} = 1$

And now for curves

Two curves

13

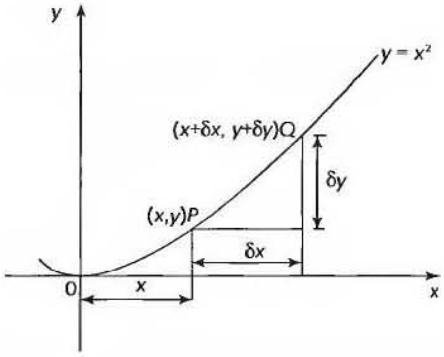
(a) $y = x^2$

At Q, $y + \delta y = (x + \delta x)^2$

$$y + \delta y = x^2 + 2x.\delta x + (\delta x)^2$$
$$y = x^2$$
$$\therefore\ \delta y = 2x.\delta x + (\delta x)^2$$
$$\therefore\ \frac{\delta y}{\delta x} = 2x + \delta x$$

If $\delta x \to 0$, $\dfrac{\delta y}{\delta x} \to \dfrac{\text{d}y}{\text{d}x}$ and $\therefore\ \dfrac{\text{d}y}{\text{d}x} = 2x$

$$\therefore\ \text{If } y = x^2,\ \frac{\text{d}y}{\text{d}x} = 2x$$



Remember that $\dfrac{\text{d}y}{\text{d}x}$ *is the gradient of the tangent to the curve.*

14

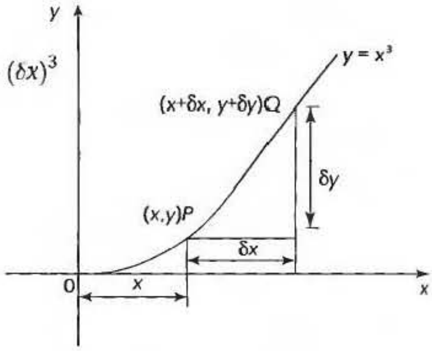
(b) $y = x^3$

At Q, $y + \delta y = (x + \delta x)^3$

$$= x^3 + 3x^2.\delta x + 3x(\delta x)^2 + (\delta x)^3$$

Now subtract $y = x^3$ from each side and finish it off as before.

$\therefore$ If $y = x^3$, $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$



15

$$\boxed{3x^2}$$

Because

$$\delta y = 3x^2.\delta x + 3x(\delta x)^2 + (\delta x)^3$$

$$\therefore \frac{\delta y}{\delta x} = 3x^2 + 3x(\delta x) + (\delta x)^2$$

Then, if $\delta x \to 0$, $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$ $\quad\therefore \dfrac{dy}{dx} = 3x^2$

$\therefore$ If $y = x^3$, $\dfrac{dy}{dx} = 3x^2$

Continuing in the same manner, we should find that:

when $y = x^4$, $\dfrac{dy}{dx} = 4x^3$

and when $y = x^5$, $\dfrac{dy}{dx} = 5x^4$

If we now collect these results together and look for a pattern, we have the evidence given in the next frame.

So move on

16

y	$\dfrac{dy}{dx}$
c	0
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$

We soon see a clear pattern emerging.

For a power of x:

In the derivative, the old index becomes a coefficient and the new index is one less than the index in the original power.

i.e. If $y = x^n$, $\dfrac{dy}{dx} = nx^{n-1}$

While this has only been demonstrated to be valid for $n = 0, 1, 2, 3$ it is in fact true for any value of n.

We have established that if $y = a$ (a constant) then $\frac{dy}{dx} = 0$. It can also be established that if $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$. For example, if $y = ax^4$ then $\frac{dy}{dx} = a \times 4x^3 = 4ax^3$.

We can prove these results by using the Binomial theorem:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots$$

If $y = x^n$, $y + \delta y = (x + \delta x)^n$

$$\therefore\ y + \delta y = x^n + nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \dots$$

$$y = x^n$$

$$\therefore\ \delta y = nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \dots$$

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}(\delta x) + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^2 + \dots$$

If $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ and all terms on the RHS, except the first $\to 0$.

$\therefore$ If $\delta x \to 0$, $\frac{dy}{dx} = nx^{n-1} + 0 + 0 + 0 + \dots$

$\therefore$ If $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

which is, of course, the general form of the results we obtained in the examples above.

Make a note of this important result

Differentiation of polynomials

17

To *differentiate a polynomial*, we differentiate each term in turn.

e.g. If $y = x^4 + 5x^3 - 4x^2 + 7x - 2$

$$\frac{dy}{dx} = 4x^3 + 5 \times 3x^2 - 4 \times 2x + 7 \times 1 - 0$$

$$\therefore\ \frac{dy}{dx} = 4x^3 + 15x^2 - 8x + 7$$

▶

Example

If $y = 2x^5 + 4x^4 - x^3 + 3x^2 - 5x + 7$, find an expression for $\frac{dy}{dx}$ and the value of $\frac{dy}{dx}$ at $x = 2$.

So, first of all, $\frac{dy}{dx} = \dots\dots\dots\dots$

18

$$\boxed{\frac{dy}{dx} = 10x^4 + 16x^3 - 3x^2 + 6x - 5}$$

Then, expressing the RHS in nested form and substituting $x = 2$, we have:

$$\text{At } x = 2,\ \frac{dy}{dx} = \dots\dots\dots\dots$$

19

$$\boxed{283}$$

Because

$\frac{dy}{dx} = f(x) = \{[(10x + 16)x - 3]x + 6\}x - 5$

$\therefore\ f(2) = 283$

Now, as an exercise, determine an expression for $\frac{dy}{dx}$ in each of the following cases and find the value of $\frac{dy}{dx}$ at the stated value of x:

1 $y = 3x^4 - 7x^3 + 4x^2 + 3x - 4$ $\quad[x = 2]$
2 $y = x^5 + 2x^4 - 4x^3 - 5x^2 + 8x - 3$ $\quad[x = -1]$
3 $y = 6x^3 - 7x^2 + 4x + 5$ $\quad[x = 3]$

20

1	$\frac{dy}{dx} = 12x^3 - 21x^2 + 8x + 3.$	At $x = 2$, $\frac{dy}{dx} = 31$
2	$\frac{dy}{dx} = 5x^4 + 8x^3 - 12x^2 - 10x + 8.$	At $x = -1$, $\frac{dy}{dx} = 3$
3	$\frac{dy}{dx} = 18x^2 - 14x + 4.$	At $x = 3$, $\frac{dy}{dx} = 124$

Now on to the next topic

Derivatives – an alternative notation

21

If $y = 2x^2 - 5x + 3$, then $\frac{dy}{dx} = 4x - 5$. This double statement can be written as a single statement by putting $2x^2 - 5x + 3$ in place of y in $\frac{dy}{dx}$.

i.e. $\frac{d}{dx}(2x^2 - 5x + 3) = 4x - 5$

In the same way, $\frac{d}{dx}(4x^3 - 7x^2 + 2x - 5) = \dots\dots\dots\dots$

22

$$\boxed{12x^2 - 14x + 2}$$

Either of the two methods is acceptable: it is just a case of which is the more convenient in any situation.

At this point let us pause and summarize derivatives of powers of x and polynomials

23

Revision summary

1 *Gradient of a straight line graph* $m = \frac{dy}{dx}$

2 *Gradient of a curve at a given point P at* (x, y)

$\frac{dy}{dx}$ = gradient of the tangent to the curve at P.

3 *Derivatives of powers of x*

(a) $y = c$ (constant), $\frac{dy}{dx} = 0$

(b) $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

(c) $y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$

4 *Differentiation of polynomials* – differentiate each term in turn.

24

Revision exercise

1 Calculate the gradient of the straight line joining:

(a) $P(4, 0)$ and $Q(7, 3)$

(b) $P(5, 6)$ and $Q(9, 2)$

(c) $P(-4, -7)$ and $Q(1, 3)$

(d) $P(0, 5)$ and $Q(5, -6)$

2 Determine, algebraically, from first principles, the gradient of the graph of $y = 5x^2 + 2$ at the point P where $x = -1{\cdot}6$.

3 If $y = -2x^4 - 3x^3 + 4x^2 - x + 5$, obtain an expression for $\frac{dy}{dx}$ and hence calculate the value of $\frac{dy}{dx}$ at $x = -3$.

25

1 (a) $P(4, 0)$ and $Q(7, 3)$:

$$dy = 3 - 0 = 3,\ dx = 7 - 4 = 3 \text{ so } \frac{dy}{dx} = \frac{3}{3} = 1$$

(b) $P(5, 6)$ and $Q(9, 2)$:

$$dy = 2 - 6 = -4,\ dx = 9 - 5 = 4 \text{ so } \frac{dy}{dx} = \frac{-4}{4} = -1$$

(c) $P(-4, -7)$ and $Q(1, 3)$:

$$dy = 3 - (-7) = 10,\ dx = 1 - (-4) = 5 \text{ so } \frac{dy}{dx} = \frac{10}{5} = 2$$

(d) $P(0, 5)$ and $Q(5, -6)$:

$$dy = -6 - 5 = -11,\ dx = 5 - 0 = 5 \text{ so } \frac{dy}{dx} = \frac{-11}{5} = -2{\cdot}2$$

2 $y = 5x^2 + 2$ so that:

$$\begin{aligned} y + \delta y &= 5(x + \delta x)^2 + 2 \\ &= 5(x^2 + 2x\delta x + [\delta x]^2) + 2 \\ &= 5x^2 + 10x\delta x + 5[\delta x]^2 + 2 \text{ so that:} \end{aligned}$$

$$\begin{aligned} y + \delta y - y &= 5x^2 + 10x\delta x + 5[\delta x]^2 + 2 - (5x^2 + 2) \\ &= 10x\delta x + 5[\delta x]^2 \\ &= \delta y \end{aligned}$$

Hence:

$\frac{\delta y}{\delta x} = \frac{10x\delta x + 5[\delta x]^2}{\delta x} = 10x + 5\delta x$ therefore $\frac{dy}{dx} = 10x$. When $x = 1{\cdot}6$:

$$\frac{dy}{dx} = -16$$

▶

3 If $y = -2x^4 - 3x^3 + 4x^2 - x + 5$ then, differentiating term by term:

$$\frac{dy}{dx} = -2 \times 4x^3 - 3 \times 3x^2 + 4 \times 2x - 1$$
$$= -8x^3 - 9x^2 + 8x - 1 = ((-8x - 9)x + 8)x - 1. \text{ When } x = -3:$$
$$\frac{dy}{dx} = ((-8(-3) - 9)(-3) + 8)(-3) - 1$$
$$= 110$$

Second derivatives

26

If $y = 2x^4 - 5x^3 + 3x^2 - 2x + 4$, then, by the previous method:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^4 - 5x^3 + 3x^2 - 2x + 4) = 8x^3 - 15x^2 + 6x - 2$$

This expression for $\frac{dy}{dx}$ is itself a polynomial in powers of x and can be differentiated in the same way as before, i.e. we can find the derivative of $\frac{dy}{dx}$.

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written $\frac{d^2y}{dx^2}$ and is the *second derivative of y with respect to x* (spoken as 'dee two y by dee x squared').

So, in this example, we have:

$$y = 2x^4 - 5x^3 + 3x^2 - 2x + 4$$
$$\frac{dy}{dx} = 8x^3 - 15x^2 + 6x - 2$$
$$\frac{d^2y}{dx^2} = 24x^2 - 30x + 6$$

We could, if necessary, find the third derivative of y in the same way:

$$\frac{d^3y}{dx^3} = \ldots\ldots\ldots\ldots$$

27

$$\boxed{48x - 30}$$

Similarly, if $y = 3x^4 + 2x^3 - 4x^2 + 5x + 1$

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots$$
$$\frac{d^2y}{dx^2} = \ldots\ldots\ldots\ldots$$

28

$$\boxed{\begin{aligned}\frac{dy}{dx} &= 12x^3 + 6x^2 - 8x + 5\\ \frac{d^2y}{dx^2} &= 36x^2 + 12x - 8\end{aligned}}$$

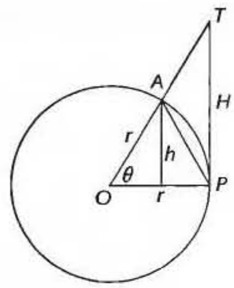
Let us now establish a limiting value that we shall need in the future.

Limiting value of $\frac{\sin\theta}{\theta}$ as $\theta \to 0$

P is a point on the circumference of a circle, centre O and radius r. PT is a tangent to the circle at P.

Use your trigonometry to show that:

$h = r\sin\theta$ and $H = r\tan\theta$



Recollect:

Area of a triangle $= \frac{1}{2} \times (\text{base}) \times (\text{height})$

Area of a circle $= \pi r^2 = \frac{1}{2}r^2.2\pi$, so

Area of a sector $= \frac{1}{2}r^2.\theta \quad (\theta \text{ in radians})$

$\triangle POA$: area $= \frac{1}{2}r.h = \frac{1}{2}r.r\sin\theta = \frac{1}{2}r^2\sin\theta$

Sector POA: area $= \frac{1}{2}r^2\theta$

$\triangle POT$: area $= \frac{1}{2}r.H = \frac{1}{2}r.r\tan\theta = \frac{1}{2}r^2\tan\theta$

In terms of area: $\triangle POA <$ sector $POA < \triangle POT$

$\therefore \ \frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$

$\therefore \ \sin\theta < \theta < \tan\theta \qquad \left[\tan\theta = \frac{\sin\theta}{\cos\theta}\right] \qquad \therefore \ \frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$

Multiplying throughout by $\sin\theta$: $1 > \frac{\sin\theta}{\theta} > \cos\theta$

When $\theta \to 0$, $\cos\theta \to 1$. $\therefore$ The limiting value of $\frac{\sin\theta}{\theta}$ is bounded on both sides by the value 1.

$$\therefore \text{ Limiting value of } \frac{\sin\theta}{\theta} \text{ as } \theta \to 0 = 1$$

Make a note of this result. We shall certainly meet it again in due course

Standard derivatives

29

So far, we have found derivatives of polynomials using the standard derivative $\dfrac{d}{dx}(x^n) = nx^{n-1}$. Derivatives of trigonometric expressions can be established by using a number of trigonometrical formulas. We shall deal with some of these in the next few frames.

1 Derivative of $y = \sin x$

If $y = \sin x$, $\quad y + \delta y = \sin(x + \delta x) \quad \therefore \ \delta y = \sin(x + \delta x) - \sin x$

We now apply the trigonometrical formula:

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \quad \text{where } A = x + \delta x \text{ and } B = x$$

$$\therefore \ \delta y = 2\cos\left(\frac{2x+\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right) = 2\cos\left(x + \frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)$$

$$\therefore \ \frac{\delta y}{\delta x} = \frac{2\cos\left(x+\frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)}{\delta x} = \frac{\cos\left(x+\frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \cos\left(x + \frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

When $\delta x \to 0$, $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$ and $\dfrac{dy}{dx} \to \cos x.1$ using the result of Frame 28

$\therefore$ If $y = \sin x$, $\dfrac{dy}{dx} = \cos x$

2 Derivative of $y = \cos x$

This is obtained in much the same way as for the previous case.

If $y = \cos x$, $y + \delta y = \cos(x + \delta x) \ \therefore \ \delta y = \cos(x + \delta x) - \cos x$

Now we use the formula $\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$$\therefore \ \delta y = -2\sin\left(\frac{2x+\delta x}{2}\right)\cdot\sin\left(\frac{\delta x}{2}\right)$$

$$= -2\sin\left(x + \frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)$$

$$\therefore \ \frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)}{\delta x} = -\sin\left(x + \frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

As $\delta x \to 0$, $\dfrac{\delta y}{\delta x} \to -\sin x.1 \quad \therefore$ If $y = \cos x$, $\dfrac{dy}{dx} = -\sin x$ using the result of Frame 28

30

At this stage, there is one more derivative that we should determine.

3 Derivative of $y = e^x$

We have already discussed the series representation of e^x in Frame 46 of Programme F.9, so:

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

If we differentiate each power of x on the RHS, this gives

$$\begin{aligned}\frac{dy}{dx} &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= e^x\end{aligned}$$

$\therefore$ If $y = e^x$, $\dfrac{dy}{dx} = e^x$

Note that e^x is a special function in which the derivative is equal to the function itself.

So, we have obtained some important standard results:

(a) If $y = x^n$, $\dfrac{dy}{dx} = n.x^{n-1}$

(b) If $y = c$ (a constant), $\dfrac{dy}{dx} = 0$

(c) A constant factor is unchanged, for example:

$$\text{if } y = a.x^n, \ \frac{dy}{dx} = a.n.x^{n-1}$$

(d) If $y = \sin x$, $\dfrac{dy}{dx} = \cos x$

(e) If $y = \cos x$, $\dfrac{dy}{dx} = -\sin x$

(f) if $y = e^x$, $\dfrac{dy}{dx} = e^x$

Now cover up the list of results above and complete the following table:

y	$\dfrac{dy}{dx}$
14	
$\cos x$	
x^n	

y	$\dfrac{dy}{dx}$
e^x	
$5x^3$	
$\sin x$	

You can check these in the next frame

31

y	$\frac{dy}{dx}$
14	0
$\cos x$	$-\sin x$
x^n	nx^{n-1}

y	$\frac{dy}{dx}$
e^x	e^x
$5x^3$	$15x^2$
$\sin x$	$\cos x$

On to the next frame

Differentiation of products of functions

32

Let $y = uv$, where u and v are functions of x.

If $x \to x + \delta x$, $u \to u + \delta u$, $v \to v + \delta v$ and, as a result, $y \to y + \delta y$.

$$y = uv \quad \therefore \; y + \delta y = (u + \delta u)(v + \delta v)$$
$$= uv + u.\delta v + v.\delta u + \delta u.\delta v$$
$$\text{Subtract} \quad y = uv$$
$$\therefore \; \delta y = u.\delta v + v.\delta u + \delta u.\delta v$$
$$\therefore \; \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u.\frac{\delta v}{\delta x}$$

If $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\frac{\delta u}{\delta x} \to \frac{du}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$, $\delta u \to 0$

$$\therefore \; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + 0\frac{dv}{dx} \qquad \therefore \text{ If } y = uv, \; \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

i.e. To differentiate a product of two functions:

Put down the first (differentiate the second) + put down the second (differentiate the first)

Example 1

$$y = x^3.\sin x$$
$$\frac{dy}{dx} = x^3(\cos x) + \sin x(3x^2)$$
$$= x^3.\cos x + 3x^2.\sin x = x^2(x\cos x + 3\sin x)$$

Example 2

$$y = x^4.\cos x$$
$$\frac{dy}{dx} = x^4(-\sin x) + \cos x.(4x^3)$$
$$= -x^4\sin x + 4x^3\cos x = x^3(4\cos x - x\sin x)$$

▶

Example 3

$$y = x^5.e^x$$

$$\frac{dy}{dx} = x^5e^x + e^x5x^4 = x^4e^x(x+5)$$

In the same way, as an exercise, now you can differentiate the following:

1 $y = e^x.\sin x$
2 $y = 4x^3.\sin x$
3 $y = e^x.\cos x$
4 $y = \cos x.\sin x$
5 $y = 3x^3.e^x$
6 $y = 2x^5.\cos x$

Finish all six and then check with the next frame

33

1 $\frac{dy}{dx} = e^x.\cos x + e^x.\sin x = e^x(\cos x + \sin x)$

2 $\frac{dy}{dx} = 4x^3.\cos x + 12x^2.\sin x = 4x^2(x\cos x + 3\sin x)$

3 $\frac{dy}{dx} = e^x(-\sin x) + e^x\cos x = e^x(\cos x - \sin x)$

4 $\frac{dy}{dx} = \cos x.\cos x + \sin x(-\sin x) = \cos^2 x - \sin^2 x$

5 $\frac{dy}{dx} = 3x^3.e^x + 9x^2.e^x = 3x^2e^x(x+3)$

6 $\frac{dy}{dx} = 2x^5(-\sin x) + 10x^4.\cos x = 2x^4(5\cos x - x\sin x)$

Now we will see how to deal with the quotient of two functions

---

Differentiation of a quotient of two functions

34

Let $y = \frac{u}{v}$, where u and v are functions of x.

$$\text{Then } y + \delta y = \frac{u + \delta u}{v + \delta v}$$

$$\therefore \ \delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$= \frac{uv + v.\delta u - uv - u.\delta v}{v(v + \delta v)} = \frac{v.\delta u - u.\delta v}{v^2 + v.\delta v}$$

$$\therefore \ \frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2 + v.\delta v}$$

▶

If $\delta x \to 0$, $\delta u \to 0$ and $\delta v \to 0$

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\therefore \quad \text{If } y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$\therefore$ *To differentiate a quotient of two functions:*

[Put down the bottom (differentiate the top) – put down the top (differentiate the bottom)] all over the bottom squared.

$$\therefore \quad \text{If } y = \frac{\sin x}{x^2}, \quad \frac{dy}{dx} = \ldots\ldots\ldots\ldots$$

35

$$\boxed{\frac{x\cos x - 2\sin x}{x^3}}$$

Because

$$y = \frac{\sin x}{x^2} \quad \therefore \quad \frac{dy}{dx} = \frac{x^2\cos x - \sin x.(2x)}{(x^2)^2}$$

$$= \frac{x^2\cos x - 2x\sin x}{x^4} = \frac{x\cos x - 2\sin x}{x^3}$$

Another example: $y = \dfrac{5e^x}{\cos x}$

$$y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{\cos x.(5e^x) - 5e^x(-\sin x)}{\cos^2 x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{5e^x(\cos x + \sin x)}{\cos^2 x}$$

Now let us deal with this one:

$$y = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x.\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad \text{But } \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{\cos^2 x} \qquad \text{Also } \frac{1}{\cos x} = \sec x$$

$$= \sec^2 x \qquad \text{and } \frac{\sin x}{\cos x} = \tan x$$

$$\therefore \quad \text{If } y = \tan x, \quad \frac{dy}{dx} = \sec^2 x$$

This is another one for our list of standard derivatives, so make a note of it. ▶

We now have the following standard derivatives:

y	$\dfrac{dy}{dx}$
x^n	
c	
$a.x^n$	
$\sin x$	
$\cos x$	
$\tan x$	
e^x	

Fill in the results in the right-hand column and then check with the next frame

36

y	$\dfrac{dy}{dx}$
x^n	$n.x^{n-1}$
c	0
$a.x^n$	$an.x^{n-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x

Also **1** If $y = uv$, $\dfrac{dy}{dx} = \dots\dots\dots\dots$

and **2** If $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dots\dots\dots\dots$

37

1 $y = uv$, $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

2 $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

▶

Now here is an exercise covering the work we have been doing. In each of the following functions, determine an expression for $\frac{dy}{dx}$:

1 $y = x^2 \cos x$ **2** $y = e^x \sin x$

3 $y = \frac{4e^x}{\sin x}$ **4** $y = \frac{\cos x}{x^4}$

5 $y = 5x^3.e^x$ **6** $y = 4x^2.\tan x$

7 $y = \frac{\cos x}{\sin x}$ **8** $y = \frac{\tan x}{e^x}$

9 $y = x^5.\sin x$ **10** $y = \frac{3x^2}{\cos x}$

All the solutions are shown in the next frame

38

1 $y = x^2.\cos x$

$$\frac{dy}{dx} = x^2(-\sin x) + \cos x.(2x)$$
$$= x(2\cos x - x\sin x)$$

2 $y = e^x.\sin x$

$$\frac{dy}{dx} = e^x.\cos x + \sin x(e^x)$$
$$= e^x(\sin x + \cos x)$$

3 $y = \frac{4e^x}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x.(4e^x) - 4e^x.\cos x}{\sin^2 x}$$
$$= \frac{4e^x(\sin x - \cos x)}{\sin^2 x}$$

4 $y = \frac{\cos x}{x^4}$

$$\frac{dy}{dx} = \frac{x^4(-\sin x) - \cos x(4x^3)}{x^8}$$
$$= \frac{-x\sin x - 4\cos x}{x^5}$$

5 $y = 5x^3e^x$

$$\frac{dy}{dx} = 5x^3.e^x + e^x.15x^2$$
$$= 5x^2e^x(x+3)$$

6 $y = 4x^2\tan x$

$$\frac{dy}{dx} = 4x^2.\sec^2 x + \tan x.(8x)$$
$$= 4x(x.\sec^2 x + 2\tan x)$$

7 $y = \frac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x(-\sin x) - \cos x.\cos x}{\sin^2 x}$$
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x} = -\mathrm{cosec}^2 x$$

8 $y = \frac{\tan x}{e^x}$

$$\frac{dy}{dx} = \frac{e^x.\sec^2 x - \tan x.e^x}{e^{2x}}$$
$$= \frac{\sec^2 x - \tan x}{e^x}$$

9 $y = x^5.\sin x$

$$\frac{dy}{dx} = x^5.\cos x + \sin x.(5x^4)$$
$$= x^4(x\cos x + 5\sin x)$$

10 $y = \frac{3x^2}{\cos x}$

$$\frac{dy}{dx} = \frac{\cos x.6x - 3x^2(-\sin x)}{\cos^2 x}$$
$$= \frac{3x(2\cos x + x\sin x)}{\cos^2 x}$$

Check the results and then move on to the next topic

Functions of a function

39

If $y = \sin x$, y is a function of the angle x, since the value of y depends on the value given to x.

If $y = \sin(2x - 3)$, y is a function of the angle $(2x - 3)$ which is itself a function of x.

Therefore, y is a function of (a function of x) and is said to be a *function of a function* of x.

Differentiation of a function of a function

To differentiate a function of a function, we must first introduce the *chain rule*.

With the example above, $y = \sin(2x - 3)$, we put $u = (2x - 3)$

i.e. $y = \sin u$ where $u = 2x - 3$.

If x has an increase δx, u will have an increase δu and then y will have an increase δy, i.e. $x \to x + \delta x$, $u \to u + \delta u$ and $y \to y + \delta y$.

At this stage, the increases δx, δu and δy are all finite values and therefore we can say that $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$ because the δu in $\dfrac{\delta y}{\delta u}$ cancels the δu in $\dfrac{\delta u}{\delta x}$.

If now $\delta x \to 0$, then $\delta u \to 0$ and $\delta y \to 0$

Also $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$, $\dfrac{\delta y}{\delta u} \to \dfrac{dy}{du}$ and $\dfrac{\delta u}{\delta x} \to \dfrac{du}{dx}$, and the previous statement now becomes $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

This is the *chain rule* and is particularly useful when determining the derivatives of functions of a function.

Example 1

To differentiate $y = \sin(2x - 3)$

Put $u = (2x - 3)$ $\quad\therefore\ y = \sin u$

$\therefore\ \dfrac{du}{dx} = 2$ and $\dfrac{dy}{du} = \cos u$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \cos u.(2) = 2\cos u = 2\cos(2x - 3)$

$\therefore$ If $y = \sin(2x - 3)$, $\dfrac{dy}{dx} = 2\cos(2x - 3)$

Further examples follow

40

Example 2

If $y = (3x+5)^4$, determine $\dfrac{dy}{dx}$

$y = (3x+5)^4$. Put $u = (3x+5)$. $\therefore\ y = u^4$

$\therefore\ \dfrac{dy}{du} = 4u^3$ and $\dfrac{du}{dx} = 3$. $\quad \dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$\therefore\ \dfrac{dy}{dx} = 4u^3.(3) = 12u^3 = 12(3x+5)^3$

$\therefore$ If $y = (3x+5)^4$, $\dfrac{dy}{dx} = 12(3x+5)^3$

And in just the same way:

Example 3

If $y = \tan(4x+1)$, $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

41

$$\boxed{4\sec^2(4x+1)}$$

Because

$y = \tan(4x+1)$ $\quad\therefore$ Put $u = 4x+1$ $\quad \therefore\ \dfrac{du}{dx} = 4$

$y = \tan u \quad \therefore\ \dfrac{dy}{du} = \sec^2 u$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \sec^2 u.(4) = 4\sec^2(4x+1)$

$\therefore$ If $y = \tan(4x+1)$, $\dfrac{dy}{dx} = 4\sec^2(4x+1)$

And now this one:

Exercise 4

If $y = e^{5x}$, $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

42

$$\boxed{5e^{5x}}$$

Because

$y = e^{5x}$ $\quad\therefore$ Put $u = 5x$ $\quad \therefore\ \dfrac{du}{dx} = 5$

$y = e^u \quad \therefore\ \dfrac{dy}{du} = e^u$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = e^u.(5) = 5e^u = 5e^{5x}$

$\therefore$ If $y = e^{5x}$, $\dfrac{dy}{dx} = 5e^{5x}$

▶

Many of these functions can be differentiated at sight by slight modification to our list of standard derivatives:

F is a function of x			
y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
F^n	$nF^{n-1}.\frac{dF}{dx}$	$\cos F$	$-\sin F.\frac{dF}{dx}$
$a.F^n$	$a.nF^{n-1}.\frac{dF}{dx}$	$\tan F$	$\sec^2 F.\frac{dF}{dx}$
$\sin F$	$\cos F.\frac{dF}{dx}$	e^F	$e^F.\frac{dF}{dx}$

Let us now apply these results

43

Here are four examples

1 $y = \cos(2x - 1)$, $\frac{dy}{dx} = -\sin(2x - 1) \times 2 = -2\sin(2x - 1)$

2 $y = e^{(3x+4)}$, $\frac{dy}{dx} = e^{(3x+4)} \times 3 = 3.e^{(3x+4)}$

3 $y = (5x - 2)^3$, $\frac{dy}{dx} = 3(5x - 2)^2 \times 5 = 15(5x - 2)^2$

4 $y = 4.e^{\sin x}$, $\frac{dy}{dx} = 4.e^{\sin x} \times \cos x = 4\cos x.e^{\sin x}$

In just the same way, as an exercise, differentiate the following:

1 $y = \sin(4x + 3)$ **4** $y = \tan 5x$

2 $y = (2x - 5)^4$ **5** $y = e^{2x-3}$

3 $y = \sin^3 x$ **6** $y = 4\cos(7x + 2)$

44

1 $\frac{dy}{dx} = 4.\cos(4x + 3)$ **4** $\frac{dy}{dx} = 5.\sec^2 5x$

2 $\frac{dy}{dx} = 8.(2x - 5)^3$ **5** $\frac{dy}{dx} = 2.e^{2x-3}$

3 $\frac{dy}{dx} = 3.\sin^2 x \cos x$ **6** $\frac{dy}{dx} = -28.\sin(7x + 2)$

▶

Now let us consider the derivative of $y = \ln x$:

If $y = \ln x$ then $x = e^y$

Differentiating with respect to x: $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(e^y)$ we find that the derivative on the LHS is equal to 1 and, by the chain rule, the derivative on the RHS is $\dfrac{d}{dy}(e^y)\dfrac{dy}{dx} = e^y\dfrac{dy}{dx}$ so that:

$1 = e^y\dfrac{dy}{dx}$ and, since $x = e^y$ this can be written as:

$1 = x\dfrac{dy}{dx}$ therefore $\dfrac{dy}{dx} = \dfrac{1}{x}$. Therefore if $y = \ln x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$

We can add this to our list of standard derivatives. Also, if F is a function of x then:

if $y = \ln F$ we have $\dfrac{dy}{dx} = \dfrac{1}{F} \cdot \dfrac{dF}{dx}$.

Here is an example:

$$\text{If } y = \ln(3x - 5),\ \frac{dy}{dx} = \frac{1}{3x - 5}.3 = \frac{3}{3x - 5}$$

$$\text{and if } y = \ln(\sin x),\ \frac{dy}{dx} = \frac{1}{\sin x}.\cos x = \cot x$$

There is one further standard derivative to be established at this stage, so move on to the next frame

---

45

Derivative of $y = a^x$

We already know that if $y = e^x$, $\dfrac{dy}{dx} = e^x$

and that if $y = e^F$, $\dfrac{dy}{dx} = e^F.\dfrac{dF}{dx}$

Then, if $y = a^x$, we can write $a = e^k$ and then $y = a^x = (e^k)^x = e^{kx}$

$$\therefore\ \frac{dy}{dx} = e^{kx}.\frac{d}{dx}(kx) = e^{kx}(k) = k.e^{kx}$$

But, $e^{kx} = a^x$ and $k = \ln a$ $\quad\therefore$ If $y = a^x$, $\dfrac{dy}{dx} = a^x \ln a$

We can add this result to our list for future reference.

That completes the topic of differentiation at this stage. However, we shall deal with it further later in the course.

46

Revision summary

1 Standard derivatives

y	$\dfrac{dy}{dx}$	y	$\dfrac{dy}{dx}$
x^n	$n.x^{n-1}$	$\tan x$	$\sec^2 x$
c	0	e^x	e^x
$\sin x$	$\cos x$	$\ln x$	$\dfrac{1}{x}$
$\cos x$	$-\sin x$	a^x	$a^x \ln a$

2 Rules of derivatives

(a) of product $y = uv$, $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

(b) of quotient $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

3 Chain rule

F is a function of x			
y	$\dfrac{dy}{dx}$	y	$\dfrac{dy}{dx}$
F^n	$nF^{n-1}.\dfrac{dF}{dx}$	$\tan F$	$\sec^2 F.\dfrac{dF}{dx}$
$\sin F$	$\cos F.\dfrac{dF}{dx}$	e^F	$e^F.\dfrac{dF}{dx}$
$\cos F$	$-\sin F.\dfrac{dF}{dx}$	$\ln F$	$\dfrac{1}{F}.\dfrac{dF}{dx}$

47

Revision exercise

1 Determine $\dfrac{dy}{dx}$ in each of the following cases:

(a) $y = x^3 \tan x$ (b) $y = x^2 e^x$

(c) $y = \dfrac{2e^x}{x^2}$ (d) $y = \dfrac{x^3}{\sin x}$

2 Differentiate the following with respect to x:

(a) $y = (4x + 3)^6$ (b) $y = \tan(2x + 3)$

(c) $y = \ln(3x - 4)$ (d) $y = -2e^{(1-3x)}$

(e) $y = e^{-3x}\sin(2x)$ (f) $y = \tan^2 x$

▶

3 (a) If $y = x^3 + 2x^2 - 3x - 4$, determine:

(i) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

(ii) the values of x at which $\dfrac{dy}{dx} = 0$.

(b) If $y = 2\cos(x+1) - 3\sin(x-1)$, obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

48

1 (a) $y = x^3 \tan x$. Applying the product rule we find that:

$$\frac{dy}{dx} = 3x^2 \tan x + x^3 \sec^2 x$$
$$= x^2(3\tan x + x\sec^2 x)$$

(b) $y = x^2e^x$. Applying the product rule we find that:

$$\frac{dy}{dx} = 2xe^x + x^2e^x$$
$$= xe^x(2+x)$$

(c) $y = \dfrac{2e^x}{x^2}$. Applying the quotient rule we find that:

$$\frac{dy}{dx} = \frac{2e^x x^2 - 2e^x 2x}{[x^2]^2}$$
$$= 2e^x\left(\frac{x^2-2x}{x^4}\right)$$
$$= 2e^x\left(\frac{x-2}{x^3}\right)$$

(d) $y = \dfrac{x^3}{\sin x}$. Applying the quotient rule we find that:

$$\frac{dy}{dx} = \frac{3x^2 \sin x - x^3 \cos x}{[\sin x]^2}$$
$$= x^2\left(\frac{3\sin x - x\cos x}{\sin^2 x}\right)$$

2 (a) $y = (4x+3)^6$. Applying the chain rule we find that:

$$\frac{dy}{dx} = 6(4x+3)^5 \times 4$$
$$= 24(4x+3)^5$$

(b) $y = \tan(2x+3)$. Applying the chain rule we find that:

$$\frac{dy}{dx} = \sec^2(2x+3) \times 2$$
$$= 2\sec^2(2x+3)$$

▶

(c) $y = \ln(3x - 4)$. Applying the chain rule we find that:

$$\frac{dy}{dx} = \frac{1}{3x-4} \times 3$$
$$= \frac{3}{3x-4}$$

(d) $y = -2e^{(1-3x)}$. Applying the chain rule we find that:

$$\frac{dy}{dx} = -2e^{(1-3x)} \times (-3)$$
$$= 6e^{(1-3x)}$$

(e) $y = e^{-3x}\sin(2x)$. Applying the chain rule and product rule combined we find that:

$$\frac{dy}{dx} = -3e^{-3x} \times \sin(2x) + e^{-3x} \times 2\cos(2x)$$
$$= e^{-3x}(2\cos(2x) - 3\sin(2x))$$

(f) $y = \tan^2 x$. Applying the chain rule we find that:

$$\frac{dy}{dx} = (2\tan x) \times \sec^2 x$$
$$= 2\tan x \sec^2 x$$

3 (a) If $y = x^3 + 2x^2 - 3x - 4$, then:

(i) $\frac{dy}{dx} = 3x^2 + 4x - 3$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x + 4$.

(ii) $\frac{dy}{dx} = 0$ when $3x^2 + 4x - 3 = 0$. That is when:

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 3 \times (-3)}}{2 \times 3}$$
$$= \frac{-4 \pm \sqrt{52}}{6} = -1{\cdot}869 \text{ or } 0{\cdot}535 \text{ to 3 dp.}$$

(b) If $y = 2\cos(x + 1) - 3\sin(x - 1)$, then:

$$\frac{dy}{dx} = -2\sin(x+1) - 3\cos(x-1) \text{ and}$$
$$\frac{d^2y}{dx^2} = -2\cos(x+1) + 3\sin(x-1) = -y$$

49

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through these at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

Can You?

Checklist F.10

50

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Determine the gradient of a straight-line graph? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 2
• Evaluate from first principles the gradient at a point on a quadratic curve? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	3 to 10
• Differentiate powers of x and polynomials? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	11 to 22
• Evaluate second derivatives and use tables of standard derivatives? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	26 to 30
• Differentiate products and quotients of expressions? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	32 to 38
• Differentiate using the chain rule for a function of a function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	38 to 44

Test exercise F.10

51

	Frames
1 Calculate the slope of the straight line joining: (a) P (3, 5) and Q (6, 9) (b) P (2, 6) and Q (7, 4) (c) P (−3, 8) and Q (4, 2) (d) P (−2, 5) and Q (3, −8)	1 to 2
2 Determine algebraically, from first principles, the slope of the graph of $y = 3x^2 + 4$ at the point P where $x = 1{\cdot}2$.	3 to 10
3 If $y = x^4 + 5x^3 - 6x^2 + 7x - 3$, obtain an expression for $\dfrac{dy}{dx}$ and hence calculate the value of $\dfrac{dy}{dx}$ at $x = -2$.	11 to 27

▶

Frames

4 (a) If $y = 2x^3 - 11x^2 + 12x - 5$, determine:

(i) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ (ii) the values of x at which $\dfrac{dy}{dx} = 0$.

(b) If $y = 3\sin(2x+1) + 4\cos(3x-1)$, obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. 11 to 27

5 Determine $\dfrac{dy}{dx}$ in each of the following cases:

(a) $y = x^2.\sin x$ (b) $y = x^3.e^x$ (c) $y = \dfrac{\cos x}{x^2}$ (d) $y = \dfrac{2e^x}{\tan x}$ 28 to 38

6 Differentiate the following with respect to x:

(a) $y = (5x+2)^4$ (d) $y = 5\cos(2x+3)$

(b) $y = \sin(3x+2)$ (e) $y = \cos^3 x$

(c) $y = e^{(4x-1)}$ (f) $y = \ln(4x-5)$ 39 to 45

Further problems F.10

52

1 Determine algebraically from first principles, the slope of the following graphs at the value of x indicated:

(a) $y = 4x^2 - 7$ at $x = -0{\cdot}5$

(b) $y = 2x^3 + x - 4$ at $x = 2$

(c) $y = 3x^3 - 2x^2 + x - 4$ at $x = -1$

2 Differentiate the following and calculate the value of $\dfrac{dy}{dx}$ at the value of x stated:

(a) $y = 2x^3 + 4x^2 - 2x + 7$ $[x = -2]$

(b) $y = 3x^4 - 5x^3 + 4x^2 - x + 4$ $[x = 3]$

(c) $y = 4x^5 + 2x^4 - 3x^3 + 7x^2 - 2x + 3$ $[x = 1]$

Differentiate the functions given in questions **3**, **4** and **5**:

3 (a) $y = x^5 \sin x$ (c) $y = x^3 \tan x$ (e) $y = 5x^2 \sin x$

(b) $y = e^x \cos x$ (d) $y = x^4 \cos x$ (f) $y = 2e^x \ln x$

4 (a) $y = \dfrac{\cos x}{x^3}$ (c) $y = \dfrac{\cos x}{\tan x}$ (e) $y = \dfrac{\tan x}{e^x}$

(b) $y = \dfrac{\sin x}{2e^x}$ (d) $y = \dfrac{4x^3}{\cos x}$ (f) $y = \dfrac{\ln x}{x^3}$

5 (a) $y = (2x-3)^3$ (e) $y = \sin(2x-3)$ (i) $y = \ln(3x^2)$

(b) $y = e^{3x+2}$ (f) $y = \tan(x^2-3)$ (j) $y = 3\sin(4-5x)$

(c) $y = 4\cos(3x+1)$ (g) $y = 5(4x+5)^2$

(d) $y = \ln(x^2+4)$ (h) $y = 6e^{x^2+2}$

Programme F.11

Integration

Frames 1 to 46

Learning outcomes

When you have completed this Programme you will be able to:

- Appreciate that integration is the reverse process of differentiation
- Recognize the need for a constant of integration
- Evaluate indefinite integrals of standard forms
- Evaluate indefinite integrals of polynomials
- Evaluate indefinite integrals of 'functions of a linear function of x'
- Integrate by partial fractions
- Appreciate that a definite integral is a measure of the area under a curve
- Evaluate definite integrals of standard forms
- Use the definite integral to find areas between a curve and the horizontal axis
- Use the definite integral to find areas between a curve and a given straight line

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.11

	Frames
1 Determine the following integrals: (a) $\int x^7 \, dx$ (b) $\int 4\cos x \, dx$ (c) $\int 2e^x \, dx$ (d) $\int 12 \, dx$ (e) $\int x^{-4} \, dx$ (f) $\int 6^x \, dx$ (g) $\int 9x^{\frac{1}{3}} \, dx$ (h) $\int 4\sec^2 x \, dx$ (i) $\int 9\sin x \, dx$ (j) $\int \frac{8}{x} \, dx$	1 to 5
2 Determine the following integrals: (a) $I = \int (x^3 - x^2 + x - 1) \, dx$ (b) $I = \int (4x^3 - 9x^2 + 8x - 2) \, dx$ given that $I = \frac{11}{16}$ when $x = \frac{1}{2}$.	6 to 11
3 Determine the following integrals: (a) $\int (5x-1)^4 \, dx$ (b) $\int \frac{\sin(6x-1)}{3} \, dx$ (c) $\int \sqrt{4-2x} \, dx$ (d) $\int 2e^{3x+2} \, dx$ (e) $\int 5^{1-x} \, dx$ (f) $\int \frac{3}{2x-3} \, dx$ (g) $\int \frac{\sec^2(2-5x)}{5} \, dx$	12 to 15
4 Integrate by partial fractions each of the following integrals: (a) $\int \frac{5x}{6x^2 - x - 1} \, dx$ (b) $\int \frac{14x+1}{2-7x-4x^2} \, dx$ (c) $\int \frac{1-9x}{1-9x^2} \, dx$	16 to 25
5 Find the area enclosed between the x-axis and the curve $y = e^x$ between $x = 1$ and $x = 2$, giving your answer in terms of e.	26 to 36
6 Find the area enclosed between the curve $y = e^x$ and the straight line $y = 1 - x$ between $x = 1$ and $x = 2$, giving your answer to 3 dp.	37 to 42

Integration

1

Integration is the reverse process of differentiation. When we differentiate we start with an expression and proceed to find its derivative. When we integrate, we start with the derivative and then find the expression from which it has been derived.

For example, $\frac{d}{dx}(x^4) = 4x^3$. Therefore, the integral of $4x^3$ with respect to x we know to be x^4. This is written:

$$\int 4x^3\,dx = x^4$$

The symbols $\int f(x)\,dx$ denote the *integral of* $f(x)$ *with respect to the variable* x; the symbol $\int$ was developed from a capital S which was used in the 17th century when the ideas of the calculus were first devised. The expression $f(x)$ to be integrated is called the *integrand* and the differential dx is usefully there to assist in the evaluation of certain integrals, as we shall see in a later Programme.

Constant of integration

$$\begin{array}{llll} \text{So} & \frac{d}{dx}(x^4) = 4x^3 & \therefore & \int 4x^3.dx = x^4 \\ \text{Also} & \frac{d}{dx}(x^4+2) = 4x^3 & \therefore & \int 4x^3.dx = x^4+2 \\ \text{and} & \frac{d}{dx}(x^4-5) = 4x^3 & \therefore & \int 4x^3.dx = x^4-5 \end{array}$$

In these three examples we happen to know the expressions from which the derivative $4x^3$ was derived. But any constant term in the original expression becomes zero in the derivative and all trace of it is lost. So if we do not know the history of the derivative $4x^3$ we have no evidence of the value of the constant term, be it 0, +2, −5 or any other value. We therefore acknowledge the presence of such a constant term of some value by adding a symbol C to the result of the integration:

i.e. $\int 4x^3.dx = x^4 + C$

C is called the *constant of integration* and must always be included.

Such an integral is called an *indefinite integral* since normally we do not know the value of C. In certain circumstances, however, the value of C might be found if further information about the integral is available.

For example, to determine $I = \int 4x^3.dx$, given that $I = 3$ when $x = 2$. As before:

$$I = \int 4x^3\,dx = x^4 + C$$

But $I = 3$ when $x = 2$ so that $3 = 2^4 + C = 16 + C \therefore C = -13$.
So, in this case $I = x^4 - 13$.

Next frame

Standard integrals

2

Every derivative written in reverse gives an integral.

e.g. $\dfrac{d}{dx}(\sin x) = \cos x \quad \therefore \quad \int \cos x.dx = \sin x + C$

It follows, therefore, that our list of standard derivatives provides a source of standard integrals.

(a) $\dfrac{d}{dx}(x^n) = nx^{n-1}$. Replacing n by $(n+1)$, $\dfrac{d}{dx}(x^{n+1}) = (n+1)x^n$

$\therefore \quad \dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = x^n \qquad \therefore \quad \int x^n.dx = \dfrac{x^{n+1}}{n+1} + C$

This is true except when $n = -1$, for then we should be dividing by 0.

(b) $\dfrac{d}{dx}(\sin x) = \cos x$

$\therefore \quad \int \cos x.dx = \sin x + C$

(c) $\dfrac{d}{dx}(\cos x) = -\sin x$

$\therefore \quad \dfrac{d}{dx}(-\cos x) = \sin x$

$\therefore \quad \int \sin x.dx = -\cos x + C$

(d) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\therefore \quad \int \sec^2 x.dx = \tan x + C$

(e) $\dfrac{d}{dx}(e^x) = e^x$

$\therefore \quad \int e^x.dx = e^x + C$

(f) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

$\therefore \quad \int \dfrac{1}{x}dx = \ln x + C$

(g) $\dfrac{d}{dx}(a^x) = a^x.\ln a$

$\therefore \quad \int a^x.dx = \dfrac{a^x}{\ln a} + C$

As with differentiation, a constant coefficient remains unchanged

e.g. $\int 5.\cos x.dx = 5\sin x + C$, etc.

Collecting the results together, we have:

$f(x)$	$\int f(x).dx$	
x^n	$\dfrac{x^{n+1}}{n+1} + C$	$(n \neq -1)$
1	$x + C$	
a	$ax + C$	
$\sin x$	$-\cos x + C$	
$\cos x$	$\sin x + C$	
$\sec^2 x$	$\tan x + C$	
e^x	$e^x + C$	
a^x	$a^x/\ln a + C$	
$\dfrac{1}{x}$	$\ln x + C$	

At this point let us pause and summarise the main facts so far

3

Revision summary

1 Integration is the reverse process of differentiation. Given a derivative we have to find the expression from which it was derived.

2 Because constants have a zero derivative we find that when we reverse the process of differentiation we must introduce an integration constant into our result.

3 Such integrals are called indefinite integrals.

4

Revision exercise

1 Determine the following integrals:

(a) $\int x^6\,dx$ (b) $\int 3e^x\,dx$ (c) $\int \frac{6}{x}\,dx$

(d) $\int 5\sin x\,dx$ (e) $\int \sec^2 x\,dx$ (f) $\int 8\,dx$

(g) $\int x^{\frac{1}{2}}\,dx$ (h) $\int 2\cos x\,dx$ (i) $\int x^{-3}\,dx$

(j) $\int 4^x\,dx$

2 (a) Determine $I = \int 4x^2.dx$, given that $I = 25$ when $x = 3$.

(b) Determine $I = \int 5.dx$, given that $I = 16$ when $x = 2$.

(c) Determine $I = \int 2.\cos x.dx$, given that $I = 7$ when $x = \frac{\pi}{2}$ (radians).

(d) Determine $I = \int 2.e^x.dx$, given that $I = 50{\cdot}2$ when $x = 3$.

Complete and then check with the next frame

5

1 (a) $\int x^6\,dx = \frac{x^7}{7} + C$

Check by differentiating the result – the derivative of the result will give the integrand

$$\frac{d}{dx}\left(\frac{x^7}{7} + C\right) = \frac{7x^6}{7} + 0 = x^6 \textit{ the integrand}$$

(b) $\int 3e^x\,dx = 3e^x + C$

The derivative of the result is $3e^x + 0 = 3e^x$ – the integrand

(c) $\int \frac{6}{x}\,dx = 6\ln x + C$

The derivative of the result is $\frac{6}{x} + 0 = \frac{6}{x}$

(d) $\int 5\sin x\,dx = -5\cos x + C$

The derivative of the result is $-5(-\sin x) + 0 = 5\sin x$

▶

(e) $\int \sec^2 x\,dx = \tan x + C$ — The derivative of the result is $\sec^2 x + 0 = \sec^2 x$

(f) $\int 8\,dx = 8x + C$ — The derivative of the result is $8 + 0 = 8$

(g) $\int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2x^{\frac{3}{2}}}{3} + C$ — The derivative of the result is $\dfrac{3}{2} \times \dfrac{2x^{\frac{3}{2}-1}}{3} + 0 = x^{\frac{1}{2}}$

(h) $\int 2\cos x\,dx = 2\sin x + C$ — The derivative of the result is $2\cos x + 0 = 2\cos x$

(i) $\int x^{-3}\,dx = \dfrac{x^{-3+1}}{-3+1} + C = -\dfrac{x^{-2}}{2} + C$ — The derivative of the result is $-(-2)\dfrac{x^{-2-1}}{2} + 0 = x^{-3}$

(j) $\int 4^x\,dx = \dfrac{4^x}{\ln 4} + C$ — Recall the standard derivative $\dfrac{d}{dx}(a^x) = a^x \ln a$

2 (a) $I = \int 4x^2.dx = 4\dfrac{x^3}{3} + C$

$\therefore\ 25 = 36 + C\ \therefore\ C = -11$

$\therefore\ \int 4x^2.dx = \dfrac{4x^3}{3} - 11$

(b) $I = \int 5.dx = 5x + C$

$16 = 10 + C\ \therefore\ C = 6$

$\therefore\ \int 5.dx = 5x + 6$

(c) $I = \int 2.\cos x.dx = 2\sin x + C$

$\therefore\ 7 = 2 + C\ \therefore\ C = 5$

$\therefore\ \int 2.\cos x.dx = 2\sin x + 5$

(d) $I = \int 2.e^x.dx = 2e^x + C$

$\therefore\ 50{\cdot}2 = 2e^3 + C = 40{\cdot}2 + C$

$\therefore\ C = 10$

$\therefore\ \int 2.e^x.dx = 2e^x + 10$

Integration of polynomial expressions

6

In the previous Programme we differentiated a polynomial expression by dealing with the separate terms, one by one. It is not surprising, therefore, that we do much the same with the integration of polynomial expressions.

Polynomial expressions are integrated term by term with the individual constants of integration consolidated into one symbol C for the whole expression. For example:

$$\int(4x^3+5x^2-2x+7)\,dx$$
$$=x^4+\frac{5x^3}{3}-x^2+7x+C$$

So, what about this one? If $I=\int(8x^3-3x^2+4x-5)\,dx$, determine the value of I when $x=3$, given that at $x=2$, $I=26$.

First we must determine the function for I, so carrying out the integration, we get

$$I=\ldots\ldots\ldots\ldots$$

7

$$\boxed{I=2x^4-x^3+2x^2-5x+C}$$

Now we can calculate the value of C since we are told that when $x=2$, $I=26$. So, expressing the function for I in nested form, we have

$$I=\ldots\ldots\ldots\ldots$$

8

$$\boxed{I=\{[(2x-1)x+2]x-5\}x+C}$$

Substituting $x=2$:

$$26=\ldots\ldots\ldots\ldots$$

9

$$\boxed{22+C}$$

We have $26=22+C \quad \therefore\; C=4$

$\therefore\; I=\{[(2x-1)x+2]x-5\}x+4$

Finally, all we now have to do is to put $x=3$ in this expression which gives us that, when $x=3$, $I=\ldots\ldots\ldots$

10

$$\boxed{142}$$

Just take one step at a time. There are no snags.

Now here is another of the same type. Determine the value of

$I=\int(4x^3-6x^2-16x+4)\,dx$ when $x=-2$, given that at $x=3$, $I=-13$.

▶

As before:

(a) Perform the integration.
(b) Express the resulting function in nested form.
(c) Evaluate the constant of integration, using the fact that when $x = 3$, $I = -13$.
(d) Determine the value of I when $x = -2$.

The method is just the same as before, so work through it.

$\therefore$ When $x = -2$. $I = \dots\dots\dots\dots$

11

12

Here is a check on the working:

(a) $I = x^4 - 2x^3 - 8x^2 + 4x + C$
(b) In nested form, $I = \{[(x-2)x - 8]x + 4\}x + C$
(c) At $x = 3$, $I = -13 = -33 + C \;\therefore\; C = 20$
$\therefore\; I = \{[(x-2)x - 8]x + 4\}x + 20$
(d) $\therefore$ When $x = -2$. $I = 12$

It is all very straightforward.

Now let us move on to something slightly different

Functions of a linear function of *x*

12

It is often necessary to integrate any one of the expressions shown in our list of standard integrals when the variable x is replaced by a linear expression in x. That is, of the form $ax + b$. For example, $y = \int (3x+2)^4\,dx$ is of the same structure as $y = \int x^4\,dx$ except that x is replaced by the linear expression $3x + 2$. Let us put $u = 3x + 2$ then:

$$\int (3x+2)^4\,dx \text{ becomes } \int u^4\,dx$$

We now have to change the variable x in dx before we can progress. Now, $u = 3x + 2$ so that:

$$\frac{du}{dx} = 3$$

That is:

$$du = 3\,dx \text{ or, alternatively } dx = \frac{du}{3}$$

We now find that our integral can be determined as:

$$y=\int u^4\,dx=\int u^4\frac{du}{3}=\frac{1}{3}\left(\frac{u^5}{5}\right)+C=\frac{1}{3}\left(\frac{(3x+2)^5}{5}\right)+C$$

That is:

$$y=\frac{(3x+2)^5}{15}+C$$

To integrate a 'function of a linear function of x', simply replace x in the corresponding standard result by the linear expression and divide by the coefficient of x in the linear expression.

Here are three examples:

(1) $\int(4x-3)^2.dx$ [Standard integral $\int x^2.dx=\frac{x^3}{3}+C$]

$\therefore\ \int(4x-3)^2.dx=\frac{(4x-3)^3}{3}\times\frac{1}{4}+C=\frac{(4x-3)^3}{12}+C$

(2) $\int\cos 3x.dx$ [Standard integral $\int\cos x.dx=\sin x+C$]

$\therefore\ \int\cos 3x.dx=\sin 3x.\frac{1}{3}+C=\frac{\sin 3x}{3}+C$

(3) $\int e^{5x+2}.dx$ [Standard integral $\int e^x.dx=e^x+C$]

$\therefore\ \int e^{5x+2}.dx=e^{5x+2}\frac{1}{5}+C=\frac{e^{5x+2}}{5}+C$

Just refer to the basic standard integral of the same form, replace x in the result by the linear expression and finally divide by the coefficient of x in the linear expression – and remember the constant of integration.

At this point let us pause and summarize the main facts dealing with the integration of polynomial expressions and 'functions of a linear function of x'

Revision summary

13

1 *Integration of polynomial expressions*
Integrate term by term and combine the individual constants of integration into one symbol.

2 *Integration of 'functions of a linear function of x'*
Replace x in the corresponding standard integral by the linear expression and divide the result by the coefficient of x in the linear expression.

Revision exercise

14

1 Determine the following integrals:

(a) $I=\int(2x^3-5x^2+6x-9)\,dx$

(b) $I=\int(9x^3+11x^2-x-3)\,dx$, given that when $x=1$, $I=2$.

▶

2 Determine the following integrals:

(a) $\int (1-4x)^2.dx$ (f) $\int \cos(1-3x).dx$

(b) $\int 4.e^{5x-2}.dx$ (g) $\int 2^{3x-1}.dx$

(c) $\int 3.\sin(2x+1).dx$ (h) $\int 6\sec^2(2+3x).dx$

(d) $\int (3-2x)^{-5}.dx$ (i) $\int \sqrt{3-4x}.dx$

(e) $\int \frac{7}{2x-5}.dx$ (j) $\int 5.e^{1-3x}.dx$

15

1 (a) $I = \int (2x^3 - 5x^2 + 6x - 9)\,dx = 2\frac{x^4}{4} - 5\frac{x^3}{3} + 6\frac{x^2}{2} - 9x + C$

$$= \frac{x^4}{2} - \frac{5}{3}x^3 + 3x^2 - 9x + C$$

(b) $I = \int (9x^3 + 11x^2 - x - 3)\,dx = 9\frac{x^4}{4} + 11\frac{x^3}{3} - \frac{x^2}{2} - 3x + C$

$$= \left(\left(\left(\frac{9x}{4} + \frac{11}{3}\right)x - \frac{1}{2}\right)x - 3\right)x + C$$

Given $I = 2$ when $x = 1$ we find that:

$$2 = \frac{29}{12} + C$$

So that $C = -\frac{5}{12}$ and $I = 9\frac{x^4}{4} + 11\frac{x^3}{3} - \frac{x^2}{2} - 3x - \frac{5}{12}$

2 (a) $\int (1-4x)^2\,dx$ [Standard integral $\int x^2\,dx = \frac{x^3}{3} + C$]

Therefore, $\int (1-4x)^2\,dx = \frac{(1-4x)^3}{3} \times \frac{1}{(-4)} + C = -\frac{(1-4x)^3}{12} + C$

(b) $\int 4e^{5x-2}\,dx$ [Standard integral $\int e^x\,dx = e^x + C$]

Therefore, $\int 4e^{5x-2}\,dx = 4e^{5x-2} \times \frac{1}{5} + C = \frac{4}{5}e^{5x-2} + C$

(c) $\int 3\sin(2x+1)\,dx$ [Standard integral $\int \sin x\,dx = -\cos x + C$]

Therefore, $\int 3\sin(2x+1)\,dx = 3(-\cos(2x+1)) \times \frac{1}{2} + C$

$= -\frac{3}{2}\cos(2x+1) + C$

(d) $\int (3-2x)^{-5}\,dx$ [Standard integral $\int x^{-5}\,dx = \frac{-x^{-4}}{4} + C$]

Therefore, $\int (3-2x)^{-5}\,dx = -\frac{(3-2x)^{-4}}{4} \times \frac{1}{(-2)} + C = \frac{(3-2x)^{-4}}{8} + C$

▶

(e) $\int \frac{7}{2x-5}\,dx$ [Standard integral $\int \frac{1}{x}\,dx = \ln x + C$]

Therefore, $\int \frac{7}{2x-5}\,dx = 7\ln(2x-5) \times \frac{1}{2} + C = \frac{7}{2}\ln(2x-5) + C$

(f) $\int \cos(1-3x)\,dx$ [Standard integral $\int \cos x\,dx = \sin x + C$]

Therefore, $\int \cos(1-3x)\,dx = \sin(1-3x) \times \frac{1}{(-3)} + C$

$$= -\frac{\sin(1-3x)}{3} + C$$

(g) $\int 2^{3x-1}\,dx$ [Standard integral $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$]

Therefore, $\int 2^{3x-1}\,dx = \frac{2^{3x-1}}{\ln 2} \times \frac{1}{3} + C = \frac{2^{3x-1}}{3\ln 2} + C$

(h) $\int 6\sec^2(2+3x)\,dx$ [Standard integral $\int \sec^2 x\,dx = \tan x + C$]

Therefore, $\int 6\sec^2(2+3x)\,dx = 6\tan(2+3x) \times \frac{1}{3} + C$

$$= 2\tan(2+3x) + C$$

(i) $\int \sqrt{3-4x}\,dx$ [Standard integral $\int \sqrt{x}\,dx = \frac{x^{\frac{3}{2}}}{3/2} + C$]

Therefore, $\int \sqrt{3-4x}\,dx = \frac{2(3-4x)^{\frac{3}{2}}}{3} \times \frac{1}{(-4)} + C = -\frac{(3-4x)^{\frac{3}{2}}}{6} + C$

(j) $\int 5e^{1-3x}\,dx$ [Standard integral $\int 5e^x\,dx = 5e^x + C$]

Therefore, $\int 5e^{1-3x}\,dx = 5e^{1-3x} \times \frac{1}{(-3)} + C = -\frac{5}{3}e^{1-3x} + C$

Integration by partial fractions

16

Expressions such as $\int \frac{7x+8}{2x^2+11x+5}\,dx$ do not appear in our list of standard integrals, but do, in fact, occur in many mathematical applications. We saw in Programme F.7 that such an expression as $\frac{7x+8}{2x^2+11x+5}$ can be expressed in partial fractions which are simpler in structure.

In fact, $\frac{7x+8}{2x^2+11x+5} = \frac{7x+8}{(x+5)(2x+1)} = \frac{3}{x+5} + \frac{1}{2x+1}$ so that

$$\int \frac{7x+8}{2x^2+11x+5}\,dx = \int \frac{3}{x+5}\,dx + \int \frac{1}{2x+1}\,dx$$

▶

These partial fractions are 'functions of a linear function of x', based on the standard integral $\int \frac{1}{x}\,dx$, so the result is clear:

$$\int \frac{7x+8}{2x^2+11x+5}\,dx = \int \frac{7x+8}{(x+5)(2x-1)}\,dx = \int \frac{3}{x+5}\,dx + \int \frac{1}{2x+1}\,dx$$
$$= 3\ln(x+5) + \frac{1}{2}\ln(2x+1) + C$$

You will recall the Rules of Partial Fractions which we listed earlier and used in Programme F.7, so let us apply them in this example. We will only deal with simple linear denominators at this stage.

Determine $\int \frac{3x^2+18x+3}{3x^2+5x-2}\,dx$ by partial fractions.

The first step is ....

17

$\boxed{\text{to divide out}}$

because the numerator is not of lower degree than that of the denominator

$$\text{So } \frac{3x^2+18x+3}{3x^2+5x-2} = \ldots\ldots\ldots\ldots$$

18

$$\boxed{1 + \frac{13x+5}{3x^2+5x-2}}$$

The denominator factorizes into $(3x-1)(x+2)$ so the partial fractions of $\frac{13x+5}{(3x-1)(x+2)} = \ldots\ldots\ldots$

19

$$\boxed{\frac{4}{3x-1} + \frac{3}{x+2}}$$

Because

$$\frac{13x+5}{(3x-1)(x+2)} = \frac{A}{3x-1} + \frac{B}{x+2}$$
$$\therefore\ 13x+5 = A(x+2) + B(3x-1)$$
$$= Ax + 2A + 3Bx - B$$
$$= (A+3B)x + (2A-B)$$

$$\begin{array}{llll} [x] & \therefore\ A+3B=13 & A+3B=13 & \\ [\mathrm{CT}] & 2A-B=5 & \underline{6A-3B=15} & \\ & & 7A \quad = 28 & \therefore\ A=4 \\ & \therefore\ 4+3B=13 & \therefore\ 3B=9 & \therefore\ B=3 \end{array}$$

$$\therefore \quad \frac{13x+5}{(3x-1)(x+2)} = \frac{4}{3x-1} + \frac{3}{x+2}$$

$$\therefore \quad \int \frac{3x^2+18x+3}{3x^2+5x-2}\,dx = \int\left(1+\frac{4}{3x-1}+\frac{3}{x+2}\right)dx$$

$$= \dots\dots\dots$$

Finish it

20

$$\boxed{I = x + \frac{4.\ln(3x-1)}{3} + 3.\ln(x+2) + C}$$

Now you can do this one in like manner:

$$\int \frac{4x^2+26x+5}{2x^2+9x+4}\,dx = \dots\dots\dots\dots$$

Work right through it and then check the solution with the next frame

21

$$\boxed{2x + 5\ln(x+4) - \ln(2x+1) + C}$$

Here is the working:

$$\frac{4x^2+26x+5}{2x^2+9x+4} = 2 + \frac{8x-3}{2x^2+9x+4}$$

$$\frac{8x-3}{2x^2+9x+4} = \frac{8x-3}{(x+4)(2x+1)} = \frac{A}{x+4} + \frac{B}{2x+1}$$

$$\therefore \; 8x-3 = A(2x+1) + B(x+4) = (2A+B)x + (A+4B)$$

$$\therefore \; 2A+B = 8 \qquad 8A+4B = 32$$

$$A+4B = -3 \qquad \underline{A+4B = -3}$$

$$\therefore \; 7A \quad = 35 \qquad \therefore \; A = 5$$

$$10+B = 8 \qquad \therefore \; B = -2$$

$$\therefore \quad \int \frac{4x^2+26x+5}{2x^2+9x+4}\,dx = \int\left(2+\frac{5}{x+4}-\frac{2}{2x+1}\right)dx$$

$$= 2x + 5\ln(x+4) - \frac{2\ln(2x+1)}{2} + C$$

$$= 2x + 5\ln(x+4) - \ln(2x+1) + C$$

And finally this one:

Determine $I = \displaystyle\int \frac{16x+7}{6x^2+x-12}\,dx$ by partial fractions.

$I = \dots\dots\dots$

22

$$\boxed{\ln(2x+3)+\frac{5}{3}\ln(3x-4)+C}$$

$$\frac{16x+7}{6x^2+x-12}=\frac{16x+7}{(2x+3)(3x-4)}=\frac{A}{2x+3}+\frac{B}{3x-4}$$

$$\therefore\ 16x+7=A(3x-4)+B(2x+3)$$
$$=(3A+2B)x-(4A-3B)$$

Equating coefficients gives $A=2$ and $B=5$

$$\therefore\ \int\frac{16x+7}{6x^2+x-12}\,\mathrm{d}x=\int\left(\frac{2}{2x+3}+\frac{5}{3x-4}\right)\mathrm{d}x$$
$$=\ln(2x+3)+\frac{5}{3}\ln(3x-4)+C$$

At this point let us pause and summarize the main facts dealing with integration by partial fractions

23

Revision summary

Integration by partial fractions

Algebraic fractions can often be expressed in terms of partial fractions. This renders integration of such algebraic fractions possible, the integration of each partial fraction

$$\int\frac{A}{ax+b}\,\mathrm{d}x=A\frac{\ln(ax+b)}{a}+C$$

24

Revision exercise

1 Integrate by partial fractions each of the following integrals:

(a) $\displaystyle\int\frac{5x+2}{3x^2+x-4}\,\mathrm{d}x$

(b) $\displaystyle\int\frac{x+1}{4x^2-1}\,\mathrm{d}x$ (c) $\displaystyle\int\frac{3x}{1+x-2x^2}\,\mathrm{d}x$

25

1 (a) $\displaystyle\frac{5x+2}{3x^2+x-4}=\frac{5x+2}{(3x+4)(x-1)}=\frac{A}{3x+4}+\frac{B}{x-1}$ therefore

$$5x+2=A(x-1)+B(3x+4)$$
$$=(A+3B)x+(-A+4B)\text{ so that}$$

$A+3B=5$

$-A+4B=2$ therefore adding we find that $7B=7$ so $B=1$ and $A=2$.

Therefore:

$$\int\frac{5x+2}{3x^2+x-4}\,\mathrm{d}x=\int\frac{2}{3x+4}\,\mathrm{d}x+\int\frac{1}{x-1}\,\mathrm{d}x$$
$$=\frac{2}{3}\ln(3x+4)+\ln(x-1)+C$$

▶

(b) $\dfrac{x+1}{4x^2-1} = \dfrac{x+1}{(2x+1)(2x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{2x-1}$ therefore

$$\begin{aligned} x+1 &= A(2x-1) + B(2x+1) \\ &= (2A+2B)x + (-A+B) \text{ so that} \end{aligned}$$

$$\begin{array}{ll} 2A+2B=1 & 2A+2B=1 \\ -A+B=1 & -2A+2B=2 \end{array}$$

therefore adding we find that $7B = 7$ so $B = 1$ and $A = 2$.

Therefore:

$$\begin{aligned} \int \frac{x+1}{4x^2-1}\,\mathrm{d}x &= -\int \frac{1/4}{2x+1}\,\mathrm{d}x + \int \frac{3/4}{2x-1}\,\mathrm{d}x \\ &= -\frac{1}{8}\ln(2x+1) + \frac{3}{8}\ln(2x-1) + C \end{aligned}$$

(c) $\dfrac{3x}{1+x-2x^2} = \dfrac{3x}{(1-x)(1+2x)} = \dfrac{A}{1-x} + \dfrac{B}{1+2x}$ therefore

$$\begin{aligned} 3x &= A(1+2x) + B(1-x) \\ &= (2A-B)x + (A+B) \text{ so that} \end{aligned}$$

$$\begin{aligned} 2A - B &= 3 \\ A + B &= 0 \end{aligned}$$

therefore, adding we find that $3A = 3$ so $A = 1$ and $B = -1$.

Therefore:

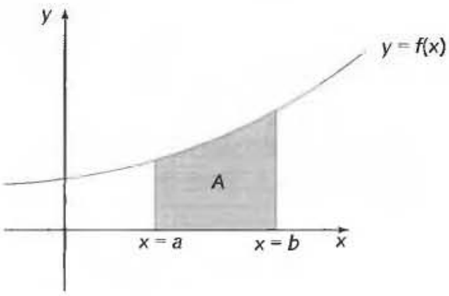
$$\begin{aligned} \int \frac{3x}{1+x-2x^2}\,\mathrm{d}x &= \int \frac{1}{1-x}\,\mathrm{d}x - \int \frac{1}{1+2x}\,\mathrm{d}x \\ &= -\ln(1-x) - \frac{1}{2}\ln(1+2x) + C \end{aligned}$$

Now on to something different

Areas under curves

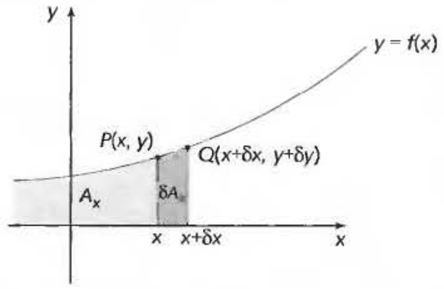
26

Consider the area A of the figure bounded by the curve $y = f(x)$, the x-axis and the two vertical lines through $x = a$ and $x = b$ (where $b > a$).



▶

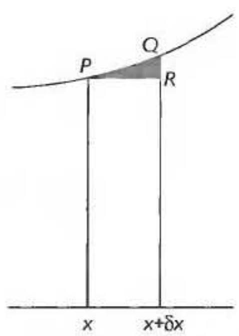
To evaluate the area A you need to consider the total area between the same curve and the x-axis from the left up to some arbitrary point P on the curve with coordinates (x, y) which we shall denote by A_x.



Area δA_x is the area enclosed by the strip under the arc PQ where Q has the coordinates $(x + \delta x,\ y + \delta y)$. If the strip is approximated by a rectangle of height y and width δx then $\delta A_x \approx y.\delta x$. This means that:

$$\frac{\delta A_x}{\delta x} \approx y$$

The error in this approximation is given by the area of PQR in the figure to the right, where the strip has been magnified.



If the width of the strip is reduced then the error is accordingly reduced. Also, if $\delta x \to 0$ then $\delta A_x \to 0$ and:

$$\frac{\delta A_x}{\delta x} \to \frac{\mathrm{d}A_x}{\mathrm{d}x} \text{ so that, in the limit, } \frac{\mathrm{d}A_x}{\mathrm{d}x} = y$$

Consequently, because integration is the reverse process of differentiation it is seen that:

$$A_x = \int y\,\mathrm{d}x$$

The total area between the curve and the x-axis up to the point P is given by the indefinite integral.

If $x = b$ then $A_b = \int_{(x=b)} y\,\mathrm{d}x$ (the value of the integral and hence the area up to b) and if $x = a$ then $A_a = \int_{(x=a)} y\,\mathrm{d}x$ (the value of the integral and hence the area up to a). Because $b > a$, the difference in these two areas $A_b - A_a$ gives the required area A. That is:

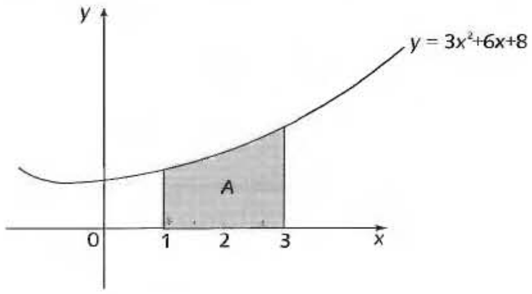
$$A = \int_{(x=b)} y\,\mathrm{d}x - \int_{(x=a)} y\,\mathrm{d}x \text{ which is written } A = \int_a^b y\,\mathrm{d}x$$

▶

The numbers a and b are called the *limits* of the integral where the right-hand limit is at the top of the integral sign and the left-hand limit is at the bottom. Such an integral with limits is called a *definite integral*. Notice that in the subtraction process when the integral is evaluated, the constant of integration disappears leaving the numerical value of the area.

Example 1

To determine the area bounded by the curve $y = 3x^2 + 6x + 8$, the x-axis and the ordinates $x = 1$ and $x = 3$.



$$A = \int_1^3 y\,dx = \int_1^3 (3x^2 + 6x + 8)\,dx = \left[x^3 + 3x^2 + 8x\right]_1^3$$

Note that we enclose the expression in square brackets with the limits attached.

Now calculate the values at the upper and lower limits and subtract the second from the first which gives $A = \ldots\ldots\ldots$

27

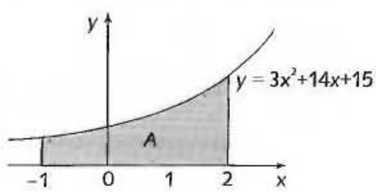
$\boxed{66 \text{ unit}^2}$

Because

$$A = \left[x^3 + 3x^2 + 8x\right]_1^3$$
$$= \{27 + 27 + 24\} - \{1 + 3 + 8\} = 78 - 12 = 66 \text{ unit}^2$$

Example 2

Find the area bounded by the curve $y = 3x^2 + 14x + 15$, the x-axis and ordinates at $x = -1$ and $x = 2$.

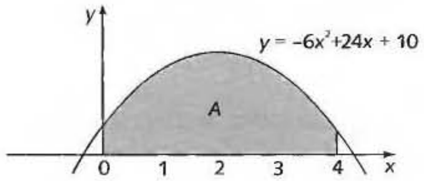


$$A = \int_{-1}^{2} y\,dx = \int_{-1}^{2} (3x^2 + 14x + 15)\,dx$$
$$= \left[x^3 + 7x^2 + 15x\right]_{-1}^{2}$$
$$\therefore\ A = \{8 + 28 + 30\} - \{-1 + 7 - 15\}$$
$$= 66 - (-9) = 75 \text{ unit}^2$$

▶

Example 3

Calculate the area bounded by the curve $y = -6x^2 + 24x + 10$, the x-axis and the ordinates $x = 0$ and $x = 4$.



$$A = \int_0^4 y\,dx = \int_0^4 (-6x^2 + 24x + 10)\,dx$$
$$= \ldots\ldots\ldots\ldots$$

28

$\boxed{104 \text{ unit}^2}$

Because

$$A = \left[-2x^3 + 12x^2 + 10x\right]_0^4 = 104 - 0 \quad \therefore \; A = 104 \text{ unit}^2$$

And now:

Example 4

Determine the area under the curve $y = e^x$ between $x = -2$ and $x = 3$.
Do this by the same method. The powers of e are available from most calculators or from tables.

$$A = \ldots\ldots\ldots\ldots$$

29

$\boxed{19{\cdot}95 \text{ unit}^2}$

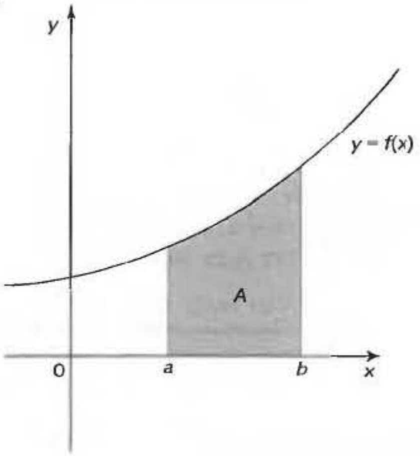
As a check:

$$A = \int_{-2}^{3} y\,dx = \int_{-2}^{3} e^x\,dx = [e^x]_{-2}^{3} = \{e^3\} - \{e^{-2}\}$$
$$e^3 = 20{\cdot}09 \text{ and } e^{-2} = 0{\cdot}14$$
$$\therefore \; A = 20{\cdot}09 - 0{\cdot}14 = 19{\cdot}95 \text{ unit}^2$$

At this point let us pause and summarize the main facts dealing with areas beneath curves

30

Revision summary



Area A, bounded by the curve $y = f(x)$, the x-axis and ordinates $x = a$ and $x = b$, is given by:

$$A = \int_a^b y\,dx$$

31

Revision exercise

1 Find the area bounded by $y = 5 + 4x - x^2$, the x-axis and the ordinates $x = 1$ and $x = 4$.

2 Calculate the area under the curve $y = 2x^2 + 4x + 3$, between $x = 2$ and $x = 5$.

3 Determine the area bounded by $y = x^2 - 2x + 3$, the x-axis and ordinates $x = -1$ and $x = 3$.

Finish all three and then check with the next frame

32

1. 24 unit^2, **2**. 129 unit^2, **3**. $13\frac{1}{3}$ unit^2

Here is the working:

1 $$A = \int_1^4 y\,dx = \int_1^4 (5 + 4x - x^2)\,dx = \left[5x + 2x^2 - \frac{x^3}{3}\right]_1^4$$
$$= \left\{20 + 32 - \frac{64}{3}\right\} - \left\{5 + 2 - \frac{1}{3}\right\}$$
$$= 30\frac{2}{3} - 6\frac{2}{3} = 24 \text{ unit}^2$$

2 $$A = \int_2^5 y\,dx = \int_2^5 (2x^2 + 4x + 3)\,dx = \left[2\frac{x^3}{3} + 2x^2 + 3x\right]_2^5$$
$$= \left\{\frac{250}{3} + 50 + 15\right\} - \left\{\frac{16}{3} + 8 + 6\right\} = 148\frac{1}{3} - 19\frac{1}{3} = 129 \text{ unit}^2$$

▶

3 $A = \int_{-1}^{3} y\,dx = \int_{-1}^{3} (x^2 - 2x + 3)\,dx = \left[\frac{x^3}{3} - x^2 + 3x\right]_{-1}^{3}$

$= \left\{9 - 9 + 9\right\} - \left\{-\frac{1}{3} - 1 - 3\right\}$

$= 9 - \left\{-4\frac{1}{3}\right\} = 13\frac{1}{3}$ unit^2.

Notice that in all these definite integrals, we omit the constant of integration because we know it will disappear at the subtraction stage.

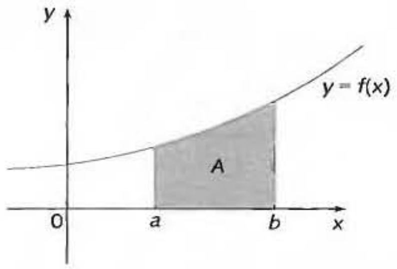
In an indefinite integral, however, it must always be included.

Now move on to the next section

Integration as a summation

33

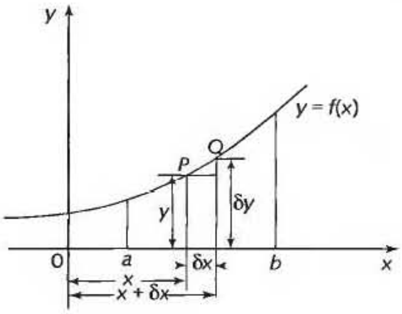
We have identified the value of a definite integral as the area beneath a curve. However, some definite integrals have a negative value so how can we link this to an area because all areas are *positive quantities*? Before we can make this link we must consider the determination of area in a slightly different manner.



We have already seen that the area A under a curve $y = f(x)$ between $x = a$ and $x = b$ is given by the definite integral:

$$A = \int_a^b y\,dx$$

Let's look at this area a little more closely:

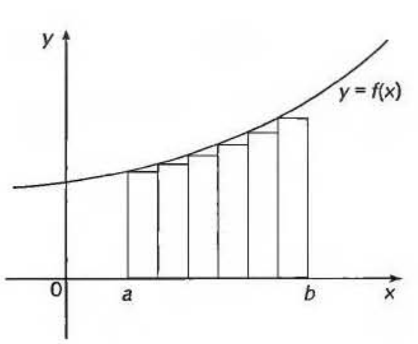


Let P be the point (x, y) on the curve and Q a similar point $(x + \delta x,\ y + \delta y)$. The approximate area δA of the strip under the arc PQ is given by

$\delta A \approx y.\delta x$

As we have indicated earlier, the error in the approximation is the area above the rectangle.

If we divide the whole figure between $x = a$ and $x = b$ into a number of such strips, the total area is approximately the sum of the areas of all rectangles $y.\delta x$.
i.e. $A \approx$ the sum of all rectangles $y.\delta x$ between $x = a$ and $x = b$. This can be written $A \approx \sum_{x=a}^{x=b} y.\delta x$ where the symbol Σ represents '*the sum of all terms of the form ...*'



If we make the strips narrower, there will be more of them to cover the whole figure, but the total error in the approximation diminishes.

If we continue the process, we arrive at an infinite number of minutely narrow rectangles, each with an area too small to exist alone.

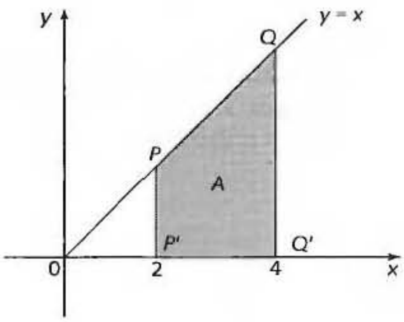
Then, in the limit as $\delta x \to 0$, $A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y.\delta x$

But we already know that $A = \int_a^b y\,dx \quad \therefore \quad \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y.\delta x = A = \int_a^b y\,dx$

Let us consider an example

34

To illustrate this, consider the area A beneath the straight line $y = x$, above the x-axis and between the values $x = 2$ and $x = 4$ as shown in the figure below.

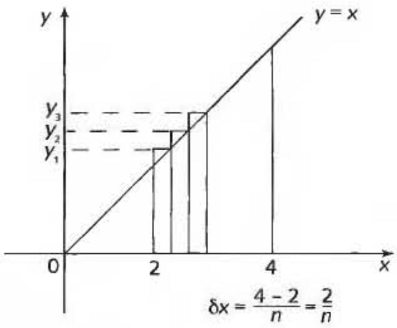


The area of a triangle is $\frac{1}{2} \times$ base $\times$ height and area A is equal to the difference in the areas of the two triangles OQQ' and OPP' so that:

$$A = \frac{1}{2} \times 4 \times 4 - \frac{1}{2} \times 2 \times 2$$
$$= 8 - 2 = 6 \text{ units}^2$$

This value will now be confirmed by dividing the area into equal strips, summing their areas and then taking the limit as the width of the strips goes to zero.

▶



In the figure above, the area has been subdivided into n strips each of width δx where:

$$\delta x = \frac{4-2}{n} = \frac{2}{n}$$

The strip heights are given as:

$$y_1 = y(2+\delta x) = y\left(2+\frac{2}{n}\right) = 2+\frac{2}{n}$$

$$y_2 = y(2+2\delta x) = y\left(2+2\times\frac{2}{n}\right) = 2+2\times\frac{2}{n}$$

$$y_3 = y(2+3\delta x) = y\left(2+3\times\frac{2}{n}\right) = 2+3\times\frac{2}{n}$$

......

$$y_r = y(2+r\delta x) = y\left(2+r\times\frac{2}{n}\right) = 2+r\times\frac{2}{n}$$

......

$$y_n = y(2+n\delta x) = y\left(2+n\times\frac{2}{n}\right) = 2+n\times\frac{2}{n} = 4$$

Consequently:

$$\begin{aligned}
\underset{\delta x\to 0}{Lim}\sum_{x=2}^{x=4} y.\delta x &= \underset{n\to\infty}{Lim}\sum_{r=1}^{n}\left(2+\frac{2r}{n}\right).\frac{2}{n} \\
&= \underset{n\to\infty}{Lim}\sum_{r=1}^{n}\left(\frac{4}{n}+\frac{4r}{n^2}\right) \\
&= \underset{n\to\infty}{Lim}\sum_{r=1}^{n}\frac{4}{n} + \underset{n\to\infty}{Lim}\sum_{r=1}^{n}\frac{4r}{n^2} \\
&= \underset{n\to\infty}{Lim}\frac{4}{n}\sum_{r=1}^{n}1 + \underset{n\to\infty}{Lim}\frac{4}{n^2}\sum_{r=1}^{n}r \\
&= \underset{n\to\infty}{Lim}\left(\frac{4}{n}\times n\right) + \underset{n\to\infty}{Lim}\left(\frac{4}{n^2}\times\frac{n(n+1)}{2}\right) \\
&= 4+2 = 6 \text{ units}^2
\end{aligned}$$

Notice that as the width of each strip decreases, that is $\delta x \to 0$, so their number n increases, $n \to \infty$

▶

Notice the use that has been made of $\sum_{r=1}^{n} 1 = n$ and $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ from Programme F.9.

Now, without following the above procedure, but by integrating normally, find the area bounded by the curve $y = x^2 - 9$ and the x-axis and between $x = -3$ and $x = 3$.

35

$\boxed{A = -36 \text{ unit}^2}$

Because

$$A = \int_{-3}^{3} (x^2 - 9)\,dx$$
$$= \left[\frac{x^3}{3} - 9x\right]_{-3}^{3}$$
$$= (9 - 27) - (-9 + 27)$$
$$= -36$$

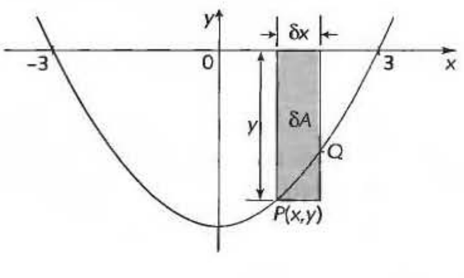
Simple enough, but what is meant by a negative area?

Have you any suggestions?

36

The area that lies beneath the x-axis

If we sketch the figure we get:



As before, the area of the strip

$$\delta A \approx y\,\delta x$$

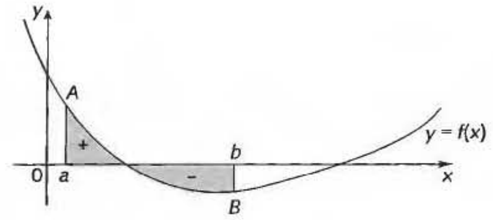
and the total area

$$A \approx \sum_{x=-3}^{x=3} y\,\delta x$$

But, for all such strips across the figure, y is negative and δx a positive value.

Therefore, in this case, $y.\delta x$ is negative and the sum of all such quantities gives a negative total, even when $\delta x \to 0$.

So $\int_{-3}^{3} y\,dx$ has a negative value, namely *minus the value of the enclosed area.*

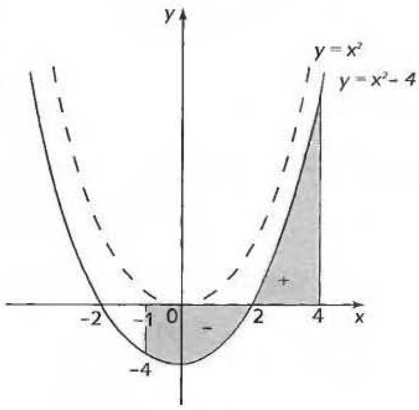


The trouble comes when part of the area to be calculated lies above the x-axis and part below. In that case, integration from $x = a$ to $x = b$ gives the algebraic sum of the two parts.

It is always wise, therefore, to sketch the figure of the problem before carrying out the integration and, if necessary, to calculate the positive and negative parts separately and to add the numerical values of each part to obtain the physical area between the limits stated.

As an example, we will determine the physical area of the figure bounded by the curve $y = x^2 - 4$, the x-axis and ordinates at $x = -1$ and $x = 4$.

The curve $y = x^2 - 4$ is, of course, the parabola $y = x^2$ lowered 4 units on the y-axis. It crosses the x-axis when $y = 0$, i.e. $x^2 - 4 = 0$, $x^2 = 4$ $\therefore$ $x = \pm 2$.



The figure of our problem extends from $x = -1$ to $x = 4$, which includes an area *beneath* the x-axis between $x = -1$ and $x = 2$ and an area *above* the x-axis between $x = 2$ and $x = 4$. Therefore, we calculate the physical area enclosed in two parts and add the results.

So let $I_1 = \int_{-1}^{2} y\,dx$ and $I_2 = \int_{2}^{4} y\,dx$. Then:

$$I_1 = \int_{-1}^{2} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-1}^{2} = \left\{\frac{8}{3} - 8\right\} - \left\{-\frac{1}{3} + 4\right\}$$

$$= -9 \text{ so } A_1 = 9 \text{ units}^2$$

and

$$I_2 = \int_{2}^{4} y\,dx = \left[\frac{x^3}{3} - 4x\right]_{2}^{4} = \left\{\frac{64}{3} - 16\right\} - \left\{\frac{8}{3} - 8\right\} = 10\frac{2}{3} \text{ units}^2$$

▶

Consequently, $A_2 = I_2 = 10\frac{2}{3}$ units^2 and so:

$$A = A_1 + A_2 = 19\frac{2}{3}\text{ units}^2$$

Had we integrated right through in one calculation, we should have obtained:

$$I = \int_{-1}^{4} y\,dx = \int_{-1}^{4}(x^2-4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-1}^{4} = \left\{21\frac{1}{3} - 16\right\} - \left\{-\frac{1}{3} + 4\right\}$$
$$= 1\frac{2}{3}\text{ units}^2$$

which, though it does give the correct value of the definite integral, does not give the correct value of the total area enclosed.

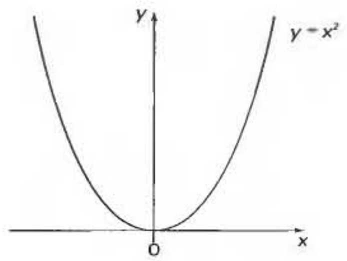
On to the next frame

The area between a curve and an intersecting line

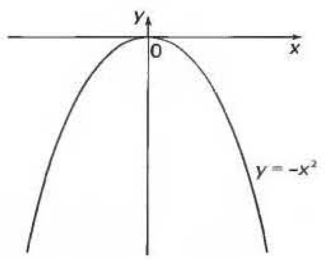
37

To find the area enclosed by the curve $y = 25 - x^2$ and the straight line $y = x + 13$.

First we must develop the figure. We know that $y = x^2$ is a normal parabola:

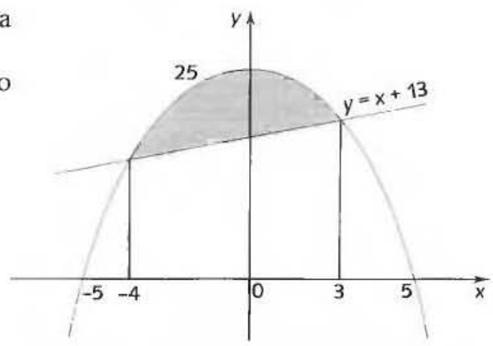


Then $y = -x^2$ is an inverted parabola:



$y = 25 - x^2$ is the inverted parabola raised 25 units on the y-scale.

Also when $y = 0$, $x^2 = 25$, so $x = \pm 5$.



$y = x + 13$ is a straight line crossing $y = 25 - x^2$ when

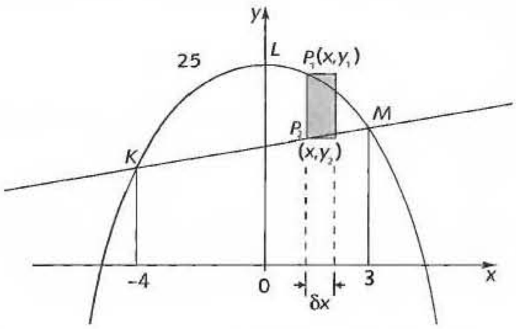
$x + 13 = 25 - x^2$, i.e. $x^2 + x - 12 = 0$

$\therefore\ (x - 3)(x + 4) = 0 \quad \therefore\ x = -4$ or 3

So the area A we need is the part shaded.

On to the next frame

38



Let $P_1(x, y_1)$ be a point on $y_1 = 25 - x^2$ and $P_2(x, y_2)$ the corresponding point on $y_2 = x + 13$.

Then area of strip

$P_1P_2 \approx (y_1 - y_2).\delta x$

Then the area of the figure

$$KLM \approx \sum_{x=-4}^{x=3} (y_1 - y_2).\delta x$$

$$\text{If } \delta x \to 0,\ A = \int_{-4}^{3} (y_1 - y_2)\,\mathrm{d}x$$

$$\therefore\ A = \int_{-4}^{3} (25 - x^2 - x - 13)\,\mathrm{d}x = \int_{-4}^{3} (12 - x - x^2)\,\mathrm{d}x$$

which you can now finish off, to give $A = \ldots\ldots\ldots$

39

$$\boxed{57{\cdot}2 \text{ unit}^2}$$

$$A = \int_{-4}^{2} (12 - x - x^2)\,\mathrm{d}x = \left[12x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-4}^{3}$$

$$= \left\{36 - \frac{9}{2} - 9\right\} - \left\{-48 - 8 + \frac{64}{3}\right\}$$

$$= 22{\cdot}5 + 34{\cdot}67 = 57{\cdot}17 \ \therefore\ A = 57{\cdot}2 \text{ unit}^2$$

At this point let us pause and summarize the main facts dealing with areas and the definite integral

Revision summary

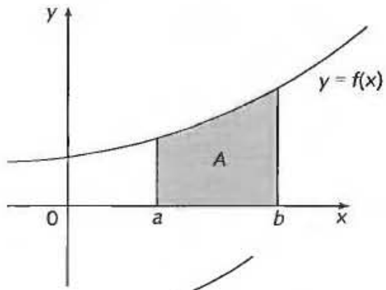
40

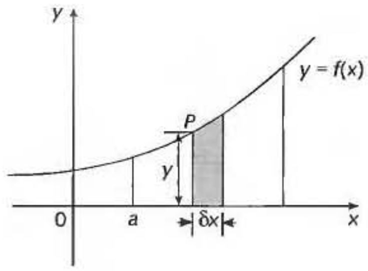
1 *Definite integral*

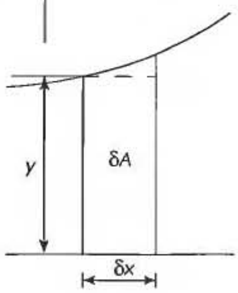
An integral with limits (e.g. $x = a$ and $x = b$) is called a definite integral. The constant of integration C in such cases will always disappear at the subtraction stage, since

$$\int_a^b y\,dx = \int_{x=b} y\,dx - \int_{x=a} y\,dx$$

2 *Integration as a summation*







$\delta A \approx y.\delta x \qquad \therefore\ A \approx \sum_{x=a}^{x=b} y.\delta x$

If $\delta x \to 0 \qquad A = \int_a^b y.dx$

Revision exercise

41

1 Evaluate each of the following definite integrals:

(a) $\displaystyle\int_2^4 3x^5\,dx$ (b) $\displaystyle\int_0^{\pi/2} (\sin x - \cos x)\,dx$

(c) $\displaystyle\int_0^1 e^{2x}\,dx$ (d) $\displaystyle\int_{-1}^0 x^3\,dx$

2 Find the area enclosed between the x-axis and the curves:

(a) $y = x^3 + 2x^2 + x + 1$ between $x = -1$ and $x = 2$

(b) $y = x^2 - 25$ for $-5 \le x \le 5$.

3 Find the area enclosed between the curve $y = x^3$ and the straight line $y = x$.

42

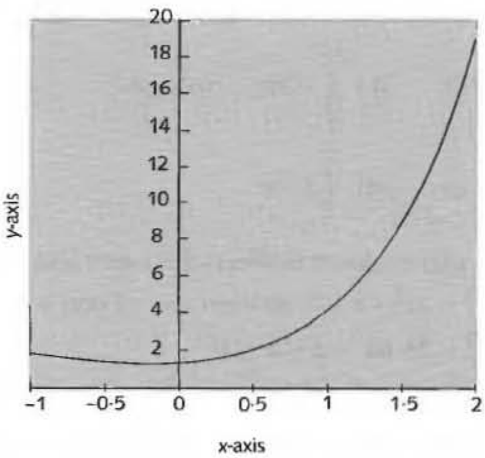
1 (a) $$\int_2^4 3x^5\,dx = \left[3\frac{x^6}{6}\right]_2^4$$
$$= \left\{3\frac{4^6}{6}\right\} - \left\{3\frac{2^6}{6}\right\}$$
$$= 2048 - 32 = 2016$$

(b) $$\int_0^{\pi/2} (\sin x - \cos x)\,dx = [-\cos x - \sin x]_0^{\pi/2}$$
$$= \left\{-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right\} - \{-\cos 0 - \sin 0\}$$
$$= \{-0 - 1\} - \{-1 - 0\}$$
$$= 0$$

(c) $$\int_0^1 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^1$$
$$= \left\{\frac{e^2}{2}\right\} - \left\{\frac{e^0}{2}\right\}$$
$$= \frac{e^2}{2} - \frac{1}{2}$$
$$= \frac{e^2 - 1}{2}$$

(d) $$\int_{-1}^0 x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^0$$
$$= \{0^4/4\} - \{(-1)^4/4\} = -1/4$$

2 (a) The graph of $y = x^3 + 2x^2 + x + 1$ between $x = -1$ and $x = 2$ lies entirely above the x-axis



so that the area enclosed between the curve, the x-axis and between $x = -1$ and $x = 2$ is given by:

$$\int_{-1}^{2} (x^3 + 2x^2 + x + 1)\,dx$$

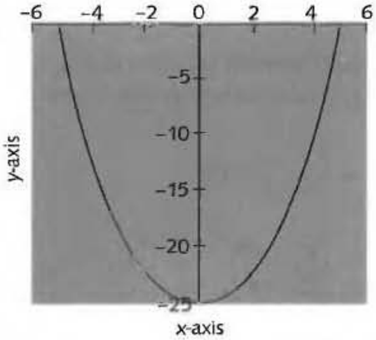
$$= \left[\frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{2}$$

$$= \left\{\frac{2^4}{4} + 2.\frac{2^3}{3} + \frac{2^2}{2} + 2\right\} - \left\{\frac{(-1)^4}{4} - 2.\frac{(-1)^3}{3} + \frac{(-1)^2}{2} - 1\right\}$$

$$= \left\{4 + \frac{16}{3} + 2 + 2\right\} - \left\{\frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1\right\}$$

$$= 13\tfrac{1}{3} + \tfrac{11}{12} = 14\tfrac{1}{4}$$

(b) The graph of $y = x^2 - 25$ for $-5 \le x \le 5$ lies entirely below the x-axis



so that the area enclosed between the curve and the x-axis for $-5 \le x \le 5$ is given by $-I$ where:

$$I = \int_{-5}^{5} (x^2 - 25)\,dx$$

$$= \left[\frac{x^3}{3} - 25x\right]_{-5}^{5}$$

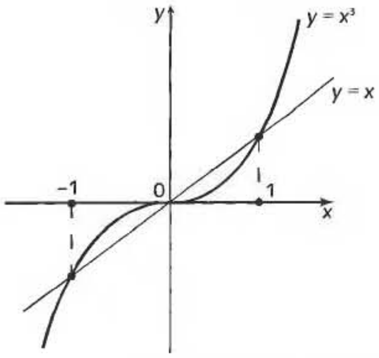
$$= \left\{\frac{5^3}{3} - 25 \times 5\right\} - \left\{\frac{(-5)^3}{3} - 25 \times (-5)\right\}$$

$$= \left\{\frac{125}{3} - 125\right\} - \left\{-\frac{125}{3} + 125\right\}$$

$$= -\frac{500}{3}$$

So the enclosed area is $-I = \dfrac{500}{3}$ units^2

3 The curve $y = x^3$ and the line $y = x$ intersect when $x^3 = x$, that is when $x^3 - x = 0$. Factorizing we see that this means $x(x^2 - 1) = 0$ which is satisfied when $x = 0$, $x = 1$ and $x = -1$.



From the figure we can see that the area enclosed between the curve $y = x^3$ and the straight line $y = x$ is in two parts, one above the x-axis and the other below. From the figure it is easily seen that both parts are the same, so we only need find the area between $x = 0$ and $x = 1$ and then double it to find the total area enclosed.

The area enclosed between $x = 0$ and $x = 1$ is equal to the area beneath the line $y = x$ minus the area beneath the curve $y = x^3$. That is:

$$\int_0^1 x\,dx - \int_0^1 x^3\,dx = \int_0^1 (x - x^3)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \left\{\frac{1^2}{2} - \frac{1^4}{4}\right\} - \{0\}$$

$$= \frac{1}{4}$$

The total area enclosed is then twice this, namely $\frac{1}{2}$ unit^2.

43

You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise.** *Work through these at your own pace, there is no need to hurry.* A set of **Further problems** provides additional valuable practice.

☑ Can You?

Checklist F.11

44

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Appreciate that integration is the reverse process of differentiation? — 1

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognize the need for a constant of integration? — 1 to 2

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate indefinite integrals of standard forms? — 2

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate indefinite integrals of polynomials? — 6 to 11

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate indefinite integrals of 'functions of a linear function of x'? — 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Integrate by partial fractions? — 16 to 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Appreciate that a definite integral is a measure of an area under a curve? — 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate definite integrals of standard forms? — 27 to 29

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the definite integral to find areas between a curve and the horizontal axis? — 33 to 36

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the definite integral to find areas between a curve and a given straight line? — 37 to 39

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.11

45

1 Determine the following integrals: Frames

(a) $\int x^4\,dx$ (f) $\int 5^x\,dx$

(b) $\int 3\sin x\,dx$ (g) $\int x^{-4}\,dx$

(c) $\int 4.e^x\,dx$ (h) $\int \frac{4}{x}\,dx$

(d) $\int 6\,dx$ (i) $\int 3\cos x\,dx$

(e) $\int 3x^{\frac{1}{2}}\,dx$ (j) $\int 2\sec^2 x\,dx$ 1 to 5

2 (a) Determine $\int(8x^3+6x^2-5x+4)\,dx$.

(b) If $I=\int(4x^3-3x^2+6x-2)\,dx$, determine the value of I when $x=4$, given that when $x=2$, $I=20$. 6 to 11

3 Determine:

(a) $\int 2\sin(3x+1)\,dx$ (e) $\int 4^{2x-3}\,dx$

(b) $\int \sqrt{5-2x}\,dx$ (f) $\int 6\cos(1-2x)\,dx$

(c) $\int 6.e^{1-3x}\,dx$ (g) $\int \frac{5}{3x-2}\,dx$

(d) $\int (4x+1)^3\,dx$ (h) $\int 3\sec^2(1+4x)\,dx$ 12 to 15

4 Express $\dfrac{2x^2+14x+10}{2x^2+9x+4}$ in partial fractions and hence determine

$$\int\frac{2x^2+14x+10}{2x^2+9x+4}\,dx$$ 16 to 25

5 (a) Evaluate the area under the curve $y=5.e^x$ between $x=0$ and $x=3$.

(b) Show that the straight line $y=-2x+28$ crosses the curve $y=36-x^2$ at $x=-2$ and $x=4$. Hence, determine the area enclosed by the curve $y=36-x^2$ and the line $y=-2x+28$ 26 to 42

Further problems F.11

46

1 Determine the following:

(a) $\int (5-6x)^2\,dx$ (e) $\int \sqrt{5-3x}\,dx$ (h) $\int 4\cos(1-2x)\,dx$

(b) $\int 4\sin(3x+2)\,dx$ (f) $\int 6.e^{3x+1}\,dx$ (i) $\int 3\sec^2(4-3x)\,dx$

(c) $\int \frac{5}{2x+3}\,dx$ (g) $\int (4-3x)^{-2}\,dx$ (j) $\int 8.e^{3-4x}\,dx$

(d) $\int 3^{2x-1}\,dx$

2 Determine $\int \left(6.e^{3x-5} + \frac{4}{3x-2} - 5^{2x+1}\right) dx$.

3 If $I = \int (8x^3 + 3x^2 - 6x + 7)\,dx$, determine the value of I when $x = -3$, given that when $x = 2$, $I = 50$.

4 Determine the following using partial fractions:

(a) $\int \frac{6x+1}{4x^2+4x-3}\,dx$ (e) $\int \frac{6x^2-2x-2}{6x^2-7x+2}\,dx$

(b) $\int \frac{x+11}{x^2-3x-4}\,dx$ (f) $\int \frac{4x^2-9x-19}{2x^2-7x-4}\,dx$

(c) $\int \frac{3x-17}{12x^2-19x+4}\,dx$ (g) $\int \frac{6x^2+2x+1}{2x^2+x-6}\,dx$

(d) $\int \frac{20x+2}{8x^2-14x-15}\,dx$ (h) $\int \frac{4x^2-2x-11}{2x^2-7x-4}\,dx$

5 Determine the area bounded by the curve $y = f(x)$, the x-axis and the stated ordinates in the following cases:

(a) $y = x^2 - 3x + 4$, $x = 0$ and $x = 4$

(b) $y = 3x^2 + 5$, $x = -2$ and $x = 3$

(c) $y = 5 + 6x - 3x^2$, $x = 0$ and $x = 2$

(d) $y = -3x^2 + 12x + 10$, $x = 1$ and $x = 4$

(e) $y = x^3 + 10$, $x = -1$ and $x = 2$

6 In each of the following, determine the area enclosed by the given boundaries:

(a) $y = 10 - x^2$ and $y = x^2 + 2$

(b) $y = x^2 - 4x + 20$, $y = 3x$, $x = 0$ and $x = 4$

(c) $y = 4e^{2x}$, $y = 4e^{-x}$, $x = 1$ and $x = 3$

(d) $y = 5e^x$, $y = x^3$, $x = 1$ and $x = 4$

(e) $y = 20 + 2x - x^2$, $y = \frac{e^x}{2}$, $x = 1$ and $x = 3$

Functions

Frames 1 to 82

Learning outcomes

When you have completed this Programme you will be able to:

- Identify a function as a rule and recognize rules that are not functions
- Determine the domain and range of a function
- Construct the inverse of a function and draw its graph
- Construct compositions of functions and de-construct them into their component functions
- Develop the trigonometric functions from the trigonometric ratios
- Find the period, amplitude and phase of a periodic function
- Distinguish between the inverse of a trigonometric function and the inverse trigonometric function
- Solve trigonometric equations using the inverse trigonometric functions and trigonometric identities
- Recognize that the exponential function and the natural logarithmic function are mutual inverses and solve indicial and logarithmic equations
- Find the even and odd parts of a function when they exist
- Construct the hyperbolic functions from the odd and even parts of the exponential function

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.12

Frames

1 Which of the following equations expresses a rule that is a function?

(a) $y = 1 - x^2$ (b) $y = -\sqrt{x^4}$ (c) $y = x^{\frac{1}{6}}$ — 1 to 4

2 Given the two functions f and g expressed by:

$$f(x) = \frac{1}{4-x} \text{ for } 0 \le x < 4 \text{ and } g(x) = x - 3 \text{ for } 0 < x \le 5$$

find the domain and range of functions h and k where:

(a) $h(x) = f(x) + 3g(x)$ (b) $k(x) = \dfrac{f(x)}{2g(x)}$ — 4 to 7

3 Use your spreadsheet to draw each of the following and their inverses. Is the inverse a function?

(a) $y = x^5$ (b) $y = -3x^2$ (c) $y = \sqrt{1 - x^2}$ — 8 to 16

4 Given that $a(x) = -2x$, $b(x) = x^3$, $c(x) = x - 1$ and $d(x) = \sqrt{x}$ find:

(a) $f(x) = a[b(c[d(x)])]$ (b) $f(x) = a(a[d(x)])$

(c) $f(x) = b[c(b[c(x)])]$ — 20 to 24

5 Given that $f(x) = (3x+4)^2 + 4$, decompose f into its component functions and find its inverse. Is the inverse a function? — 25 to 28

6 Use a calculator to find the value of each of the following:

(a) $\cos(-5\pi/3)$ (b) $\sec(115^\circ)$ (c) $\tan(-13^\circ)$ — 29 to 36

7 Find the period, amplitude and phase of each of the following:

(a) $f(\theta) = 2\cos 6\theta$ (b) $f(\theta) = -2\tan(2\theta - 2)$

(c) $f(\theta) = \cos(\pi - \theta)$ — 37 to 48

8 A function is defined by the following prescription:

$$f(x) = 4 - x^2, \quad 0 \le x < 2, \quad f(x \pm 2) = f(x)$$

Plot a graph of this function for $-6 \le x \le 6$ and find:

(a) the period

(b) the amplitude

(c) the phase of $f(x) + 2$ with respect to $f(x)$ — 37 to 48

9	Solve the following trigonometric equations: (a) $\cos(x+\pi/3)+\cos(x-\pi/4)=0$ (b) $\sin^2 x - 4\sin x\cos x + 3\cos^2 x = 0$ (c) $9\cos x - 4\sin x = 2\sqrt{2}$ for $0 \le x \le \pi$	**Frames** 49 to 56
10	Find the value of x corresponding to each of the following: (a) $2^{-2x}=1$ (b) $\exp(-4x)=e^3$ (c) $e^{\frac{1}{x}}=2{\cdot}34$ (d) $\log_2 x = 0{\cdot}4$ (e) $\ln x = 2$	60 to 67
11	Solve for x: (a) $2^{x-5}5^{x+1}=62{\cdot}5$ (b) $e^{2x}-e^{x+3}-e^{x+1}+e^4=0$ (c) $\log(x^2)=\log\left(\frac{1}{x}\right)-\log 5$ (d) $\log_4(3-x^2)=-6{\cdot}2$	60 to 67
12	Find the even and odd parts of $f(x)=x(x^2+x+1)$.	68 to 75

Processing numbers

1

The equation that states that y *is equal to some expression in* x, written as:

$y = f(x)$

has been described with the words 'y *is a function of* x'. Despite being widely used and commonly accepted, this description is not strictly correct as will be seen in Frame 3. Put simply, for all the functions that you have considered so far, both x and y are *numbers*.

Take out your calculator and enter the number:

5 this is x, the *input* number

Now press the x^2 key and the display changes to:

25 this is y, the *output* number where $y = x^2$

The *function* is a *rule* embodied in a *set of instructions* within the calculator that changed the 5 to 25, activated by you pressing the x^2 key. A diagram can be constructed to represent this:

The box labelled f represents the function. The notation ^2 inside the box means *raising to the power 2* and describes the rule – what the set of instructions will do when activated. The diagram tells you that the input number x is *processed* by the function f to produce the output number $y = f(x)$. So that $y = f(x)$ is the *result* of function f acting on x.

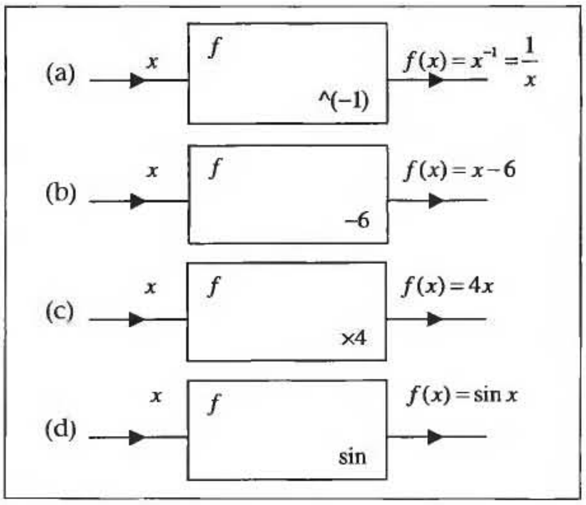
So, use diagrams and describe the functions appropriate to each of the following equations:

(a) $y = \dfrac{1}{x}$ (b) $y = x - 6$

(c) $y = 4x$ (d) $y = \sin x$

Just follow the reasoning above, the answers are in the next frame

2



(a) Function f produces the reciprocal of the input
(b) Function f subtracts 6 from the input
(c) Function f multiplies the input by 4
(d) Function f produces the sine of the input

Let's now expand this idea

Functions are rules but not all rules are functions

3

A function of a variable x is a *rule* that describes how a value of the variable x is manipulated to generate a value of the variable y. The rule is often expressed in the form of an equation $y = f(x)$ with the proviso that for any input x there is a unique value for y. Different outputs are associated with different inputs – the function is said to be *single valued* because for a given input there is only one output. For example, the equation:

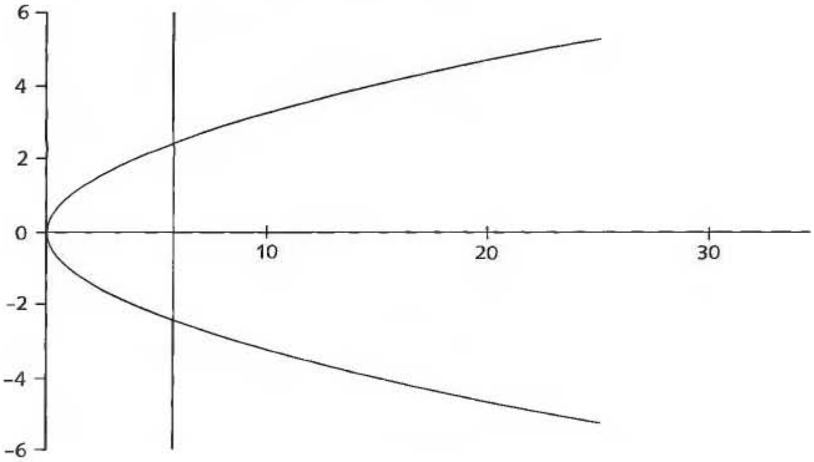
$$y = 2x + 3$$

expresses the rule '*multiply the value of x and add three*' and this rule is the function. On the other hand, the equation:

$y = x^{\frac{1}{2}}$ which is the same as $y = \pm\sqrt{x}$

expresses the rule '*take the positive and negative square roots of the value of x*'. This rule is not a function because to each value of the input $x > 0$ there are two different values of output y.

The graph of $y = \pm\sqrt{x}$ illustrates this quite clearly:



If a vertical line is drawn through the x-axis (for $x > 0$) it intersects the graph at more than one point. The fact that for $x = 0$ the vertical line intersects the graph at only *one* point does not matter – that there are other points where the vertical line intersects the graph in more than one point is sufficient to bar this from being the graph of a function. Notice that $y = x^{\frac{1}{2}}$ has no real values of y for $x < 0$.

Also note that your calculator only gives a single answer to $x^{\frac{1}{2}}$ because it is, in fact, calculating $\sqrt{x}$.

So, which of the following equations express rules that are functions?

(a) $y = 5x^2 + 2x^{-\frac{1}{4}}$

(b) $y = 7x^{\frac{1}{3}} - 3x^{-1}$

Next frame

4

(a) $y = 5x^2 + 2x^{-\frac{1}{4}}$ does not
(b) $y = 7x^{\frac{1}{3}} - 3x^{-1}$ does

(a) $y = 5x^2 + 2x^{-\frac{1}{4}}$ does not express a function because to each value of x $(x > 0)$ there are two values of $x^{-\frac{1}{4}}$, positive and negative because $x^{-\frac{1}{4}} \equiv (x^{-\frac{1}{2}})^{\frac{1}{2}} \equiv \pm\sqrt{x^{-\frac{1}{2}}}$. Indeed, *any* even root produces two values.

(b) $y = 7x^{\frac{1}{3}} - 3x^{-1}$ does express a function because to each value of x $(x \neq 0)$ there is just one value of y.

All the input numbers x that a function can process are collectively called the function's *domain*. The complete collection of numbers y that correspond to the numbers in the domain is called the *range* (or *co-domain*) of the function. For example, if:

$y = \sqrt{1 - x^2}$ where both x and y are real numbers

the domain is $-1 \leq x \leq 1$ because these are the only values of x for which y has a real value. The range is $0 \leq y \leq 1$ because 0 and 1 are the minimum and maximum values of y over the domain. Other functions may, for some purpose or other, be defined on a restricted domain. For example, if we specify:

$y = x^3, \quad -2 \leq x < 3$ (the function is defined only for the restricted set of x-values given)

the domain is given as $-2 \leq x < 3$ and the range as $-8 \leq y < 27$ because -8 and 27 are the minimum and maximum values of y over the domain.
So the domains and ranges of each of the following are:

(a) $y = x^3 \quad -5 \leq x < 4$ (b) $y = x^4$ (c) $y = \dfrac{1}{(x-1)(x+2)} \quad 0 \leq x \leq 6$

The answers are in the next frame

5

(a) $y = x^3 \quad -5 \leq x < 4$
domain $-5 \leq x < 4$, range $-125 \leq y < 64$
(b) $y = x^4$
domain $-\infty < x < \infty$, range $0 \leq y < \infty$
(c) $y = \dfrac{1}{(x-1)(x+2)}, \quad 0 \leq x \leq 6$
domain $0 \leq x < 1$ and $1 < x \leq 6$,
range $-\infty < y \leq -0{\cdot}5$, $0{\cdot}025 \leq y < \infty$

Because

(a) The domain is given as $-5 \leq x < 4$ and the range as $-125 \leq y < 64$ because -125 and 64 are the minimum and maximum values of y over the domain.
(b) The domain is not given and is assumed to consist of all finite values of x, that is, $-\infty < x < \infty$. The range values are all positive because of the even power.
(c) The domain is $0 \leq x < 1$ and $1 < x \leq 6$ since y is not defined when $x = 1$ where there is a vertical asymptote. To the left of the asymptote $(0 \leq x < 1)$ the y-values range from $y = -0{\cdot}5$ when $x = 0$ and increase negatively towards $-\infty$ as $x \to 1$. To the right of the asymptote $1 < y \leq 6$ the y-values range from infinitely large and positive to $0{\cdot}025$ when $x = 6$. If you plot the graph on your spreadsheet this will be evident.

Next frame

6

Functions and the arithmetic operations

Functions can be combined under the action of the arithmetic operators provided care is taken over their common domains. For example:

If $f(x) = x^2 - 1$, $-2 \leq x < 4$ and $g(x) = \dfrac{2}{x+3}$, $0 < x \leq 5$ then, for example

(a) $h(x) = f(x) + g(x) = x^2 - 1 + \dfrac{2}{x+3}$, $0 < x < 4$

because $g(x)$ is not defined for $-2 \leq x \leq 0$ and $f(x)$ is not defined for $4 \leq x \leq 5$ so $0 < x < 4$ is the common domain between them.

(b) $k(x) = \dfrac{g(x)}{f(x)} = \dfrac{2}{(x+3)(x^2-1)}$, $0 < x < 4$ and $x \neq 1$

because $g(x)$ is not defined for $-2 \leq x \leq 0$, $f(x)$ is not defined for $4 \leq x \leq 5$ and $k(x)$ is not defined when $x = 1$.

So if:

$f(x) = \dfrac{2x}{x^3 - 1}$, where $-3 < x < 3$ and $x \neq 1$ and

$g(x) = \dfrac{4x-8}{x+5}$, $0 < x \leq 6$ then $h(x) = \dfrac{f(x)}{g(x)}$ is ....

The answer is in the next frame

7

$$\boxed{h(x) = \frac{f(x)}{g(x)} = \frac{2x(x+5)}{(x^3-1)(4x-8)} \text{ where } 0 < x < 3,\ x \neq 1 \text{ and } x \neq 2}$$

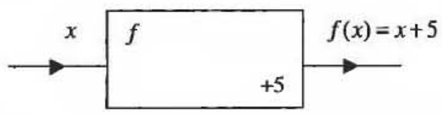
Because when $x = 1$ or 2, $h(x)$ is not defined; when $-3 < x \leq 0$, $g(x)$ is not defined; and when $3 \leq x \leq 6$, $f(x)$ is not defined.

8

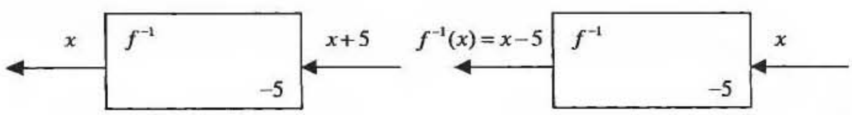
Inverses of functions

The process of generating the output of a function is assumed to be reversible so that what has been constructed can be de-constructed. The effect can be described by reversing the flow of information through the diagram so that, for example, if:

$y = f(x) = x + 5$



the flow is reversed by making the output the input and *retrieving the original input as the new output*:



The reverse process is different because instead of adding 5 to the input, 5 is now subtracted from the input. The rule that describes the reversed process is called the *inverse of the function* which is labelled as either f^{-1} or *arcf*. That is:

$f^{-1}(x) = x - 5$

The notation f^{-1} is very commonly used but care must be taken to remember that the -1 does not mean that it is in any way related to the reciprocal of f.

Try some. Find $f^{-1}(x)$ in each of the following cases:

(a) $f(x) = 6x$ (b) $f(x) = x^3$ (c) $f(x) = \dfrac{x}{2}$

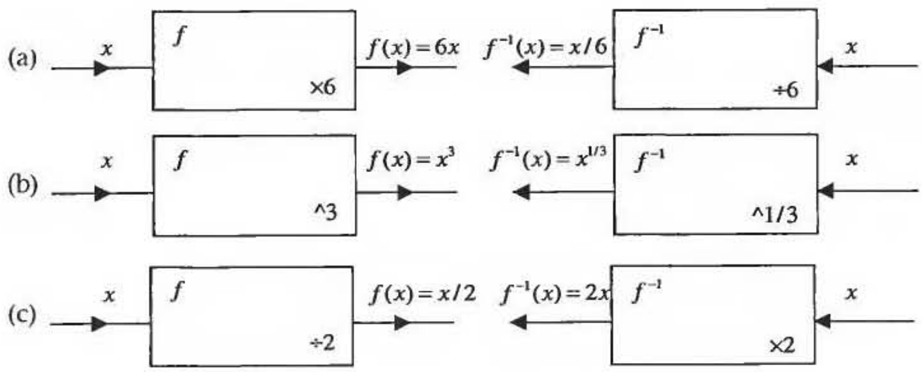
Draw the diagram, reverse the flow and find the inverse of the function in each case

---

9

(a) $f^{-1}(x) = \dfrac{x}{6}$
(b) $f^{-1}(x) = x^{\frac{1}{3}}$
(c) $f^{-1}(x) = 2x$

Because



The inverses of the arithmetic operations are just as you would expect:

addition and subtraction are inverses of each other
multiplication and division are inverses of each other
raising to the power k and raising to a power 1/k are inverses of each other

Now, can you think of two functions that are each identical to their inverse?

Think carefully

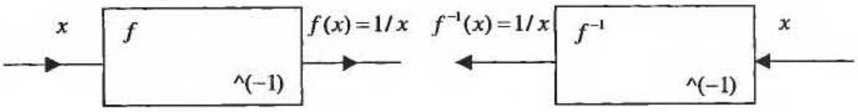
10

$f(x) = x$ and $f(x) = \dfrac{1}{x}$

Because the function with output $f(x) = x$ does not alter the input at all so the inverse does not either, and the function with output $f(x) = \dfrac{1}{x}$ is its own ▶

inverse because the reciprocal of the reciprocal of a number is the number:

$$\frac{1}{1/x} = x$$



Let's progress

11 Graphs of inverses

The diagram of the inverse of a function can be drawn by reversing the flow of information and this is the same as interchanging the contents of each ordered pair generated by the function. As a result, when the ordered pairs generated by the inverse of a function are plotted, the graph takes up the shape of the original function but reflected in the line $y = x$. Let's try it. Use your spreadsheet to plot $y = x^3$ and the inverse $y = x^{\frac{1}{3}}$. If you are unfamiliar with the use of a spreadsheet, read Programme F.4 first where the spreadsheet is introduced.

What you are about to do is a little involved, so follow the instructions to the letter and take it slowly and carefully

12 The graph of $y = x^3$

Open up your spreadsheet

Enter −1·1 in cell **A1**
Highlight **A1** to **A24**
Click **Edit-Fill-Series** and enter the **step value as** 0·1

The cells **A1** to **A24** then fill with the numbers −1·1 to 1·2.

In cell **B1** enter the formula **=A1^3** and press **Enter**

Cell **B1** now contains the cube of the contents of cell **A1**

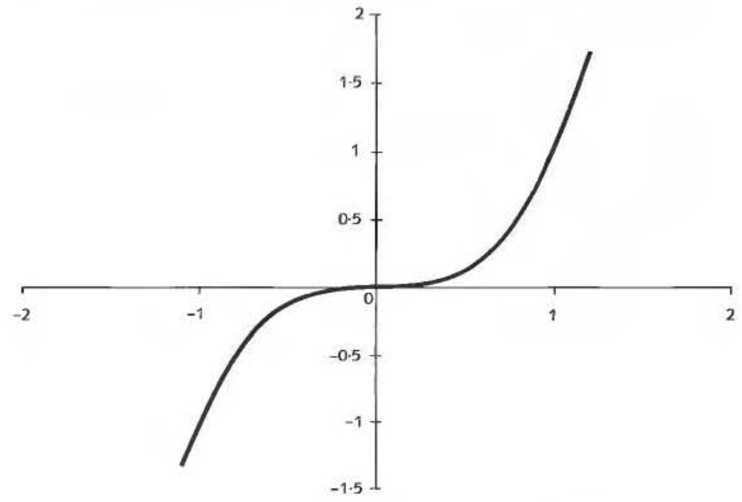
Make **B1** the active cell	
Click **Edit-Copy**	This copies the contents of B1 to the Clipboard
Highlight **B2** to **B24**	
Click **Edit-Paste**	This pastes the contents of the Clipboard to B2 to B24

Each of the cells **B1** to **B24** contains the cube of the contents of the adjacent cell in the **A** column.

Highlight the block of cells **A1** to **B24**
Click the *Chart Wizard* button to create an **XY (Scatter)** graph with joined-up points

▶

The graph you obtain will look like that depicted below:



Now for the graph of $y = x^{\frac{1}{3}}$

The graph of $y = x^{\frac{1}{3}}$

13

Keep the data you already have on the spreadsheet, you are going to use it:

Highlight cells **A1** to **A24**	
Click **Edit-Copy**	This copies the contents of A1 to A24 to the Clipboard
Place the cursor in cell **B26**	
Click **Edit-Paste**	This pastes the contents of the Clipboard from B26 to B49

The cells **B26** to **B49** then fill with the same values as those in cells **A1** to **A24**

Highlight cells **B1** to **B24**	
Click **Edit-Copy**	This copies the contents of B1 to B24 to the Clipboard
Place the cursor in cell **A26**	
Click **Edit-Paste Special**	
In the *Paste Special* window select **Values** and click **OK**	

The cells **A26** to **A49** then fill with the same values as those in cells **B1** to **B24**. Because the cells **B1** to **B24** contain formulas, using **Paste Special** rather than simply **Paste** ensures that you copy the values rather than the formulas.

What you now have are the original ordered pairs for the first function reversed in readiness to draw the graph of the inverse of the function.

Notice that row 25 is empty. This is essential because later on you are going to obtain a plot of two curves on the same graph.

▶

For now you must first clear away the old graph:

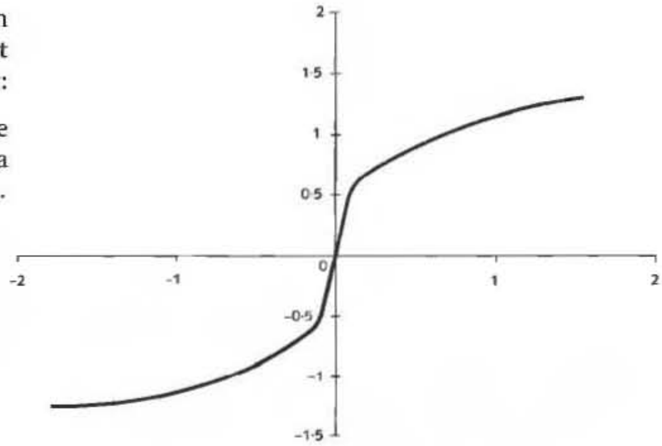
Click the boundary of the graph to display the handles
Click **Edit-Clear-All**

and the graph disappears. Now, to draw the new graphs:

Highlight the block of cells **A26** to **B49**
Click the *Chart Wizard* button to create an **XY (Scatter)** graph with joined-up points

The graph you obtain will look like that depicted to the right:

Same shape as the previous one but a different orientation.



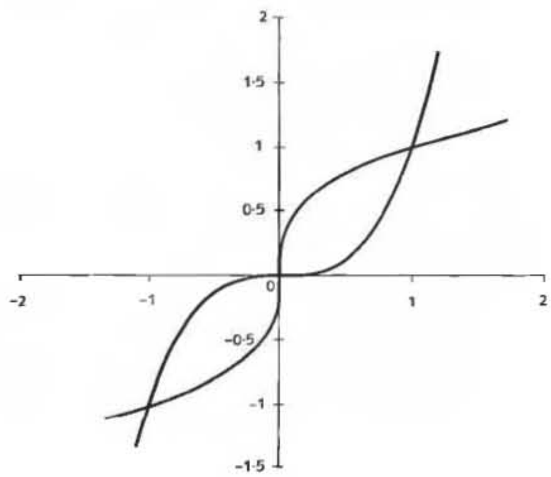
Now for both the graph of $y = x^3$ and $y = x^{\frac{1}{3}}$ together

14 The graphs of $y = x^3$ and $y = x^{\frac{1}{3}}$ plotted together

Clear away the graph you have just drawn. Then:

Highlight the block of cells **A1** to **B49**
Click the *Chart Wizard* button to create an **XY (Scatter)** graph with joined-up points

The graph you obtain will look like that depicted to the right:



Now you can see that the two graphs are each a reflection of the other in the line $y = x$. To firmly convince yourself of this:

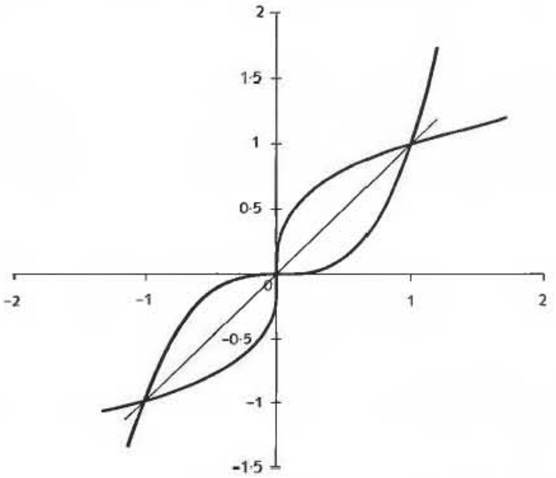
Place the cursor in cell **A51** and enter the number $-1{\cdot}1$
Enter the number $-1{\cdot}1$ in cell **B51**
Enter the number $1{\cdot}2$ in cell **A52**
Enter the number $1{\cdot}2$ in cell **B52**

You now have two points with which to plot the straight line $y = x$. *Notice again, row 50 this time is empty.*

Clear away the last graph. Then:

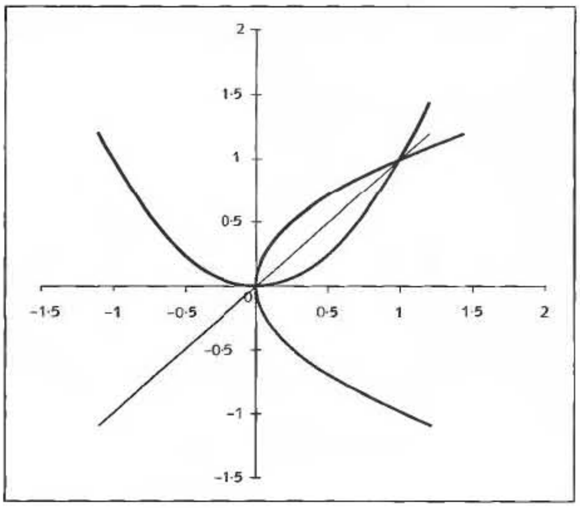
Highlight the block of cells **A1** to **B52**
Click the *Chart Wizard* button to create an **XY (Scatter)** graph with joined-up points

The graph you obtain will look like that depicted below:



Now you try one. Use the spreadsheet to plot the graphs of $y = x^2$ and its inverse $y = x^{\frac{1}{2}}$. You do not need to start from scratch, just used the sheet you have already used and change the contents of cell **B1** to the formula **=A1^2**, copy this down the **B** column to **B24** and then **Paste Special** these values into cells **A26** to **A49**.

15



The graph of the inverse of the square function is a parabola on its side. However, as you have seen earlier, this is not a graph of a function. If, however, the bottom branch of this graph is removed, what is left is the graph of the function expressed by $y = \sqrt{x}$ which is called the *inverse function* because it is single valued.

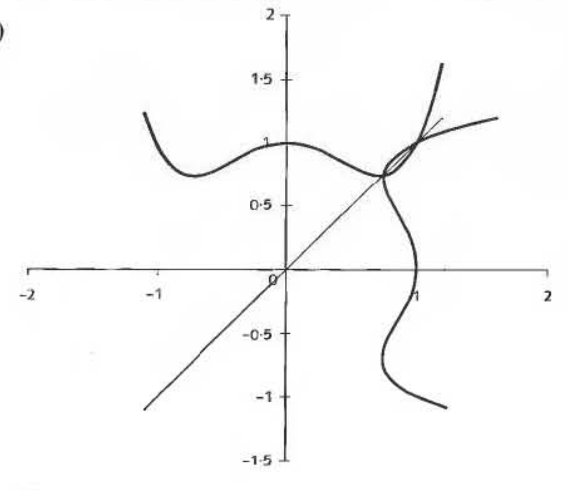
Plot the graph of $y = x^4 - x^2 + 1$ by simply changing the formula in **B1** and copying it into cells **B2** to **B24**. So:

(a) What does the inverse of the function look like?

(b) Is the inverse of the function the inverse function?

16

(a)



(b) No

Because

(b) The inverse of the function is not single valued so it cannot be a function. The inverse function would have to be obtained by removing parts of the inverse of the function to obtain a function that was single valued.

At this point let us pause and summarize the main facts so far on functions and their inverses

Revision summary 17

1 A function is a rule expressed in the form $y = f(x)$ with the proviso that for each value of x there is a unique value of y.

2 The collection of permitted input values to a function is called the *domain* of the function and the collection of corresponding output values is called the *range*.

3 The inverse of a function is a rule that associates range values to domain values of the original function.

Revision exercise 18

1 Which of the following equations expresses a rule that is a function:

(a) $y = 6x - 2$

(b) $y = \sqrt{x^3}$

(c) $y = \left(\dfrac{3x}{x^2 + 3}\right)^{\frac{5}{2}}$

2 Given the two functions f and g expressed by:

$$f(x) = 2x - 1 \text{ for } -2 < x < 4 \text{ and } g(x) = \frac{4}{x - 2} \text{ for } 3 < x < 5,$$

find the domain and range of:

(a) $h(x) = f(x) - g(x)$

(b) $k(x) = -\dfrac{2f(x)}{g(x)}$

3 Use your spreadsheet to draw each of the following and their inverses. Is the inverse a function?

(a) $y = x^6$ Use the data from the text and just change the formula

(b) $y = -3x$ Use the data from the text and just change the formula

(c) $y = \sqrt{x^3}$ Enter 0 in cell **A1** and **Edit-Fill-Series** with *step value* 0·1

19

1 (a) $y = 6x - 2$ expresses a rule that is a function because to each value of x there is only one value of y.

(b) $y = \sqrt{x^3}$ expresses a rule that is a function because to each value of x there is only one value of y. The surd sign $\sqrt{\ }$ stands for the positive square root.

(c) $y = \left(\dfrac{3x}{x^2+3}\right)^{\frac{5}{2}}$ expresses a rule that is not a function because to each positive value of the bracket there are two values of y. The power 5/2 represents raising to the power 5 and taking the square root, and there are always two square roots to each positive number.

2 (a) $h(x) = f(x) - g(x) = 2x - 1 - \dfrac{4}{x-2}$ for $3 < x < 4$ because $g(x)$ is not defined for $-2 < x \leq 3$ and $f(x)$ is not defined for $4 \leq x < 5$. Range $1 < h(x) < 5$.

(b) $k(x) = -\dfrac{2f(x)}{g(x)} = -\dfrac{(2x-1)(x-2)}{2}$ for $3 < x < 4$.

Range $-7 < k(x) < -5/2$.

3 (a) $y = x^6$ has an inverse $y = x^{\frac{1}{6}}$. This does not express a function because there are always two values to an even root (see Frame 4).

(b) $y = -3x$ has an inverse $y = -\dfrac{x}{3}$. This does express a function because there is only one value of y to each value of x.

(c) $y = \sqrt{x^3}$ has an inverse $y = x^{\frac{2}{3}}$ because $\sqrt{x^3}$ represents the positive value of $y = x^{\frac{3}{2}}$. The inverse does express a function.

Now let's move on

20

Composition – 'function of a function'

Chains of functions can be built up where the output from one function forms the input to the next function in the chain. Take out your calculator again and this time enter the number:

$\boxed{4}$

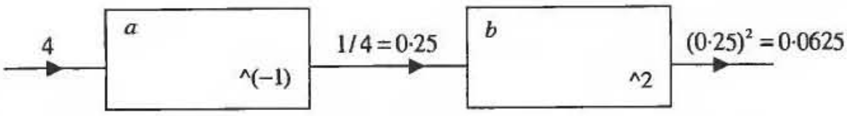
Now press the $\boxed{\frac{1}{x}}$ key – the reciprocal key – and the display changes to:

0·25 the reciprocal of 4

Now press the $\boxed{x^2}$ key and the display changes to:

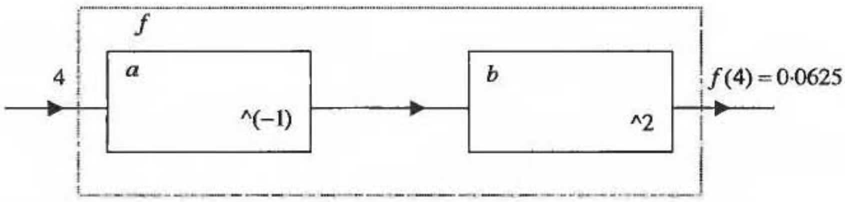
0·0625 the square of 0·25

Here, the number 4 was the input to the reciprocal function and the number 0·25 was the output. This same number 0·25 was then the input to the squaring function with output 0·0625. This can be represented by the following diagram:



Notice that the two functions have been named a and b, but any letter can be used to label a function.

At the same time the *total* processing by f could be said to be that the number 4 was input and the number 0·0625 was output:



So the function f is *composed* of the two functions a and b where $a(x) = \frac{1}{x}$, $b(x) = x^2$ and $f(x) = \left(\frac{1}{x}\right)^2$. It is said that f is the *composition* of a and b, written as:

$$f = b \circ a$$

and read as b *of* a. Notice that the functions a and b are written down algebraically in the reverse order from the order in which they are given in the diagram. This is because in the diagram the input to the composition enters on the left, whereas algebraically the input is placed to the right:

$$f(x) = b \circ a(x)$$

So that $f(x) = b \circ a(x)$, which is read as f *of* x *equals* b *of* a *of* x. An alternative notation, more commonly used, is:

$$f(x) = b[a(x)]$$

and f is described as being a *function of a function*.

Now you try. Given that $a(x) = x + 3$, $b(x) = 4x$ find the functions f and g where:

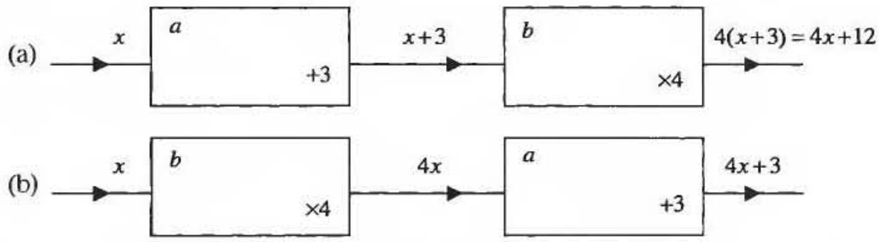
(a) $f(x) = b[a(x)]$
(b) $g(x) = a[b(x)]$

Stick with what you know, draw the boxes and see what you find

21

(a) $f(x) = 4x + 12$
(b) $g(x) = 4x + 3$

Because



Notice how these two examples show that $b[a(x)]$ is different from $a[b(x)]$. That is, the order of composition matters.

Now, how about something a little more complicated? Given the three functions a, b and c where $a(x) = x^3$, $b(x) = 2x$ and $c(x) = \tan x$, find each of the following as expressions in x:

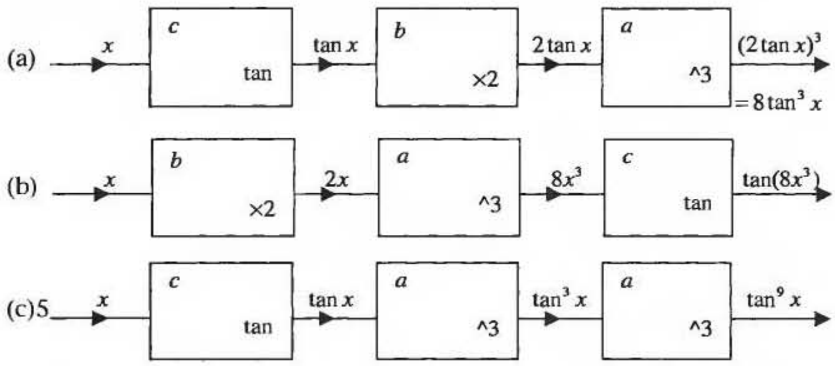
(a) $f(x) = a(b[c(x)])$ (b) $g(x) = c(a[b(x)])$ (c) $h(x) = a(a[c(x)])$

Remember, draw the boxes and follow the logic

22

(a) $f(x) = 8\tan^3 x$
(b) $g(x) = \tan(8x^3)$
(c) $h(x) = \tan^9 x$

Because



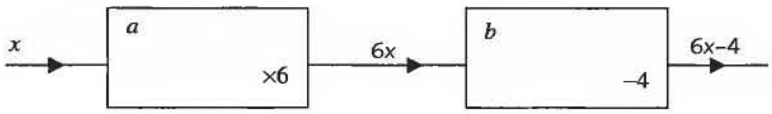
How about working the other way? Given the expression $f(x)$ for the output from a composition of functions, how do you decompose it into its component functions? This is particularly easy because you already know how to do it even though you may not yet realize it.

Let's look at a specific example first ▶

Given the output from a composition of functions as $f(x) = 6x - 4$ ask yourself how, given a calculator, would you find the value of $f(2)$? You would:

enter the number 2	this is the input	x	
multiply by 6 to give 12	this is the first function	$a(x) = 6x$	input times 6
subtract 4 to give 8	this is the second function	$b(x) = 6x - 4$	input minus 4

so that $f(x) = b[a(x)]$. The very act of using a calculator to enumerate the output from a composition requires you to decompose the composition automatically as you go.



Try it yourself. Decompose the composition with output $f(x) = (x + 5)^4$.

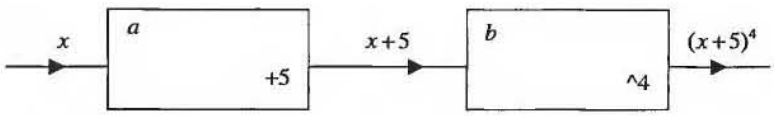
Get your calculator out and find the output for a specific input

---

23

$f(x) = b[a(x)]$ where $a(x) = x + 5$ and $b(x) = x^4$

Because



Notice that this decomposition is not unique. You could have defined $b(x) = x^2$ in which case the composition would have been $f(x) = b(b[a(x)])$.

Just to make sure you are clear about this, decompose the composition with output $f(x) = 3\sin(2x + 7)$.

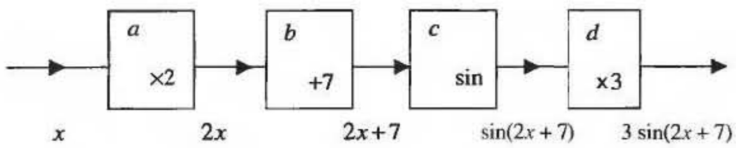
Use your calculator and take it steady, there are four functions here

---

24

$f(x) = d[c(b[a(x)])]$ where $a(x) = 2x$
$b(x) = x + 7$, $c(x) = \sin x$ and $d(x) = 3x$

Because

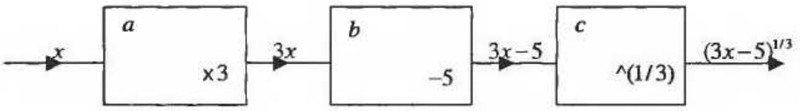


Let's keep going

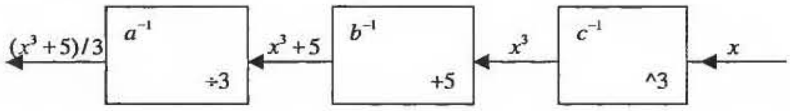
25

Inverses of compositions

As has been stated before, the diagram of the inverse of a function can be drawn as the function with the information flowing through it in the reverse direction:



The diagram permits the inverse of a composition of functions to be found. For example, consider the function f with output $f(x) = (3x-5)^{\frac{1}{3}}$. By decomposing f you find that:



$f(x) = c(b[a(x)])$ where $a(x) = 3x$, $b(x) = x - 5$ and $c(x) = x^{\frac{1}{3}}$. Each of the three functions a, b and c has its respective inverse as:

$$a^{-1}(x) = \frac{x}{3}$$
$$b^{-1}(x) = x + 5$$
$$c^{-1}(x) = x^3$$

and from the diagram you can see that $f^{-1}(x) = a^{-1}(b^{-1}[c^{-1}(x)]) = (x^3+5)/3$.
Notice the *reversal* of the order of the components:

$$f(x) = c(b[a(x)]), \qquad f^{-1}(x) = a^{-1}(b^{-1}[c^{-1}(x)])$$

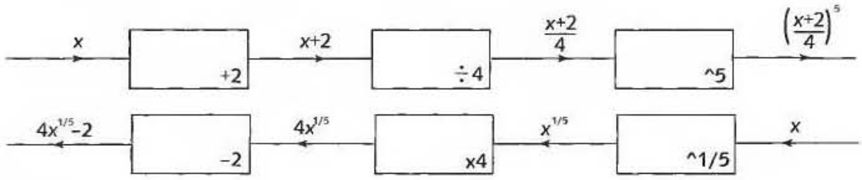
Now you try this one. Find the inverse of the function f with output $f(x) = \left(\frac{x+2}{4}\right)^5$.

Answer in the next frame

26

$$\boxed{f^{-1}(x) = 4x^{\frac{1}{5}} - 2}$$

Because



Now try the following exercises

1 Given that $a(x) = 4x$, $b(x) = x^2$, $c(x) = x - 5$ and $d(x) = \sqrt{x}$ find: 27

(a) $f(x) = a[b(c[d(x)])]$

(b) $f(x) = a(a[d(x)])$

(c) $f(x) = b[c(b[c(x)])]$

2 Given that $f(x) = (2x - 3)^3 - 3$, decompose f into its component functions and find its inverse. Is the inverse a function?

Take it steady – you will find the solutions in the next frame

---

1 (a) $f(x) = a[b(c[d(x)])] = 4(\sqrt{x} - 5)^2$ 28

(b) $f(x) = a(a[d(x)]) = 16\sqrt{x}$

(c) $f(x) = b[c(b[c(x)])] = ((x - 5)^2 - 5)^2 = x^4 - 20x^3 + 140x^2 - 400x + 400$

2 $f = b \circ c \circ b \circ a$ so that $f(x) = b[c(b[a(x)])]$ where $a(x) = 2x$, $b(x) = x - 3$ and $c(x) = x^3$. The inverse is $f^{-1}(x) = a^{-1}[b^{-1}(c^{-1}[b^{-1}(x)])]$ so that:

$$f^{-1}(x) = a^{-1}[b^{-1}(c^{-1}[b^{-1}(x)])] = \frac{(x+3)^{\frac{1}{3}} + 3}{2} \text{ where } a^{-1}(x) = x/2,$$

$b^{-1}(x) = x + 3$ and $c^{-1}(x) = x^{\frac{1}{3}}$. The inverse is a function.

So far our work on functions has centred around *algebraic functions*. This is just one category of function. We shall now move on and consider other types of function and their specific properties.

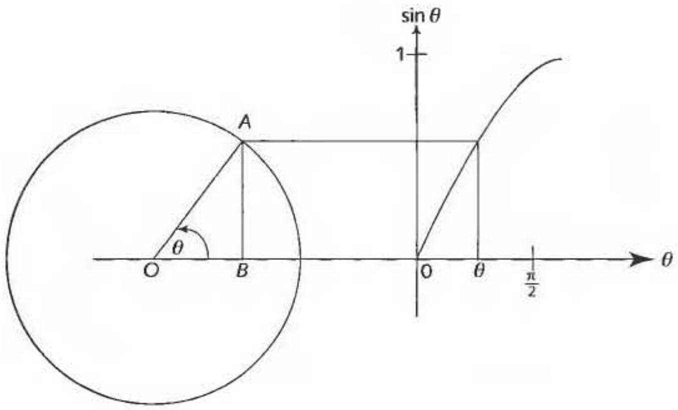
Next frame

Trigonometric functions

Rotation 29

In Programme F.8 the trigonometric ratios were defined for the two acute angles in a right-angled triangle. These definitions can be extended to form *trigonometric functions* that are valid for *any* angle and yet retain all the properties of the original ratios. Start with the circle generated by the end point A of a straight line OA of unit length rotating anticlockwise about the end O as shown in the diagram over the page:

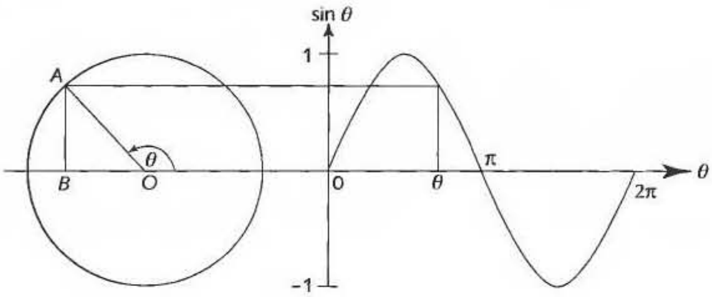
▶



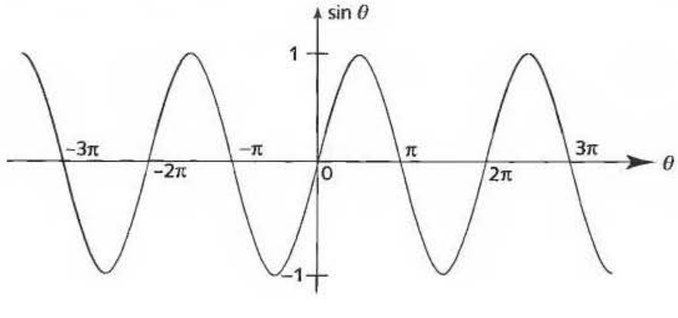
For angles θ where $0 < \theta < \pi/2$ radians you already know that:

$$\sin\theta = \frac{AB}{OA} = AB \text{ since } OA = 1$$

That is, the value of the trigonometric ratio $\sin\theta$ is equal to the height of A above B. The *sine* function with output $\sin\theta$ is now defined as the height of A above B *for any angle* θ $(0 \le \theta < \infty)$.



Notice that when A is *below* B the height is *negative*. The definition of the sine function can be further extended by taking into account negative angles, which represent a clockwise rotation of the line OA giving the complete graph of the sine function as in the diagram below:



As you can see from this diagram, the value of $\sin\theta$ ranges from $+1$ to -1 depending upon the value of θ. You can reproduce this graph using a spreadsheet – in cells **A1** to **A21** enter the numbers -10 to 10 in steps of 1 and in cell **B1** enter the formula **=sin(A1)** and copy this into cells **B2** to **B21**. Use the *Chart Wizard* to draw the graph.

Just as before, you can use a calculator to find the values of the sine of an angle. So the sine of 153° is ....

Remember to put your calculator in degree mode

$\boxed{0{\cdot}4540 \text{ to } 4 \text{ dp}}$ 30

and the sine of $-\pi/4$ radians is ....

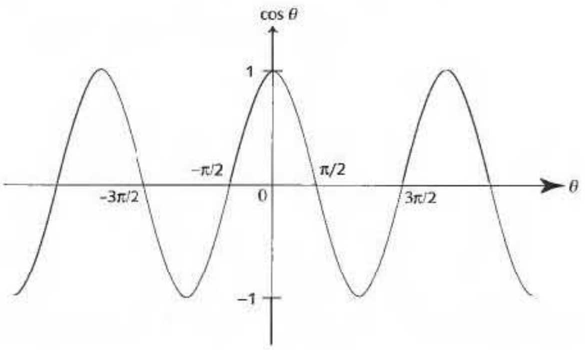
Remember to put your calculator in radian mode

$\boxed{-0{\cdot}7071 \text{ to } 4 \text{ dp}}$ 31

By the same reasoning, referring back to the first diagram in Frame 29, for angles θ where $0 < \theta < \pi/2$ radians you already know that:

$$\cos\theta = \frac{OB}{OA} = OB \text{ since } OA = 1$$

This time, the value of the trigonometric ratio $\cos\theta$ is equal to the distance from O to B. The *cosine* function with output $\cos\theta$ is now defined as the distance from O to B *for any angle* θ $(-\infty < \theta < \infty)$.



Notice that when B is to the left of O the distance from O to B is negative.

Again, you can reproduce this graph using a spreadsheet – in cells **A1** to **A21** enter the numbers -10 to 10 in steps of 1 and in cell **B1** enter the formula **=cos (A1)** and copy this into cells **B2** to **B21**. Use the *Chart Wizard* to draw the graph.

A calculator is used to find the values of the cosine of an angle. So the cosine of -272° is ....

Remember to put your calculator in degree mode

32

0·0349 to 4 dp

and the cosine of $2\pi/3$ radians is ....

Remember to put your calculator in radian mode

33

$-0{\cdot}5$

Now to put these two functions together

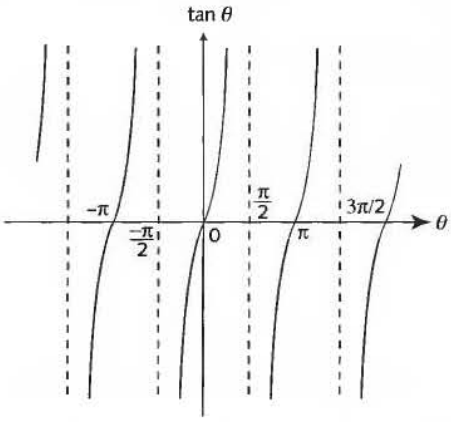
34

The tangent

The third basic trigonometric function, the tangent, is defined as the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Because $\cos\theta = 0$ whenever θ is an odd multiple of $\pi/2$, the tangent is not defined at these points. Instead the graph has vertical asymptotes as seen below:



You can plot a single branch of the tangent function using your spreadsheet. Enter $-1{\cdot}5$ in cell **A1** and **Edit-Fill-Series** down to **A21** with *step value* 0·15, then use the function **=tan (A1)** in cell **B1** and copy down to **B21**. Use the *Chart Wizard* to create the graph.

Make a note of the diagram in this frame and the diagrams in Frames 29 and 31. It is essential that you are able to draw sketch graphs of these functions.

For the sine and cosine functions the repeated sinusoidal wave pattern is easily remembered, all you then have to remember is that each rises and falls between $+1$ and -1 and crosses the horizontal axis:

(a) every whole multiple of π for the sine function

(b) every odd multiple of $\pi/2$ for the cosine function

The repeated *branch* pattern of the tangent function is also easily remembered, all you then have to remember is that it rises from $-\infty$ to $+\infty$, crosses the horizontal axis every even multiple of $\pi/2$ and has a vertical asymptote every odd multiple of $\pi/2$.

Just as before, you can use a calculator to find the values of the tangent of an angle. So the tangent of 333° is ....

Remember to put your calculator in degree mode

35

$$\boxed{-0{\cdot}5095 \text{ to 4 dp}}$$

and the tangent of $-6\pi/5$ radians is ....

Remember to put your calculator in radian mode

36

$$\boxed{-0{\cdot}7265 \text{ to 4 dp}}$$

Now to look at some common properties of these trigonometric functions

---

Period

37

Any function whose output repeats itself over a regular interval of the input is called a *periodic function*, the regular interval of the input being called the *period* of the function. From the graphs of the trigonometric functions you can see that:

Both the sine and cosine functions repeat themselves every 2π radians so both are periodic with period 2π radians. The tangent function repeats itself every π radians so it is periodic with period π radians. Finding the periods of trigonometric functions with more involved outputs requires some manipulation. For example, to find the period of $\sin 3\theta$ note that:

$$\sin 3\theta = \sin(3\theta + 2\pi)$$

It is tempting to say that the period is, therefore, 2π but this is not the case because there is a smaller interval of θ over which the basic sinusoidal shape repeats itself:

$$\sin 3\theta = \sin(3\theta + 2\pi) = \sin 3\left(\theta + \frac{2\pi}{3}\right) \text{ so the period is } \frac{2\pi}{3}$$

$\sin 3\theta$ certainly repeats itself over 2π but within 2π the basic sinusoidal shape is repeated three times.

So the period of $\cos 4\theta$ is ....

Answer in the next frame

38

$$\boxed{\frac{\pi}{2}}$$

Because

$$\cos 4\theta = \cos(4\theta + 2\pi) = \cos 4\left(\theta + \frac{2\pi}{4}\right) = \cos 4\left(\theta + \frac{\pi}{2}\right)$$

And the period of $\tan 5\theta = \ldots\ldots\ldots\ldots$

Answer in the next frame

39

$$\boxed{\frac{\pi}{5}}$$

Because

$$\tan 5\theta = \tan(5\theta + \pi) = \tan 5(\theta + \pi/5)$$

Now, try another one. The period of $\sin(\theta/3) = \ldots\ldots\ldots\ldots$

Just follow the same procedure. The answer may surprise you

40

$$\boxed{6\pi}$$

Because

$$\sin(\theta/3) = \sin(\theta/3 + 2\pi) = \sin\frac{1}{3}(\theta + 6\pi)$$

The answer is not 2π because the basic sinusoidal shape is only completed over the interval of 6π radians. If you are still not convinced of all this, use the spreadsheet to plot their graphs. Just one more before moving on.

The period of $\cos(\theta/2 + \pi/3) = \ldots\ldots\ldots\ldots$

Just follow the procedure

41

$$\boxed{4\pi}$$

Because

$$\cos(\theta/2 + \pi/3) = \cos(\theta/2 + \pi/3 + 2\pi) = \cos\left(\frac{1}{2}[\theta + 4\pi] + \pi/3\right)$$

The $\pi/3$ has no effect on the period, it just shifts the basic sinusoidal shape $\pi/3$ radians to the left.

Move on

Amplitude

42

Every periodic function possesses an *amplitude* that is given as *the difference between the maximum value and the average value of the output taken over a single period*. For example, the average value of the output from the cosine function is zero (it ranges between $+1$ and -1) and the maximum value of the output is 1, so the amplitude is $1 - 0 = 1$.

So the amplitude of $4\cos(2\theta - 3)$ is ....

Next frame

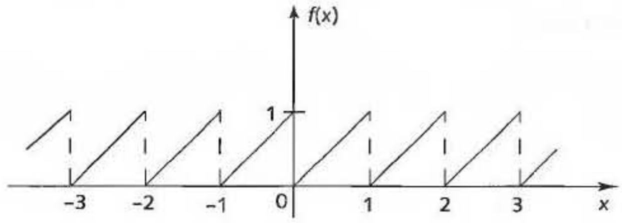
43

$\boxed{4}$

Because

The maximum and minimum values of the cosine function are $+1$ and -1 respectively, so the output here ranges from $+4$ to -4 with an average of zero. The maximum value is 4 so that the amplitude is $4 - 0 = 4$.

Periodic functions are not always trigonometric functions. For example, the function with the graph shown in the diagram below is also periodic:



The straight line branch between $x = 0$ and $x = 1$ repeats itself indefinitely. For $0 \leq x < 1$ the output from f is given as $f(x) = x$. The output from f for $1 \leq x < 2$ matches the output for $0 \leq x < 1$. That is:

$f(x + 1) = f(x)$ for $0 \leq x < 1$

So for example, $f(1{\cdot}5) = f(0{\cdot}5 + 1) = f(0{\cdot}5) = 0{\cdot}5$

The output from f for $2 \leq x < 3$ also matches the output for $0 \leq x < 1$. That is:

$f(x + 2) = f(x)$ for $0 \leq x < 1$

So that, for example, $f(2{\cdot}5) = f(0{\cdot}5 + 2) = f(0{\cdot}5) = 0{\cdot}5$

This means that we can give the prescription for the function as:

$f(x) = x$ for $0 \leq x < 1$

$f(x + n) = f(x)$ for any integer n

▶

For a periodic function of this type with period P where the first branch of the function is given for $a \leq x < a+P$ we can say that:

$$f(x) = \text{ some expression in } x \text{ for } a \leq x < a+P$$
$$f(x+nP) = f(x)$$

Because of its shape, the specific function we have considered is called a *sawtooth wave.*

The amplitude of this sawtooth wave is ....

Remember the definition of amplitude

44

$$\boxed{\frac{1}{2}}$$

Because

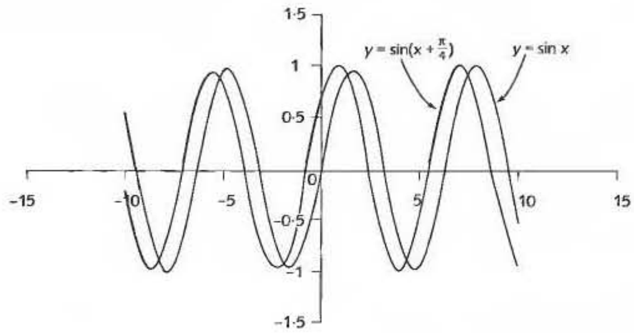
The amplitude is given as the *difference between the maximum value and the average value of the output taken over a single period.* Here the maximum value of the output is 1 and the average output is $\frac{1}{2}$, so the amplitude is $1 - 1/2 = 1/2$.

Next frame

45

Phase difference

The phase difference of a periodic function is the interval of the input by which the output leads of lags behind the *reference function.* For example, the plots of $y = \sin x$ and $y = \sin(x + \pi/4)$ on the same graph are shown below:



The diagram shows that $y = \sin(x + \pi/4)$ has the identical shape to $y = \sin x$ but is *leading* it by $\pi/4$ radians. It might appear to lag behind when you look at the diagram but it is, in fact, leading because when $x = 0$ then $\sin(x + \pi/4)$ already has the value $\sin \pi/4$, whereas $\sin x$ only has the value $\sin 0$. It is said that $y = \sin(x + \pi/4)$ leads with a *phase difference* of $\pi/4$ radians relative to the reference function $y = \sin x$. A function with a negative phase difference is said to *lag behind* the reference function. So that $y = \sin(x - \pi/4)$ lags behind $y = \sin x$ with a phase difference of $-\pi/4$.

So the phase difference of $y = \sin(x - \pi/6)$ relative to $y = \sin x$ is ....

Next frame

46

$\boxed{-\pi/6 \text{ radians}}$

Because

The graph of $y = \sin(x - \pi/6)$ *lags behind* $y = \sin x$ by $\pi/6$ radians.

The phase difference of $y = \cos x$ relative to the reference function $y = \sin x$ is ....

Think how $\cos x$ *relates to* $\sin x$

47

$\boxed{\pi/2 \text{ radians}}$

Because

$\cos x = \sin(x + \pi/2)$ and $\sin(x + \pi/2)$ *leads* $\sin x$ by $\pi/2$ radians.

Finally, try this. The phase difference of $y = \sin(3x + \pi/8)$ relative to $y = \sin 3x$ is ....

Take care to compare like with like – plot the graph if necessary

48

$\boxed{\pi/24 \text{ radians}}$

Because

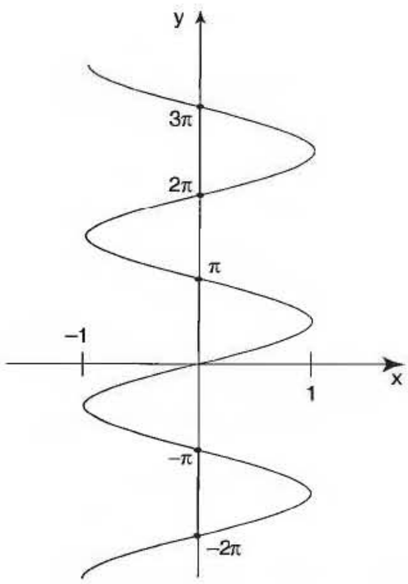
$\sin(3x + \pi/8) = \sin(3[x + \pi/24])$ and $\sin(3[x + \pi/24])$ *leads* $\sin 3x$ by $\pi/24$ radians.

Now for inverse trigonometric functions

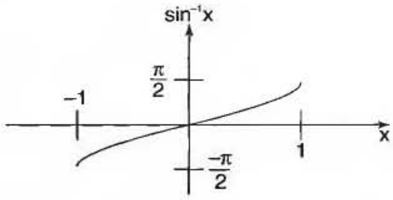
Inverse trigonometric functions

49

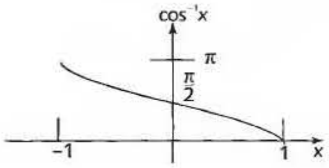
If the graph of $y = \sin x$ is reflected in the line $y = x$, the graph of the inverse of the sine function is what results:

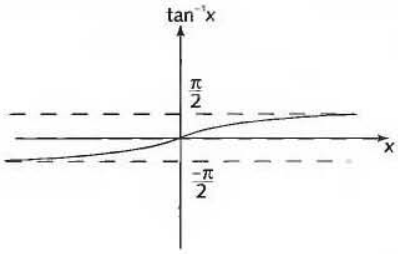


However, as you can see, this is not a function because there is more than one value of y corresponding to a given value of x. If you cut off the upper and lower parts of the graph you obtain a single-valued function and it is this that is the *inverse sine function*:



In a similar manner you can obtain the *inverse cosine function* and the *inverse tangent function*:





As in the case of the trigonometric functions, the values of these inverse functions are found using a calculator.

So that:

(a) $\sin^{-1}(0{\cdot}5) = \ldots\ldots\ldots$

(b) $\tan^{-1}(-3{\cdot}5) = \ldots\ldots\ldots$

(c) $\sec^{-1}(10) = \ldots\ldots\ldots$ (refer to Programme F.8 for the definition of $\sec\theta$)

Next frame for the answer

50

(a) 30°	(b) −74·05°	(c) 84·26°

Because

(c) If $\sec^{-1}(10) = \theta$ then $\sec\theta = 10 = \dfrac{1}{\cos\theta}$ so that $\cos\theta = 0{\cdot}1$ and

$\theta = \cos^{-1}(0{\cdot}1) = 84{\cdot}26°$

So remember $\sec^{-1}\theta = \cos^{-1}\dfrac{1}{\theta}$.

Similar results are obtained for $\text{cosec}^{-1}\theta$ and $\cot^{-1}\theta$:

$$\text{cosec}^{-1}\theta = \sin^{-1}\frac{1}{\theta} \text{ and } \cot^{-1}\theta = \tan^{-1}\frac{1}{\theta}$$

Now to use these functions and their inverses to solve equations

Trigonometric equations

51

A simple trigonometric equation is one that involves just a single trigonometric expression. For example, the equation:

$\sin 3x = 0$ is a simple trigonometric equation

The solution of this equation can be found from inspecting the graph of the sine function $\sin\theta$ which crosses the θ-axis whenever θ is an integer multiple of π. That is, $\sin n\pi = 0$ where n is an integer. This means that the solutions to $\sin 3x = 0$ are found when:

$3x = n\pi$ so that $x = \dfrac{n\pi}{3}$, $n = 0, \ \pm 1, \ \pm 2, \ \ldots$

So the values of x that satisfy the simple trigonometric equation:

$\cos 2x = 1$ are ....

Next frame

52

$$\boxed{x = n\pi, \ n = 0, \ \pm 1, \ \pm 2, \ \ldots}$$

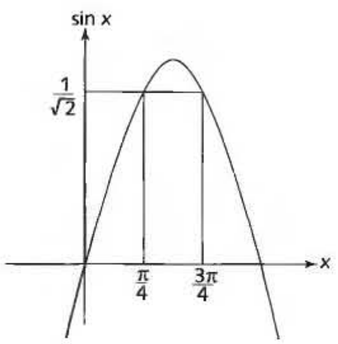
Because

From the graph of the cosine function you can see that it rises to its maximum $\cos\theta = 1$ whenever θ is an even multiple of π, that is $\theta = 0, \ \pm 2\pi, \ \pm 4\pi$. Consequently, $\cos 2x = 1$ when $2x = 2n\pi$ so that $x = n\pi, \ n = 0, \ \pm 1, \ \pm 2, \ \ldots$

Just look at another. Consider the equation:

$2\sin 3x = \sqrt{2}$

This can be rewritten as $\sin 3x = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$. From what you know about the right-angled isosceles triangle, you can say that when $\theta = \dfrac{\pi}{4}$ then $\sin\theta = \dfrac{1}{\sqrt{2}}$. However if you look at the graph of the sine function you can see that between $\theta = 0$ and $\theta = 2\pi$ there are two values of θ where $\sin\theta = \dfrac{1}{\sqrt{2}}$, namely $\theta = \dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$.



▶

Consequently:

$$\sin 3x = \frac{1}{\sqrt{2}} \text{ when } 3x = \frac{\pi}{4} \pm 2n\pi \text{ and when } 3x = \frac{3\pi}{4} \pm 2n\pi,\ n = 0,\ \pm 1,\ \pm 2,\ \dots$$

So the values of x that satisfy $2\sin 3x = \sqrt{2}$ are:

$$x = \frac{\pi}{12} \pm \frac{2n\pi}{3} \text{ and } x = \frac{\pi}{4} \pm \frac{2n\pi}{3} \text{ where } n = 0,\ \pm 1,\ \pm 2,\ \dots$$

So, the values of x that satisfy $\cos 4x = \dfrac{1}{2}$ are ....

Next frame

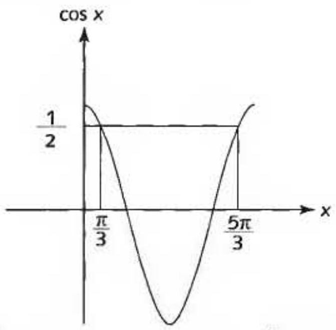
53

$$\boxed{x = \frac{\pi}{12} \pm \frac{n\pi}{2},\ \text{and}\ \frac{5\pi}{12} \pm \frac{n\pi}{2},\ n = 0,\ \pm 1,\ \pm 2,\ \dots}$$

Because

From the graph of the cosine function you see that when $\cos\theta = \dfrac{1}{2}$

$$\theta = \frac{\pi}{3} \pm 2n\pi \text{ or } \theta = \frac{5\pi}{3} \pm 2n\pi \text{ where } n = 0,\ \pm 1,\ \pm 2,\ \dots$$



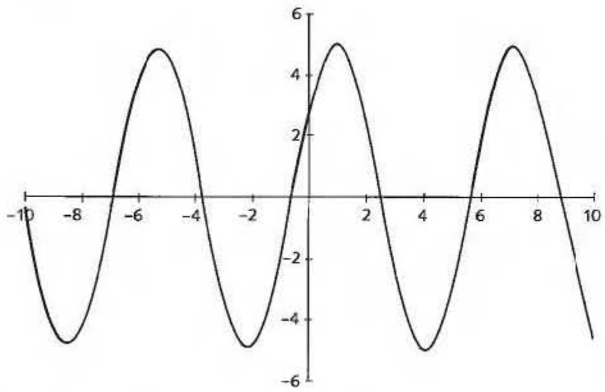
So that when $\cos 4x = \dfrac{1}{2}$, $4x = \dfrac{\pi}{3} \pm 2n\pi$ or $4x = \dfrac{5\pi}{3} \pm 2n\pi$. That is:

$$x = \frac{\pi}{12} \pm \frac{n\pi}{2} \text{ and } x = \frac{5\pi}{12} \pm \frac{n\pi}{2} \text{ where } n = 0,\ \pm 1,\ \pm 2,\ \dots$$

Equations of the form $a\cos x + b\sin x = c$ 54

A plot of $f(x) = a\cos x + b\sin x$ against x will produce a sinusoidal graph. Try it. Use your spreadsheet to plot:

$f(x) = 3\cos x + 4\sin x$ against x for $-10 \le x \le 10$ with *step value* 1. The result is shown below:



This sinusoidal shape possesses an amplitude and a phase, so its equation must be of the form:

$f(x) = R\sin(x + \theta)$ or $f(x) = R\cos(x + \phi)$

Either form will suffice – we shall select the first one to find solutions to the equation:

$3\cos x + 4\sin x = 5$

That is:

$R\sin(x + \theta) = 5$

The left-hand side can be expanded to give:

$R\sin\theta\cos x + R\cos\theta\sin x = 5$

Comparing this equation with the equation $3\cos x + 4\sin x = 5$ enables us to say that:

$3 = R\sin\theta$ and $4 = R\cos\theta$

Now:

$R^2\sin^2\theta + R^2\cos^2\theta = R^2 = 3^2 + 4^2 = 25 = 5^2$

so that $R = 5$. This means that $5\sin(\theta + x) = 5$ so that:

$\sin(\theta + x) = 1$ with solution $\theta + x = \dfrac{\pi}{2} \pm 2n\pi$

Thus:

$$x = \frac{\pi}{2} - \theta \pm 2n\pi$$

▶

Now, $\dfrac{R\sin\theta}{R\cos\theta} = \tan\theta = \dfrac{3}{4}$ so that $\theta = \tan^{-1}\left(\dfrac{3}{4}\right) = 0{\cdot}64$ rad. This gives the solution to the original equation as:

$$x = \frac{\pi}{2} - 0{\cdot}64 \pm 2n\pi = 0{\cdot}93 \pm 2n\pi \text{ radians to 2 dp}$$

Notice that if we had assumed a form $f(x) = R\cos(x + \phi)$ the end result would have been the same.

55

Try this one yourself.

The solutions to the equation $\sin x - \sqrt{2}\cos x = 1$ are ....

Next frame for the answer

56

$$\boxed{x = 1{\cdot}571 \pm 2n\pi \text{ radians}}$$

Because

Letting $R\sin(x + \theta) = 1$ we find, by expanding the sine, that:

$$R\sin x\cos\theta + R\sin\theta\cos x = 1$$

Comparing this equation with $\sin x - \sqrt{2}\cos x = 1$ enables us to say that:

$$R\cos\theta = 1 \text{ and } R\sin\theta = -\sqrt{2}$$

where $R^2\cos^2\theta + R^2\sin^2\theta = R^2 = 1^2 + \left(-\sqrt{2}\right)^2 = 3$ so that $R = \sqrt{3}$.

This means that $\sqrt{3}\sin(x + \theta) = 1$ so that $\sin(x + \theta) = \dfrac{1}{\sqrt{3}}$ giving:

$$x + \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0{\cdot}6155 \pm 2n\pi \text{ radians}$$

That is, $x = 0{\cdot}6155 \pm 2n\pi - \theta$ rad. Now:

$$\frac{R\sin\theta}{R\cos\theta} = \tan\theta = -\sqrt{2} \text{ so that } \theta = \tan^{-1}\left(-\sqrt{2}\right) = -0{\cdot}9553 \text{ rad},$$

giving the final solution as $x = 1{\cdot}571 \pm 2n\pi$ radians.

At this point let us pause and summarise the main facts so far on trigonometric functions and equations

57

Revision summary

1 The definitions of the trigonometric ratios, valid for angles greater than 0° and less than 90°, can be extended to the trigonometric functions valid for any angle.

2 The trigonometric functions possess periods, amplitudes and phases.

3 The inverse trigonometric functions have restricted ranges.

58

Revision exercise

1 Use a calculator to find the value of each of the following (take care to ensure that your calculator is in the correct mode):

(a) $\sin(3\pi/4)$ (b) $\operatorname{cosec}(-\pi/13)$ (c) $\tan(125^\circ)$
(d) $\cot(-30^\circ)$ (e) $\cos(-5\pi/7)$ (f) $\sec(18\pi/11)$

2 Find the period, amplitude and phase (in radians) of each of the following:

(a) $f(\theta) = 3\sin 9\theta$ (b) $f(\theta) = -7\cos(5\theta - 3)$
(c) $f(\theta) = \tan(2 - \theta)$ (d) $f(\theta) = -\cot(3\theta - 4)$

3 A function is defined by the following prescription:

$$f(x) = -x + 4, \quad 0 \le x < 3, \quad f(x+3) = f(x)$$

Plot a graph of this function for $-9 \le x < 9$ and find:

(a) the period

(b) the amplitude

(c) the phase of $f(x) + 2$ with respect to $f(x)$

4 Solve the following trigonometric equations:

(a) $\tan 4x = 1$

(b) $\sin(x + 2\pi) + \sin(x - 2\pi) = \dfrac{1}{2}$

(c) $2\cot\theta + 3\cot\phi = 1{\cdot}4$
$\cot\theta - \cot\phi = 0{\cdot}2$

(d) $18\cos^2 x + 3\cos x - 1 = 0$

(e) $3\sin^2 x - \cos^2 x = \sin 2x$

(f) $\cos x + \sqrt{3}\sin x = \sqrt{2}$ for $0 \le x \le 2\pi$

59

1 Using your calculator you will find:

(a) $0{\cdot}7071$ (b) $-4{\cdot}1786$ (c) $-1{\cdot}4281$
(d) $-1{\cdot}7321$ (e) $-0{\cdot}6235$ (f) $2{\cdot}4072$

2 (a) $3\sin 9\theta = 3\sin(9\theta + 2\pi) = 3\sin 9(\theta + 2\pi/9)$ so the period of $f(\theta)$ is $2\pi/9$ and the phase is 0. The maximum value of $f(\theta)$ is 3 and the average value is 0, so the amplitude of $f(\theta)$ is 3.

(b) $-7\cos(5\theta - 3) = -7\cos(5\theta - 3 + 2\pi) = -7\sin 5(\theta + 2\pi/5 - 3/5)$ so the period of $f(\theta)$ is $2\pi/5$, the phase is $-3/5$ and the amplitude is 7.

(c) $\tan(2 - \theta) = \tan(2 - \theta + \pi) = \tan(-\theta + 2 + \pi)$ so the period of $f(\theta)$ is π and $f(\theta)$ leads $\tan(-\theta)$ by the phase 2 with an infinite amplitude.

(d) $-\cot(3\theta - 4) = \cot(-3\theta + 4 + \pi) = \cot 3(-\theta + 4/3 + \pi/3)$ so the period of $f(\theta)$ is $\pi/3$ and $f(\theta)$ leads $\cot(-3\theta)$ by the phase 4/3 with an infinite amplitude.

3 (a) 3 (b) $1{\cdot}5$ (c) 0

▶

4 (a) If $\tan\theta = 1$ then $\theta = \frac{\pi}{4} \pm n\pi$ radians. Since $\tan 4x = 1$ then $4x = \frac{\pi}{4} \pm n\pi$ so that $x = \frac{\pi}{16} \pm n\frac{\pi}{4}$.

(b) $\text{LHS} = (\sin x \cos 2\pi + \sin 2\pi \cos x) + (\sin x \cos 2\pi - \sin 2\pi \cos x)$
$= 2\sin x \cos 2\pi$
$= 2\sin x$ because $\cos 2\pi = 1$
$= 1/2$ right-hand side

Therefore $\sin x = 1/4$ so $x = \sin^{-1}\left(\frac{1}{4}\right) = 0{\cdot}2526 \pm 2n\pi$ radians.

(c) Multiplying the second equation by 3 yields:
$2\cot\theta + 3\cot\phi = 1{\cdot}4$
$3\cot\theta - 3\cot\phi = 0{\cdot}6$ adding yields $5\cot\theta = 2{\cdot}0$ so that
$\theta = \cot^{-1}\left(\frac{2}{5}\right) = 1{\cdot}1903 \pm n\pi$ radians.

Also, substituting $\cot\theta = 0{\cdot}4$ into the first equation gives
$\cot\phi = \frac{1{\cdot}4 - 0{\cdot}8}{3} = 0{\cdot}2$ so that $\phi = \cot^{-1}(0{\cdot}2) = 1{\cdot}3734 \pm n\pi$ radians.

(d) This equation factorizes as $(6\cos x - 1)(3\cos x + 1) = 0$ so that:
$\cos x = 1/6$ or $-1/3$. Thus $x = \pm 1{\cdot}4033 \pm 2n\pi$ radians or
$x = \pm 1{\cdot}9106 \pm 2n\pi$ radians.

(e) This equation can be written as $3\sin^2 x - \cos^2 x = 2\sin x \cos x$.
That is: $3\sin^2 x - 2\sin x \cos x - \cos^2 x = 0$.
That is $(3\sin x + \cos x)(\sin x - \cos x) = 0$
so that $3\sin x + \cos x = 0$ or $\sin x - \cos x = 0$.
If $3\sin x + \cos x = 0$ then $\tan x = -1/3$ and so $x = -0{\cdot}3218 \pm n\pi$
and if $\sin x - \cos x = 0$ then $\tan x = 1$ and so $x = \pi/4 \pm n\pi$.

(f) To solve $\cos x + \sqrt{3}\sin x = \sqrt{2}$, write $R\sin\theta = 1$ and $R\cos\theta = \sqrt{3}$. The equation then becomes:

$R\sin\theta\cos x + R\cos\theta\sin x = \sqrt{2}$

That is:

$R\sin(\theta + x) = \sqrt{2}$

Now:

$R^2\sin^2\theta + R^2\cos^2\theta = R^2 = 1^2 + (\sqrt{3})^2 = 4$

so that $R = 2$. This means that $2\sin(\theta + x) = \sqrt{2}$ so that:

$\sin(\theta + x) = \frac{1}{\sqrt{2}}$ with solution $\theta + x = \frac{\pi}{4} \pm 2n\pi$ or $\theta + x = \frac{3\pi}{4} \pm 2n\pi$

Thus:

$\theta = \frac{\pi}{4} - x \pm 2n\pi$ or $\theta = \frac{3\pi}{4} - x \pm 2n\pi$

▶

Now $\dfrac{R\sin\theta}{R\cos\theta} = \tan\theta = \dfrac{1}{\sqrt{3}}$ so that $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$ rad. This gives the solution to the original equation as:

$$x = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \text{ or } x = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12} \text{ within the range } 0 \le x \le 2\pi.$$

Exponential and logarithmic functions

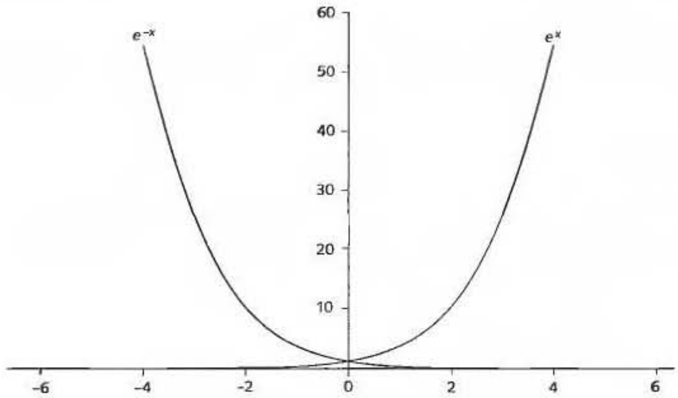
Exponential functions

60

The exponential function is expressed by the equation:

$y = e^x$ or $y = \exp(x)$

where e is the exponential number 2·7182818... . The graph of this function lies entirely above the x-axis as does the graph of its reciprocal $y = e^{-x}$, as can be seen in the diagram:



The value of e^x can be found to any level of precision desired from the series expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

In practice a calculator is used. The general exponential function is given by:

$y = a^x$ where $a > 0$

and because $a = e^{\ln a}$ the general exponential function can be written in the form:

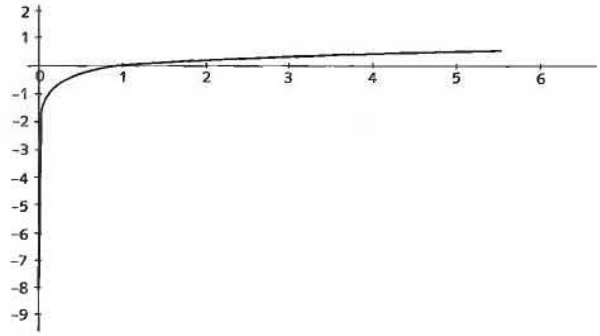
$y = e^{x \ln a}$

Because $\ln a < 1$ when $a < e$ you can see that the graph increases less quickly than the graph of e^x and if $a > e$ it grows faster.

The inverse exponential function is the logarithmic function expressed by the equation:

$y = \log_a x$

with the graph shown in the diagram:



When the base a of the logarithmic function takes on the value of the exponential number e the notation $y = \ln x$ is used.

Indicial equations

An *indicial equation* is an equation where the variable appears as an index and the solution of such an equation requires the application of logarithms.

Example 1

Here is a simple case. We have to find the value of x, give that $12^{2x} = 35{\cdot}4$.

Taking logs of both sides – and using $\log(A^n) = n \log A$ we have

$$(2x)\log 12 = \log 35{\cdot}4$$

i.e. $$(2x)1{\cdot}0792 = 1{\cdot}5490$$

$$2{\cdot}1584x = 1{\cdot}5490$$

$$\therefore\ x = \frac{1{\cdot}5490}{2{\cdot}1584} = 0{\cdot}71766$$

$$\therefore\ x = 0{\cdot}7177 \text{ to 4 sig fig}$$

Example 2

Solve the equation $4^{3x-2} = 26^{x+1}$

The first line in this solution is ....

61

$$\boxed{(3x-2)\log 4 = (x+1)\log 26}$$

$\therefore\ (3x-2)0{\cdot}6021 = (x+1)1{\cdot}4150$

Multiplying out and collecting up, we eventually get

$x =$ .... to 4 sig fig

62

$$\boxed{6{\cdot}694}$$

Because we have

$$\begin{aligned}(3x-2)0{\cdot}6021 &= (x+1)1{\cdot}4150\\ 1{\cdot}8063x - 1{\cdot}2042 &= 1{\cdot}4150x + 1{\cdot}4150\\ (1{\cdot}8063 - 1{\cdot}4150)x &= (1{\cdot}4150 + 1{\cdot}2042)\\ 0{\cdot}3913x &= 2{\cdot}6192\\ \therefore\ x = \frac{2{\cdot}6192}{0{\cdot}3913} &= 6{\cdot}6936\\ \therefore\ x &= 6{\cdot}694 \text{ to 4 sig fig}\end{aligned}$$

Care must be taken to apply the rules of logarithms rigidly.

Now we will deal with another example

63

Example 3

Solve the equation $5{\cdot}4^{x+3} \times 8{\cdot}2^{2x-1} = 4{\cdot}8^{3x}$

We recall that $\log(A \times B) = \log A + \log B$

Therefore, we have $\log\{5{\cdot}4^{x+3}\} + \log\{8{\cdot}2^{2x-1}\} = \log\{4{\cdot}8^{3x}\}$

i.e. $(x+3)\log 5{\cdot}4 + (2x-1)\log 8{\cdot}2 = 3x\log 4{\cdot}8$

You can finish it off, finally getting

$x =$ .... to 4 sig fig

64

$$\boxed{2{\cdot}485}$$

Here is the working as a check:

$$\begin{aligned}&(x+3)0{\cdot}7324 + (2x-1)0{\cdot}9138 = (3x)0{\cdot}6812\\ &0{\cdot}7324x + 2{\cdot}1972 + 1{\cdot}8276x - 0{\cdot}9138 = 2{\cdot}0436x\\ &(0{\cdot}7324 + 1{\cdot}8276)x + (2{\cdot}1972 - 0{\cdot}9138) = 2{\cdot}0436x\\ &\therefore\ 2{\cdot}5600x + 1{\cdot}2834 = 2{\cdot}0436x\\ &(2{\cdot}5600 - 2{\cdot}0436)x = -1{\cdot}2834\\ &0{\cdot}5164x = -1{\cdot}2834\end{aligned}$$

$$x = -\frac{1{\cdot}2834}{0{\cdot}5164} = -2{\cdot}4853 \quad \therefore\ x = -2{\cdot}485 \text{ to 4 sig fig}$$

Finally, here is one to do all on your own. ▶

Example 4

Solve the equation $7(14{\cdot}3^{x+5}) \times 6{\cdot}4^{2x} = 294$

Work right through it, giving the result to 4 sig fig.

$x = \ldots\ldots\ldots\ldots$

65

$\boxed{-1{\cdot}501}$

Check the working:

$$7(14{\cdot}3^{x+5}) \times 6{\cdot}4^{2x} = 294$$

$$\therefore \quad \log 7 + (x+5)\log 14{\cdot}3 + (2x)\log 6{\cdot}4 = \log 294$$

$$0{\cdot}8451 + (x+5)1{\cdot}1553 + (2x)0{\cdot}8062 = 2{\cdot}4683$$

$$0{\cdot}8451 + 1{\cdot}1553x + 5{\cdot}7765 + 1{\cdot}6124x = 2{\cdot}4683$$

$$(1{\cdot}1553 + 1{\cdot}6124)x + (0{\cdot}8451 + 5{\cdot}7765) = 2{\cdot}4683$$

$$2{\cdot}7677x + 6{\cdot}6216 = 2{\cdot}4683$$

$$2{\cdot}7677x = 2{\cdot}4683 - 6{\cdot}6216 = -4{\cdot}1533$$

$$\therefore \quad x = -\frac{4{\cdot}1533}{2{\cdot}7677} = -1{\cdot}5006$$

$$x = -1{\cdot}501 \text{ to 4 sig fig}$$

66

Some indicial equations may need a little manipulation before the rules of logarithms can be applied.

Example 5

Solve the equation $2^{2x} - 6 \times 2^x + 8 = 0$

Because $2^{2x} = (2^x)^2$ this is an equation that is quadratic in 2^x. We can, therefore, write $y = 2^x$ and substitute y for 2^x in the equation to give:

$$y^2 - 6y + 8 = 0$$

which factorizes to give:

$$(y-2)(y-4) = 0$$

so that $y = 2$ or $y = 4$. That is $2^x = 2$ or $2^x = 4$ so that $x = 1$ or $x = 2$.

Try this one. Solve $2 \times 3^{2x} - 6 \times 3^x + 4 = 0$.

The answer is in Frame 67

67

$$\boxed{x = 0{\cdot}631 \text{ to 3 dp or } x = 0}$$

Because $3^{2x} = (3^x)^2$ this is an equation that is quadratic in 3^x. We can, therefore, write $y = 3^x$ and substitute y for 3^x in the equation to give:

$$2y^2 - 6y + 4 = 0$$

which factorizes to give:

$$(2y - 4)(y - 1) = 0$$

so that $y = 2$ or $y = 1$. That is $3^x = 2$ or $3^x = 1$ so that $x \log 3 = \log 2$ or $x = 0$. That is $x = \dfrac{\log 2}{\log 3} = 0{\cdot}631$ to 3 dp or $x = 0$.

And now to the final topic of Part I

Odd and even functions

68

If, by replacing x by $-x$ in $f(x)$ the expression does not change its value, f is called an *even* function. For example, if:

$f(x) = x^2$ then $f(-x) = (-x)^2 = x^2 = f(x)$ so that f is an even function.

On the other hand, if $f(-x) = -f(x)$ then f is called an *odd* function. For example, if:

$f(x) = x^3$ then $f(-x) = (-x)^3 = -x^3 = -f(x)$ so that f is an odd function .

Because $\sin(-\theta) = -\sin\theta$ the sine function is an odd function and because $\cos(-\theta) = \cos\theta$ the cosine function is an even function. Notice how the graph of the cosine function is reflection symmetric about the vertical axis through $\theta = 0$ in the diagram in Frame 31. All even functions possess this type of symmetry. The graph of the sine function is rotation symmetric about the origin as it goes into itself under a rotation of 180° about this point as can be seen from the third diagram in Frame 29. All odd functions possess this type of antisymmetry. Notice also that $\tan(-\theta) = -\tan\theta$ so that the tangent function, like the sine function, is odd and has an antisymmetric graph (see the diagram in Frame 34).

69

Odd and even parts

Not every function is either even or odd but many can be written as the sum of an even part and an odd part. If, given $f(x)$ where $f(-x)$ is also defined then:

$$f_e(x) = \frac{f(x) + f(-x)}{2} \text{ is even and } f_o(x) = \frac{f(x) - f(-x)}{2} \text{ is odd.}$$

Furthermore $f_e(x)$ is called the *even part* of $f(x)$ and

$f_o(x)$ is called the *odd part* of $f(x)$.

For example, if $f(x) = 3x^2 - 2x + 1$ then $f(-x) = 3(-x)^2 - 2(-x) + 1 = 3x^2 + 2x + 1$ so that the even and odd parts of $f(x)$ are:

$$f_e(x) = \frac{(3x^2 - 2x + 1) + (3x^2 + 2x + 1)}{2} = 3x^2 + 1 \text{ and}$$

$$f_o(x) = \frac{(3x^2 - 2x + 1) - (3x^2 + 2x + 1)}{2} = -2x$$

So, the even and odd parts of $f(x) = x^3 - 2x^2 - 3x + 4$ are ....

Apply the two formulas; the answer is in the next frame

70

$$\boxed{\begin{aligned} f_e(x) &= -2x^2 + 4 \\ f_o(x) &= x^3 - 3x \end{aligned}}$$

Because

$$\begin{aligned} f_e(x) &= \frac{f(x) + f(-x)}{2} \\ &= \frac{(x^3 - 2x^2 - 3x + 4) + ((-x)^3 - 2(-x)^2 - 3(-x) + 4)}{2} \\ &= \frac{x^3 - 2x^2 - 3x + 4 - x^3 - 2x^2 + 3x + 4}{2} = \frac{-2x^2 + 4 - 2x^2 + 4}{2} \\ &= -2x^2 + 4 \end{aligned}$$

and

$$\begin{aligned} f_o(x) &= \frac{f(x) - f(-x)}{2} \\ &= \frac{(x^3 - 2x^2 - 3x + 4) - ((-x)^3 - 2(-x)^2 - 3(-x) + 4)}{2} \\ &= \frac{x^3 - 2x^2 - 3x + 4 + x^3 + 2x^2 - 3x - 4}{2} = \frac{x^3 - 3x + x^3 - 3x}{2} \\ &= x^3 - 3x \end{aligned}$$

Make a note here that *even polynomial functions consist of only even powers* and *odd polynomial functions consist of only odd powers*.

So the odd and even parts of $f(x) = x^3(x^2 - 3x + 5)$ are ....

Answer in next frame

71

$$\boxed{f_e(x) = -3x^4 \qquad f_o(x) = x^5 + 5x^3}$$

Because

Even polynomial functions consist of only even powers and odd polynomial functions consist of only odd powers. Consequently, the even part of $f(x)$ consists of even powers only and the odd part of $f(x)$ consists of odd powers only.

Now try this. The even and odd parts of $f(x) = \dfrac{1}{x-1}$ are ....

Next frame

72

$$\boxed{\begin{aligned} f_e(x) &= \frac{1}{(x-1)(x+1)} \\ f_o(x) &= \frac{x}{(x-1)(x+1)} \end{aligned}}$$

Because

$$\begin{aligned} f_e(x) &= \frac{f(x)+f(-x)}{2} \\ &= \frac{1}{2}\left(\frac{1}{(x-1)} + \frac{1}{(-x-1)}\right) = \frac{1}{2}\left(\frac{1}{(x-1)} - \frac{1}{(x+1)}\right) \\ &= \frac{1}{2}\frac{x+1-(x-1)}{(x-1)(x+1)} = \frac{1}{2}\frac{2}{(x-1)(x+1)} \\ &= \frac{1}{(x-1)(x+1)} \end{aligned}$$

and

$$\begin{aligned} f_o(x) &= \frac{f(x)-f(-x)}{2} \\ &= \frac{1}{2}\left(\frac{1}{(x-1)} - \frac{1}{(-x-1)}\right) = \frac{1}{2}\left(\frac{1}{(x-1)} + \frac{1}{(x+1)}\right) \\ &= \frac{1}{2}\frac{x+1+x-1}{(x-1)(x+1)} = \frac{1}{2}\frac{2x}{(x-1)(x+1)} \\ &= \frac{x}{(x-1)(x+1)} \end{aligned}$$

Next frame

73

Now show that if $f(x)$ is even then:

$$\int_{x=-a}^{a} f(x)\,dx = 2\int_{x=0}^{a} f(x)\,dx$$

74

$$\int_{x=-a}^{a} f(x)\,dx$$

$$= \int_{x=-a}^{0} f(x)\,dx + \int_{x=0}^{a} f(x)\,dx$$ integrating first from $-a$ to 0 and then from 0 to $+a$

$$= \int_{x=a}^{0} f(-x)d(-x) + \int_{x=0}^{a} f(x)\,dx$$ changing the variable of integration from x to $-x$ in the first integral so changing the sign of the limits

$$= -\int_{x=a}^{0} f(x)\,dx + \int_{x=0}^{a} f(x)\,dx$$ because $f(-x) = f(x)$ and $d(-x) = -dx$

$$= \int_{x=0}^{a} f(x)\,dx + \int_{x=0}^{a} f(x)\,dx$$ interchanging the limits on the first integral

$$= 2\int_{x=0}^{a} f(x)\,dx$$

75 Odd and even parts of the exponential function

The exponential function is neither odd nor even but it can be written as a sum of an odd part and an even part.

That is, $\exp_e(x) = \dfrac{\exp(x) + \exp(-x)}{2}$ and $\exp_o(x) = \dfrac{\exp(x) - \exp(-x)}{2}$. These two functions are known as the *hyperbolic cosine* and the *hyperbolic sine* respectively:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Using these two functions the hyperbolic tangent can also be defined:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

No more will be said about these hyperbolic trigonometric functions here. Instead they will be looked at in more detail in Programme 3 of Part II.

The logarithmic function $y = \log_a x$ is neither odd nor even and indeed does not possess even and odd parts because $\log_a(-x)$ is not defined.

At this point let us pause and summarize the main facts so far on exponential and logarithmic functions as well as on odd and even functions

Revision summary

76

1 Any function of the form $f(x) = a^x$ is called an *exponential* function.

2 The exponential function $f(x) = a^x$ and the logarithmic function $g(x) = \log_a x$ are *mutual inverses*: $f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$.

3 If $f(-x) = f(x)$ then f is called an *even* function and if $f(-x) = -f(x)$ then f is called an *odd* function.

4 If $f_e(x) = \dfrac{f(x) + f(-x)}{2}$ can be defined it if called the *even* part of $f(x)$ and if $f_o(x) = \dfrac{f(x) - f(-x)}{2}$ can be defined it is called the *odd* part of $f(x)$.

Revision exercise

77

1 Solve the following indicial equations giving the results to 4 dp:

(a) $13^{3x} = 8{\cdot}4$ (b) $2{\cdot}8^{2x+1} \times 9{\cdot}4^{2x-1} = 6{\cdot}3^{4x}$

(c) $7^{2x} - 9 \times 7^x + 14 = 0$

2 Show that:

$$\int_{x=-a}^{a} f(x)\,\mathrm{d}x = 0 \text{ if } f(x) \text{ is odd.}$$

78

1 (a) $13^{3x} = 8{\cdot}4$

Taking logs of both sides gives: $3x \log 13 = \log 8{\cdot}4$ so that:

$$x = \frac{\log 8{\cdot}4}{3 \log 13} = \frac{0{\cdot}9242\ldots}{3{\cdot}3418\ldots} = 0{\cdot}277 \text{ to 3 dp}$$

(b) $2{\cdot}8^{2x+1} \times 9{\cdot}4^{2x-1} = 6{\cdot}3^{4x}$

Taking logs:

$$(2x + 1)\log 2{\cdot}8 + (2x - 1)\log 9{\cdot}4 = 4x \log 6{\cdot}3$$

Factorizing x gives:

$$x(2\log 2{\cdot}8 + 2\log 9{\cdot}4 - 4\log 6{\cdot}3) = \log 9{\cdot}4 - \log 2{\cdot}8$$

So that:

$$x = \frac{\log 9{\cdot}4 - \log 2{\cdot}8}{2\log 2{\cdot}8 + 2\log 9{\cdot}4 - 4\log 6{\cdot}3}$$
$$= \frac{0{\cdot}9731\ldots - 0{\cdot}4471\ldots}{2(0{\cdot}4471\ldots) + 2(0{\cdot}9731\ldots) - 4(0{\cdot}7993\ldots)} = -1{\cdot}474 \text{ to 3 dp}$$

▶

(c) $7^{2x} + 9 \times 7^x - 14 = 0$

This equation is quadratic in 7^x so let $y = 7^x$ and rewrite the equation as:

$y^2 - 9y + 14 = 0$ which factorizes to $(y - 2)(y - 7) = 0$
with solution: $y = 2$ or $y = 7$ that is $7^x = 2$ or $7^x = 7$

so that $x = \dfrac{\log 2}{\log 7} = 0{\cdot}356$ to 3 dp or $x = 1$

2 $\displaystyle\int_{x=-a}^{a} f(x)\,dx = \int_{x=-a}^{0} f(x)\,dx + \int_{x=0}^{a} f(x)\,dx$ from the rules of integrals

Changing the variable of integration in the first integral on the right-hand side by substituting $x = -u$ so that $dx = -du$ you find that:

$$\int_{x=-a}^{a} f(x)\,dx = \int_{u=a}^{0} f(-u)d(-u) + \int_{x=0}^{a} f(x)\,dx$$

$$= -\int_{u=a}^{0} f(-u)du + \int_{x=0}^{a} f(x)\,dx$$

$$= \int_{u=0}^{a} f(-u)du + \int_{x=0}^{a} f(x)\,dx \quad \text{by interchanging the limits in the first integral}$$

$$= -\int_{u=0}^{a} f(u)du + \int_{x=0}^{a} f(x)\,dx \quad \text{since } f(-u) = -f(u)$$

$$= 0$$

79 You have now come to the end of this Programme. A list of **Can You?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through them at your own pace, there is no need to hurry*. A set of **Further problems** provides additional valuable practice.

☑ Can You?

Checklist F.12

80

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Identify a function as a rule and recognize rules that are not functions? 1 to 4
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine the domain and range of a function? 4 to 7
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Construct the inverse of a function and draw its graph? 8 to 16
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Construct compositions of functions and de-construct them into their component functions? 20 to 28
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Develop the trigonometric functions from the trigonometric ratios? 29 to 36
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Find the period, amplitude and phase of a periodic function? 37 to 48
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Distinguish between the inverse of a trigonometric function and the inverse trigonometric function? 49 to 50
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Solve trigonometric equations using the inverse trigonometric functions? 51 to 56
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognize that the exponential function and the natural logarithmic function are mutual inverses and solve indicial and logarithmic equations? 60 to 67
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Find the even and odd parts of a function when they exist? 68 to 74
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Construct the hyperbolic functions from the odd and even parts of the exponential function 75
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.12

81

Frames

1 Which of the following equations expresses a rule that is a function?

(a) $y = -x^{\frac{3}{2}}$ (b) $y = x^2 + x + 1$ (c) $y = (\sqrt{x})^3$ — Frames 1 to 4

2 Given the two functions f and g expressed by:

$f(x) = \dfrac{1}{x-2}$ for $2 < x \leq 4$ and $g(x) = x - 1$ for $0 \leq x < 3$

find the domain and range of functions h and k where:

(a) $h(x) = 2f(x) - 3g(x)$ (b) $k(x) = -\dfrac{3f(x)}{5g(x)}$ — Frames 4 to 7

3 Use your spreadsheet to draw each of the following and their inverses. Is the inverse a function?

(a) $y = 3x^4$ (b) $y = -x^3$ (c) $y = 3 - x^2$ — Frames 8 to 16

4 Given that $a(x) = 5x$, $b(x) = x^4$, $c(x) = x + 3$ and $d(x) = \sqrt{x}$ find:

(a) $f(x) = a[b(c[d(x)])]$ (b) $f(x) = a(a[d(x)])$

(c) $f(x) = b[c(b[c(x)])]$ — Frames 20 to 24

5 Given that $f(x) = (5x - 4)^3 - 4$ decompose f into its component functions and find its inverse. Is the inverse a function? — Frames 25 to 28

6 Use a calculator to find the value of each of the following:

(a) $\sin(-320°)$ (b) $\operatorname{cosec}(\pi/11)$ (c) $\cot(-\pi/2)$ — Frames 29 to 36

7 Find the period, amplitude and phase of each of the following:

(a) $f(\theta) = 4\cos 7\theta$ (b) $f(\theta) = -2\sin(2\theta - \pi/2)$

(c) $f(\theta) = \sec(3\theta + 4)$ — Frames 37 to 48

8 A function is defined by the following prescription:

$f(x) = 9 - x^2, \quad 0 \leq x < 3$

$f(x + 3) = f(x)$

Plot a graph of this function for $-6 \leq x \leq 6$ and find:

(a) the period

(b) the amplitude

(c) the phase of $f(x) + 2$ with respect to $f(x)$ — Frames 37 to 48

9 Solve the following trigonometric equations:

(a) $\tan(x + \pi) + \tan(x - \pi) = 0$

(b) $6\sin^2 x - 3\sin^2 2x + \cos^2 x = 0$

(c) $3\sin x - 5\cos x = 2$ — Frames 49 to 56

Frames

10 Find the value of x corresponding to each of the following:

(a) $4^{-x} = 1$ (b) $\exp(3x) = e$
(c) $e^x = 54{\cdot}32$ (d) $\log_5 x = 4$
(e) $\log_{10}(x - 3) = 0{\cdot}101$ (f) $\ln 100 = x$ 60 to 67

11 Solve for x:

(a) $4^{x+2}5^{x+1} = 32\,000$ (b) $e^{2x} - 5e^x + 6 = 0$
(c) $\log_x 36 = 2$ (d) $\frac{1}{2}\log(x^2) = 5\log 2 - 4\log 3$
(e) $\ln(x^{1/2}) = \ln x + \ln 3$ 60 to 67

12 Find the even and odd parts of $f(x) = a^x$. 68 to 75

Further problems F.12

82

1 Do the graphs of $f(x) = 3\log x$ and $g(x) = \log(3x)$ intersect?

2 Let $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$.

(a) Find the domain and range of f.
(b) Show that the new function g formed by replacing x in $f(x)$ by $\dfrac{2x}{1+x^2}$ is given by $g(x) = 2f(x)$.

3 Describe the graph of $x^2 - 9y^2 = 0$.

4 Two functions C and S are defined as the even and odd parts of f where $f(x) = a^x$. Show that:

(a) $[C(x)]^2 - [S(x)]^2 = 1$
(b) $S(2x) = 2S(x)C(x)$
(c) $C'(x) = \ln aS(x)$

5 Show by using a diagram that, for functions f and g, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

6 Is it possible to find a value of x such that $\log_a(x) = a^x$ for $a > 1$?

7 Given the three functions a, b and c where $a(x) = 6x$, $b(x) = x - 2$ and $c(x) = x^3$ find the inverse of:

(a) $f(x) = a(b[c(x)])$ (b) $f = c \circ b \circ c$ (c) $f = b \circ c \circ a \circ b \circ c$

8 Use your spreadsheet to plot $\sin\theta$ and $\sin 2\theta$ on the same graph. Plot from $\theta = -5$ to $\theta = +5$ with a step value of 0·5.

▶

9 The square sine wave with period 2 is given by the prescription:

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ -1 & 1 \le x < 2 \end{cases}$$

Plot this wave for $-4 \le x \le 4$ on a sheet of graph paper.

10 The absolute value of x is given as:

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

(a) Plot the graph of $y = |x|$ for $-2 \le x \le 2$.

(b) Find the derivative of y.

(c) Does the derivative exist at $x = 0$?

11 Use the spreadsheet to plot the rectified sine wave $f(x) = |\sin x|$ for $-10 \le x \le 10$ with step value 1.

12 Use your spreadsheet to plot $f(x) = \dfrac{\sin x}{x}$ for $-40 \le x \le 40$ with step value 4. (You will have to enter the value $f(0) = 1$ specifically in cell **B11**.)

13 Solve the following giving the results to 4 sig fig:

(a) $6\{8^{3x+2}\} = 5^{2x-7}$

(b) $4{\cdot}5^{1-2x} \times 6{\cdot}2^{3x+4} = 12{\cdot}7^{5x}$

(c) $5\{17{\cdot}2^{x+4}\} \times 3\{8{\cdot}6^{2x}\} = 4{\cdot}7^{x-1}$

PART II

Programme 1

Complex numbers 1

Frames 1 to 63

Learning outcomes

When you have completed this Programme you will be able to:

- Recognize j as standing for $\sqrt{-1}$ and be able to reduce powers of j to $\pm j$ or ± 1
- Recognize that all complex numbers are in the form (real part) + j(imaginary part)
- Add, subtract and multiply complex numbers
- Find the complex conjugate of a complex number
- Divide complex numbers
- State the conditions for the equality of two complex numbers
- Draw complex numbers and recognize the parallelogram law of addition
- Convert a complex number from Cartesian to polar form and vice versa
- Write a complex number in its exponential form
- Obtain the logarithm of a complex number

Introduction

1 Ideas and symbols

Engineers and scientists use *ideas* about the physical world to explain observed events.

A force is observed being applied to a mass and the mass is observed to accelerate.

Isaac Newton said that:

Force applied = mass × acceleration produced

Linking the force applied with the acceleration produced in this way involves ideas about space, time, mass, force, rates of change – a whole stack of ideas. In mathematics, symbols are used to represent these ideas. If symbol F represents the force, m the mass and a the acceleration then we write:

$F = m \times a$

So when you manipulate symbols you are often manipulating ideas. Take, for instance, the everyday numbers that are used for counting and measuring. The numerals were originally devised to record ideas of quantity so when a symbol such as $\sqrt{-1}$ arises to which there is no corresponding quantity we must ask ourselves why? Why does the symbol arise if there is no quantity associated with it? Often, the only way to answer a question such as this is to accept the symbol and carry on manipulating with it to see if any new ideas are forthcoming.

In an earlier Programme it was stated that there are some quadratics that cannot be factorized into two linear factors. In fact all quadratics can be factorized into two linear factors but with the present state of your knowledge you cannot always find them. This will now be remedied.

Read on and see how

The symbol *j*

2 Quadratic equations

The solution of a quadratic equation $ax^2 + bx + c = 0$ can, of course, be obtained by the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For example, if $2x^2 + 9x + 7 = 0$, then we have

$$x = \frac{-9 \pm \sqrt{81 - 56}}{4} = \frac{-9 \pm \sqrt{25}}{4} = \frac{-9 \pm 5}{4}$$

$$\therefore\ x = -\frac{4}{4} \text{ or } -\frac{14}{4}$$

$$\therefore\ x = -1 \text{ or } -3{\cdot}5$$

▶

That was straightforward enough, but if we solve the equation $5x^2 - 6x + 5 = 0$ in the same way, we get

$$x = \frac{6 \pm \sqrt{36 - 100}}{10} = \frac{6 \pm \sqrt{-64}}{10}$$

and the next stage is now to determine the square root of -64.

Is it (a) 8, (b) -8, (c) neither?

3

$\boxed{\text{neither}}$

It is, of course, neither, since $+8$ and -8 are the square roots of 64 and not of -64. In fact $\sqrt{-64}$ cannot be represented by an ordinary number, for there is no real number whose square is a negative quantity.

However, $-64 = -1 \times 64$ and therefore we can write

$$\sqrt{-64} = \sqrt{-1 \times 64} = \sqrt{-1}\sqrt{64} = 8\sqrt{-1}$$
$$\text{i.e. } \sqrt{-64} = 8\sqrt{-1}$$

Of course, we are still faced with $\sqrt{-1}$, which cannot be evaluated as a real number, for the same reason as before, but, if we write the letter j to stand for $\sqrt{-1}$, then $\sqrt{-64} = \sqrt{-1}.8 = j8$.

So although we cannot evaluate $\sqrt{-1}$, we can denote it by j and this makes our working a lot neater.

$$\sqrt{-64} = \sqrt{-1}\sqrt{64} = j8$$
$$\text{Similarly, } \sqrt{-36} = \sqrt{-1}\sqrt{36} = j6$$
$$\sqrt{-7} = \sqrt{-1}\sqrt{7} = j2{\cdot}646$$

So $\sqrt{-25}$ can be written as ....

4

$\boxed{j5}$

We now have a way of finishing off the quadratic equation we started in Frame 2.

$$5x^2 - 6x + 5 = 0 \quad \therefore \; x = \frac{6 \pm \sqrt{36 - 100}}{10} = \frac{6 \pm \sqrt{-64}}{10}$$
$$\therefore \; x = \frac{6 \pm j8}{10} \quad \therefore \; x = 0{\cdot}6 \pm j0{\cdot}8$$
$$\therefore \; x = 0{\cdot}6 + j0{\cdot}8 \text{ or } x = 0{\cdot}6 - j0{\cdot}8$$

We will talk about results like these later.

For now, on to Frame 5

Powers of j

5

Because j stands for $\sqrt{-1}$, let us consider some powers of j.

$$\begin{array}{ll} j = \sqrt{-1} & j = \sqrt{-1} \\ j^2 = -1 & j^2 = -1 \\ j^3 = (j^2)j = -1.j = -j & j^3 = -j \\ j^4 = (j^2)^2 = (-1)^2 = 1 & j^4 = 1 \end{array}$$

Note especially the last result: $j^4 = 1$. Every time a factor j^4 occurs, it can be replaced by the factor 1, so that the power of j is reduced to one of the four results above:

$$\begin{aligned} \text{e.g.}\quad j^9 &= (j^4)^2 j = (1)^2 j = 1.j = j \\ j^{20} &= (j^4)^5 = (1)^5 = 1 \\ j^{30} &= (j^4)^7 j^2 = (1)^7(-1) = 1(-1) = -1 \\ \text{and}\quad j^{15} &= (j^4)^3 j^3 = 1(-j) = -j \end{aligned}$$

So, in the same way, $j^5 = \dots\dots\dots\dots$

6

$$\boxed{j}$$

Because $j^5 = (j^4)j = 1.j = j$

Every one is done in the same way:

$$\begin{aligned} j^6 &= (j^4)j^2 = 1(j^2) = 1(-1) = -1 \\ j^7 &= (j^4)j^3 = 1(-j) = -j \\ j^8 &= (j^4)^2 = (1)^2 = 1 \end{aligned}$$

So

(a) $j^{42} = \dots\dots\dots\dots$ (b) $j^{12} = \dots\dots\dots\dots$

(c) $j^{11} = \dots\dots\dots\dots$ (d) If $x^2 - 6x + 34 = 0$, $x = \dots\dots\dots\dots$

7

$$\boxed{\text{(a) } -1 \quad \text{(b) } 1 \quad \text{(c) } -j \quad \text{(d) } x = 3 \pm j5}$$

The working in (d) is as follows:

$$x^2 - 6x + 34 = 0 \;\therefore\; x = \frac{6 \pm \sqrt{36 - 136}}{2} = \frac{6 \pm \sqrt{-100}}{2}$$

$$\therefore\; x = \frac{6 \pm j10}{2} = 3 \pm j5$$

i.e. $x = 3 + j5$ or $x = 3 - j5$

So remember, to simplify powers of j, we take out the highest power of j^4 that we can, and the result must then simplify to one of the four results: $j, -1, -j, 1$.

Move on now to Frame 8

Complex numbers

8

The result $x = 3 + j5$ that we obtained consists of two separate terms, 3 and $j5$. These terms cannot be combined any further, since the second is not a real number (owing to its having the factor j).

In such as expression as $x = 3 + j5$:

3 is called the *real part* of x

5 is called the *imaginary part* of x

and the two together form what is called a *complex number*.

So, a Complex number = (Real part) + j(Imaginary part)

In the complex number $2 + j7$, the real part = ...

and the imaginary part = ...

9

real part = 2; imaginary part = 7 (*not* $j7$!)

Complex numbers have many applications in engineering. To use them, we must know how to carry out the usual arithmetical operations.

1 Addition and subtraction of complex numbers

This is easy, as a few examples will show:

$(4 + j5) + (3 - j2)$

Although the real and imaginary parts cannot be combined, we can remove the brackets and total up terms of the same kind:

$$\begin{aligned}(4 + j5) + (3 - j2) &= 4 + j5 + 3 - j2 = (4 + 3) + j(5 - 2)\\ &= 7 + j3\end{aligned}$$

Another example:

$$\begin{aligned}(4 + j7) - (2 - j5) &= 4 + j7 - 2 + j5 = (4 - 2) + j(7 + 5)\\ &= 2 + j12\end{aligned}$$

So in general, $(a + jb) + (c + jd) = (a + c) + j(b + d)$

Now you do this one:

$(5 + j7) + (3 - j4) - (6 - j3) =$ ....

10

$2 + j6$

since $(5 + j7) + (3 - j4) - (6 - j3)$

$= 5 + j7 + 3 - j4 - 6 + j3$

$= (5 + 3 - 6) + j(7 - 4 + 3) = 2 + j6$

Now you do these in the same way:

(a) $(6 + j5) - (4 - j3) + (2 - j7) =$ ....

and (b) $(3 + j5) - (5 - j4) - (-2 - j3) =$ ....

11

$\boxed{\text{(a) } 4 + j \qquad \text{(b) } j12}$

Here is the working:

(a) $(6 + j5) - (4 - j3) + (2 - j7)$
$= 6 + j5 - 4 + j3 + 2 - j7$
$= (6 - 4 + 2) + j(5 + 3 - 7) = 4 + j$

(b) $(3 + j5) - (5 - j4) - (-2 - j3)$
$= 3 + j5 - 5 + j4 + 2 + j3$ (Take care with signs!)
$= (3 - 5 + 2) + j(5 + 4 + 3)$
$= 0 + j12 = j12$

This is very easy then, so long as you remember that the real and the imaginary parts must be treated quite separately – just like x's and y's in an algebraic expression.

On to Frame 12

12

2 Multiplication of complex numbers

Take as an example: $(3 + j4)(2 + j5)$

These are multiplied together in just the same way as you would determine the product $(3x + 4y)(2x + 5y)$.

$$(3 + j4)(2 + j5)$$

Form the product terms of

(a) the two left-hand terms
(b) the two inner terms
(c) the two outer terms
(d) the two right-hand terms

$$= 6 + j8 + j15 + j^2 20$$
$$= 6 + j23 - 20 \quad (\text{since } j^2 = -1)$$
$$= -14 + j23$$

Likewise, $(4 - j5)(3 + j2)$ . . . .

13

$\boxed{22 - j7}$

Because

$$(4 - j5)(3 + j2) = 12 - j15 + j8 - j^2 10$$
$$= 12 - j7 + 10 \quad (j^2 = -1)$$
$$= 22 - j7$$

If the expression contains more than two factors, we multiply the factors together in stages:

$$\begin{aligned}(3+j4)(2-j5)(1-j2) &= (6+j8-j15-j^2 20)(1-j2)\\ &= (6-j7+20)(1-j2)\\ &= (26-j7)(1-j2)\\ &= \dots\dots\dots\dots\end{aligned}$$

Finish it off

14

$$\boxed{12-j59}$$

Because

$$\begin{aligned}&(26-j7)(1-j2)\\ &\quad = 26-j7-j52+j^2 14\\ &\quad - 26-j59-14-12-j59\end{aligned}$$

Note that when we are dealing with complex numbers, the result of our calculations is also, in general, a complex number.

Now you do this one on your own.

$$(5+j8)(5-j8) = \dots\dots\dots\dots$$

15

$$\boxed{89}$$

Here it is:

$$\begin{aligned}(5+j8)(5-j8) &= 25+j40-j40-j^2 64\\ &= 25+64\\ &= 89\end{aligned}$$

In spite of what we said above, here we have a result containing no j term. The result is therefore entirely real.

This is rather an exceptional case. Look at the two complex numbers we have just multiplied together. Can you find anything special about them? If so, what is it?

When you have decided, move on to the next frame

16

They are identical except for the middle sign in the brackets, i.e. $(5 + j8)$ and $(5 - j8)$

A pair of complex numbers like these are called *conjugate* complex numbers and *the product of two conjugate complex numbers is always entirely real.*

Look at it this way:

$(a + b)(a - b) = a^2 - b^2$ Difference of two squares

Similarly

$$\begin{aligned}(5 + j8)(5 - j8) &= 5^2 - (j8)^2 = 5^2 - j^2 8^2 \\ &= 5^2 + 8^2 \quad (j^2 = -1) \\ &= 25 + 64 = 89\end{aligned}$$

Without actually working it out, will the product of $(7 - j6)$ and $(4 + j3)$ be

(a) a real number
(b) an imaginary number
(c) a complex number?

17

A complex number

since $(7 - j6)(4 + j3)$ is a product of two complex numbers which are *not* conjugate complex numbers or multiples of conjugates.

Remember: Conjugate complex numbers are identical except for the signs in the middle of the brackets.

	$(4 + j5)$ and $(4 - j5)$	*are*	conjugate complex numbers
	$(a + jb)$ and $(a - jb)$	*are*	conjugate complex numbers
but	$(6 + j2)$ and $(2 + j6)$	*are not*	conjugate complex numbers
	$(5 - j3)$ and $(-5 + j3)$	*are not*	conjugate complex numbers

So what must we multiply $(3 - j2)$ by, to produce a result that is entirely real?

18

$(3 + j2)$ or a multiple of it.

because the conjugate of $(3 - j2)$ is identical to it, except for the middle sign, i.e. $(3 + j2)$, and we know that the product of two *conjugate* complex numbers is always real.

Here are two examples:

$$\begin{aligned}(3 - j2)(3 + j2) &= 3^2 - (j2)^2 = 9 - j^2 4 \\ &= 9 + 4 = 13\end{aligned}$$

$$\begin{aligned}(2 + j7)(2 - j7) &= 2^2 - (j7)^2 = 4 - j^2 49 \\ &= 4 + 49 = 53\end{aligned}$$

... and so on.

Complex numbers of the form $(a + jb)$ and $(a - jb)$ are called .... complex numbers.

19

$\boxed{\text{conjugate}}$

Now you should have no trouble with these:

(a) Write down the following products

(i) $(4 - j3)(4 + j3)$ (ii) $(4 + j7)(4 - j7)$

(iii) $(a + jb)(a - jb)$ (iv) $(x - jy)(x + jy)$

(b) Multiply $(3 - j5)$ by a suitable factor to give a product that is entirely real.

When you have finished, move on to Frame 20

20

Here are the results in detail:

(a) (i) $(4 - j3)(4 + j3) = 4^2 - j^2 3^2 = 16 + 9 = \boxed{25}$

(ii) $(4 + j7)(4 - j7) = 4^2 - j^2 7^2 = 16 + 49 = \boxed{65}$

(iii) $(a + jb)(a - jb) = a^2 - j^2 b^2 = \boxed{a^2 + b^2}$

(iv) $(x - jy)(x + jy) = x^2 - j^2 y^2 = \boxed{x^2 + y^2}$

(b) To obtain a real product, we can multiply $(3 - j5)$ by its conjugate, i.e. $(3 + j5)$, giving:

$(3 - j5)(3 + j5) = 3^2 - j^2 5^2 = 9 + 25 = \boxed{34}$

Now move on to the next frame for a short revision exercise

Revision exercise

21

1 Simplify (a) j^{12} (b) j^{10} (c) j^{23}

2 Simplify:

(a) $(5 - j9) - (2 - j6) + (3 - j4)$

(b) $(6 - j3)(2 + j5)(6 - j2)$

(c) $(4 - j3)^2$

(d) $(5 - j4)(5 + j4)$

3 Multiply $(4 - j3)$ by an appropriate factor to give a product that is entirely real. What is the result?

When you have completed this exercise, move on to Frame 22

22

Here are the results. Check yours.

1 (a) $j^{12} = (j^4)^3 = 1^3 = \boxed{1}$

(b) $j^{10} = (j^4)^2 j^2 = 1^2(-1) = \boxed{-1}$

(c) $j^{23} = (j^4)^5 j^3 = j^3 = \boxed{-j}$

▶

2 (a) $(5 - j9) - (2 - j6) + (3 - j4)$

$= 5 - j9 - 2 + j6 + 3 - j4$

$= (5 - 2 + 3) + j(6 - 9 - 4) = \boxed{6 - j7}$

(b) $(6 - j3)(2 + j5)(6 - j2)$

$= (12 - j6 + j30 - j^2 15)(6 - j2)$

$= (27 + j24)(6 - j2)$

$= 162 + j144 - j54 + 48 = \boxed{210 + j90}$

(c) $(4 - j3)^2 = 16 - j24 - 9$

$= \boxed{7 - j24}$

(d) $(5 - j4)(5 + j4)$

$= 25 - j^2 16 = 25 + 16 = \boxed{41}$

3 A suitable factor is the conjugate of the given complex number:

$(4 - j3)(4 + j3) = 16 + 9 = \boxed{25}$

All correct? Right.

Now move on to the next frame to continue the Programme

23

Now let us deal with division.

Division of a complex number by a real number is easy enough:

$$\frac{5 - j4}{3} = \frac{5}{3} - j\frac{4}{3} = 1{\cdot}67 - j1{\cdot}33$$

But how do we manage with $\dfrac{7 - j4}{4 + j3}$?

If we could, somehow, convert the denominator into a real number, we could divide out as in the example above. So our problem is really, how can we convert $(4 + j3)$ into a completely real denominator – and this is where our last piece of work comes in.

We know that we can convert $(4 + j3)$ into a completely real number by multiplying it by its ....

24

$\boxed{\text{conjugate}}$

i.e. the same complex number but with the opposite sign in the middle, in this case $(4 - j3)$.

But if we multiply the denominator by $(4-j3)$, we must also multiply the numerator by the same factor:

$$\frac{7-j4}{4+j3}=\frac{(7-j4)(4-j3)}{(4+j3)(4-j3)}=\frac{28-j37-12}{16+9}=\frac{16-j37}{25}$$
$$\frac{16}{25}-j\frac{37}{25}=0{\cdot}64-j1{\cdot}48$$

and the job is done. To divide one complex number by another, therefore, we multiply numerator and denominator by the conjugate of the denominator. This will convert the denominator into a real number and the final step can then be completed.

Thus, to simplify $\dfrac{4-j5}{1+j2}$, we shall multiply top and bottom by ....

25

the conjugate of the denominator, i.e. $(1-j2)$

If we do that, we get:

$$\frac{4-j5}{1+j2}=\frac{(4-j5)(1-j2)}{(1+j2)(1-j2)}=\frac{4-j13-10}{1+4}$$
$$=\frac{-6-j13}{5}=\frac{-6}{5}-j\frac{13}{5}=-1{\cdot}2-j2{\cdot}6$$

Now here is one for you to do: Simplify $\dfrac{3+j2}{1-j3}$

When you have done it, move on to the next frame

26

$-0{\cdot}3+j1{\cdot}1$

Because

$$\frac{3+j2}{1-j3}=\frac{(3+j2)(1+j3)}{(1-j3)(1+j3)}=\frac{3+j11-6}{1+9}=\frac{-3+j11}{10}=-0{\cdot}3+j1{\cdot}1$$

Now do these in the same way:

(a) $\dfrac{4-j5}{2-j}$ (b) $\dfrac{3+j5}{5-j3}$ (c) $\dfrac{(2+j3)(1-j2)}{3+j4}$

When you have worked these, move on to Frame 27 to check your results

---

27

Here are the solutions in detail:

(a) $\dfrac{4-j5}{2-j}=\dfrac{(4-j5)(2+j)}{(2-j)(2+j)}=\dfrac{8-j6+5}{4+1}=\dfrac{13-j6}{5}=\boxed{2{\cdot}6-j1{\cdot}2}$

(b) $\dfrac{3+j5}{5-j3}=\dfrac{(3+j5)(5+j3)}{(5-j3)(5+j3)}=\dfrac{15+j34-15}{25+9}=\dfrac{j34}{34}=\boxed{j}$

▶

(c) $\dfrac{(2+j3)(1-j2)}{(3+j4)} = \dfrac{2-j+6}{3+j4} = \dfrac{8-j}{3+j4}$

$$= \frac{(8-j)(3-j4)}{(3+j4)(3-j4)}$$

$$= \frac{24-j35-4}{9+16} = \frac{20-j35}{25}$$

$$= \boxed{0{\cdot}8 - j1{\cdot}4}$$

And now you know how to apply the four rules to complex numbers

Equal complex numbers

28

Now let us see what we can find out about two complex numbers which we are told are equal.

Let the numbers be	$a+jb$ and $c+jd$
Then we have	$a+jb=c+jd$
Rearranging terms, we get	$a-c=j(d-b)$

In this last statement, the quantity on the left-hand side is entirely real, while that on the right-hand side is entirely imaginary, i.e. a real quantity equals an imaginary quantity! This seems contradictory and in general it just cannot be true. But there is *one* special case for which the statement can be true. That is when ....

29

$$\boxed{\text{each side is zero}}$$

$a-c=j(d-b)$

can be true only if $a-c=0$, i.e. $a=c$
and if $d-b=0$, i.e. $b=d$

So we get this important result:
If two complex numbers are equal

(a) the two real parts are equal

(b) the two imaginary parts are equal

For example, if $x+jy=5+j4$, then we know $x=5$ and $y=4$
and if $a+jb=6-j3$, then $a=$ .... and $b=$ ....

$\boxed{a = 6 \text{ and } b = -3}$ 30

Be careful to include the sign!

Now what about this one?

If $(a+b)+j(a-b) = 7 + j2$, find the values of a and b.

Well now, following our rule about two equal complex numbers, what can we say about $(a+b)$ and $(a-b)$?

$\boxed{a + b = 7 \text{ and } a - b = 2}$ 31

since the two real parts are equal and the two imaginary parts are equal.

This gives you two simultaneous equations, from which you can determine the values of a and b.

So what are they?

$\boxed{a = 4{\cdot}5;\ b = 2{\cdot}5}$ 32

For $\left.\begin{array}{l} a+b=7 \\ a-b=2 \end{array}\right\} \begin{array}{ll} 2a = 9 & \therefore\ a = 4{\cdot}5 \\ 2b = 5 & \therefore\ b = 2{\cdot}5 \end{array}$

We see then that an equation involving complex numbers leads to a pair of simultaneous equations by putting

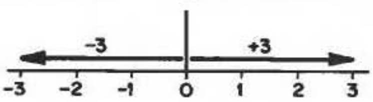
(a) the two real parts equal
(b) the two imaginary parts equal

This is quite an important point to remember.

Graphical representation of a complex number

Although we cannot evaluate a complex number as a real number, we can represent it diagrammatically, as we shall now see. 33

In the usual system of plotting numbers, the number 3 could be represented by a line from the origin to the point 3 on the scale. Likewise, a line to represent (-3) would be drawn from the origin to the point (-3). These two lines are equal in length but are drawn in opposite directions. Therefore, we put an arrowhead on each to distinguish between them.



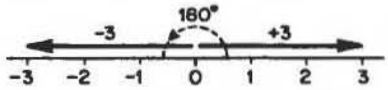
A line which represents a magnitude (by its length) and direction (by the arrowhead) is called a vector. We shall be using this word quite a lot.

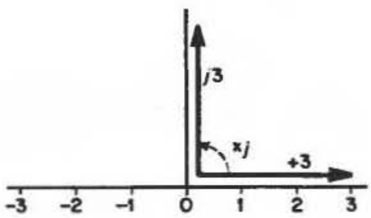
Any vector therefore must include both magnitude (or size) and ....

34

direction

If we multiply (+3) by the factor (−1), we get (−3), i.e. the factor (−1) has the effect of turning the vector through 180°



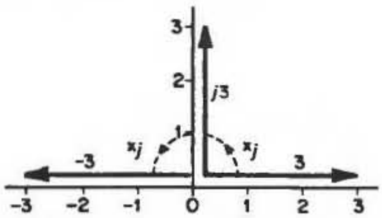


Multiplying by (−1) is equivalent to multiplying by j^2, i.e. by the factor j twice. Therefore multiplying by a single factor j will have half the effect and rotate the vector through only ....°

35

90°

The factor j always turns a vector through 90° in the positive direction of measuring angles, i.e. anticlockwise.



If we now multiply $j3$ by a further factor j, we get $j^2 3$, i.e. (−3) and the diagram agrees with this result.

If we multiply (−3) by a further factor j, sketch the new position of the vector on a similar diagram.

The result is given in the next frame

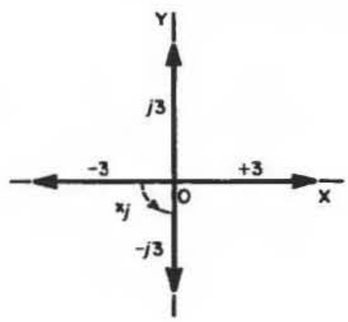
36

Let us denote the two reference lines by the X-axis and the Y-axis, as usual.

You will see that

(i) The scale on the X-axis represents real numbers. This is therefore called the *real axis*.

(ii) The scale on the Y-axis represents imaginary numbers. This is therefore called the *imaginary axis*.

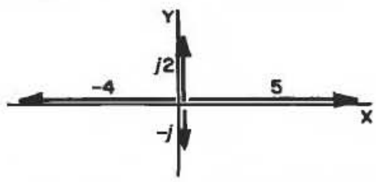


On a similar diagram, sketch vectors to represent:

(a) 5 (b) − 4 (c) $j2$ (d) $-j$

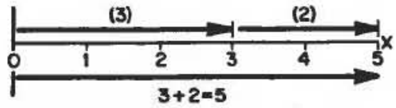
Results in the next frame

37



Check that each of your vectors carries an arrowhead to show direction.

If we now wish to represent $3+2$ as the sum of two vectors, we must draw them as a chain, the second vector starting where the first one finishes.

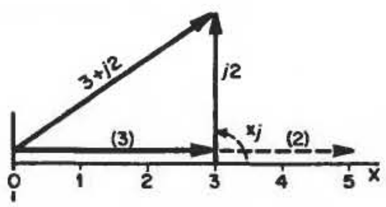


The two vectors, 3 and 2, are together equivalent to a single vector drawn from the origin to the end of the final vector (giving naturally that $3+2=5$).

Continue to the next frame

38

If we wish to represent the complex number $(3+j2)$, then we add together the vectors which represent 3 and $j2$.



Notice that the 2 is now multiplied by a factor j which turns that vector through 90°.

The equivalent single vector to represent $(3+j2)$ is therefore the vector from the beginning of the first vector (origin) to the end of the last one.

This graphical representation constitutes an *Argand diagram.*

Draw an Argand diagram to represent the vectors:

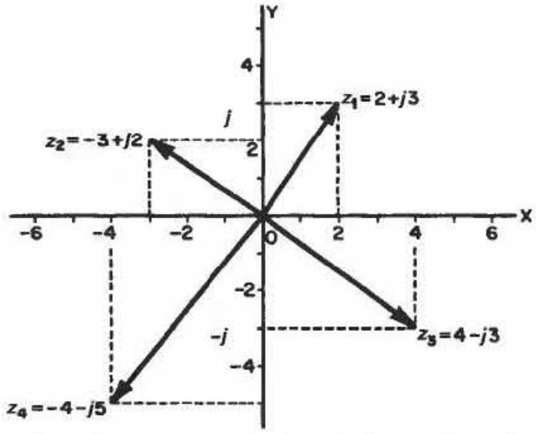
(a) $z_1 = 2 + j3$ (b) $z_2 = -3 + j2$

(c) $z_3 = 4 - j3$ (d) $z_4 = -4 - j5$

Label each one clearly

39

Here they are. Check yours.



Note once again that the end of each vector is plotted very much like plotting x and y coordinates.

The real part corresponds to the x-value.

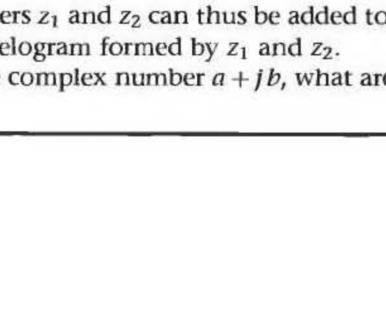
The imaginary part corresponds to the y-value.

Move on to Frame 40

Graphical addition of complex numbers

40

Let us find the sum of $z_1 = 5 + j2$ and $z_2 = 2 + j3$ by an Argand diagram. If we are adding vectors, they must be drawn as a chain. We therefore draw at the end of z_1, a vector AP representing z_2 in magnitude and direction, i.e. AP = OB and is parallel to it. Therefore OAPB is a parallelogram. Thus the sum of z_1 and z_2 is given by the vector joining the starting point to the end of the last vector, i.e. OP.



The complex numbers z_1 and z_2 can thus be added together by drawing the diagonal of the parallelogram formed by z_1 and z_2.

If OP represents the complex number $a + jb$, what are the values of a and b in this case?

41

$$\boxed{a = 5 + 2 = 7 \qquad b = 2 + 3 = 5}$$

$\therefore$ OP $= z = 7 + j5$

You can check this result by adding $(5 + j2)$ and $(2 + j3)$ algebraically.

So the sum of the two vectors on an Argand diagram is given by the .... of the parallelogram of vectors.

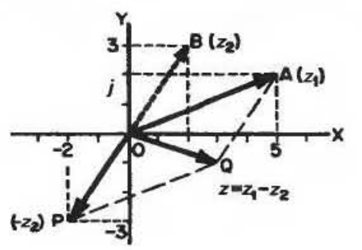
42

$$\boxed{\text{diagonal}}$$

How do we do subtraction by similar means? We do this rather craftily without learning any new methods. The trick is simply this:

$z_1 - z_2 = z_1 + (-z_2)$

That is, we draw the vector representing z_1 and the *negative* vector of z_2 and add them as before. The negative vector of z_2 is simply a vector with the same magnitude (or length) as z_2 but pointing in the opposite direction.



e.g. If $z_1 = 5 + j2$ and $z_2 = 2 + j3$

vector OA $= z_1 = 5 + j2$

OP $= -z_2 = -(2 + j3)$

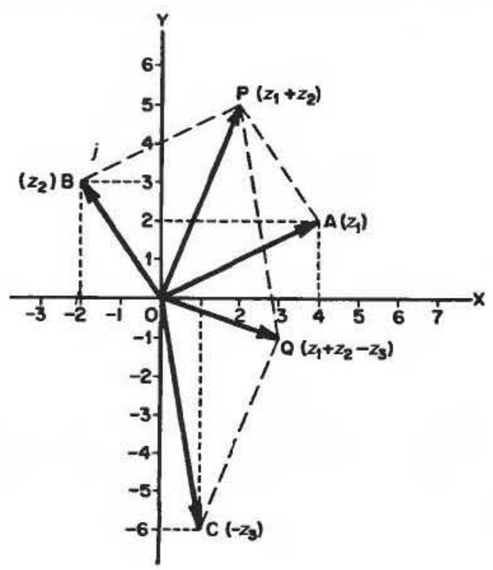
Then OQ $= z_1 + (-z_2)$

$= z_1 - z_2$

Determine on an Argand diagram

$(4 + j2) + (-2 + j3) - (-1 + j6)$

43



OA $= z_1 = 4 + j2$

OB $= z_2 = -2 + j3$

OC $= -z_3 = 1 - j6$

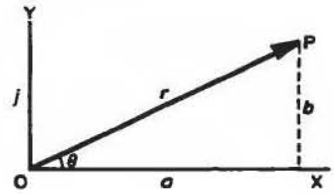
Then OP $= z_1 + z_2$

OQ $= z_1 + z_2 - z_3 = 3 - j$

Polar form of a complex number

44

It is convenient sometimes to express a complex number $a + jb$ in a different form. On an Argand diagram, let OP be a vector $a + jb$. Let r = length of the vector and θ the angle made with OX.



Then $\quad r^2 = a^2 + b^2 \qquad r = \sqrt{a^2 + b^2}$

and $\quad \tan\theta = \frac{b}{a} \qquad \theta = \tan^{-1}\frac{b}{a}$

Also $\quad a = r\cos\theta$ and $b = r\sin\theta$

Since $z = a + jb$, this can be written

$$z = r\cos\theta + jr\sin\theta \qquad \text{i.e. } z = r(\cos\theta + j\sin\theta)$$

This is called the *polar form* of the complex number $a + jb$, where

$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\frac{b}{a}$$

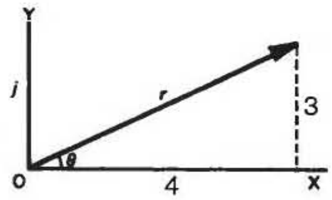
Let us take a numerical example.

Next frame

45

To express $z = 4 + j3$ in polar form.

First draw a sketch diagram (that always helps).



We can see that:

(a) $r^2 = 4^2 + 3^2 = 16 + 9 = 25$

$r = 5$

(b) $\tan\theta = \frac{3}{4} = 0{\cdot}75$

$\theta = 36°52'$

$$z = a + jb = r(\cos\theta + j\sin\theta)$$

So in this case $z = 5(\cos 36°52' + j\sin 36°52')$

Now here is one for you to do.

Find the polar form of the complex number $(2 + j3)$.

When you have finished it, consult the next frame

46

$\boxed{z = 3{\cdot}606(\cos 56°19' + j\sin 56°19')}$

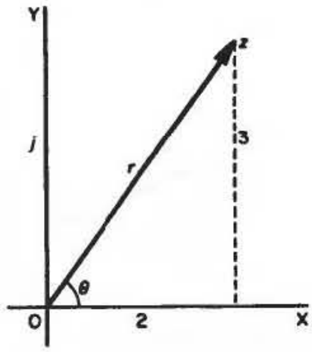
Here is the working:

$z = 2 + j3 = r(\cos\theta + j\sin\theta)$

$r^2 = 4 + 9 = 13 \qquad r = 3{\cdot}606$

$\tan\theta = \dfrac{3}{2} = 1{\cdot}5 \qquad \theta = 56°19'$

$z = 3{\cdot}606(\cos 56°19' + j\sin 56°19')$



We have special names for the values of r and θ:

$z = a + jb = r(\cos\theta + j\sin\theta)$

(a) r is called the *modulus* of the complex number z and is often abbreviated to 'mod z' or indicated by $|z|$.

Thus if $z = 2 + j5$, then $|z| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$

(b) θ is called the *argument* of the complex number and can be abbreviated to 'arg z'.

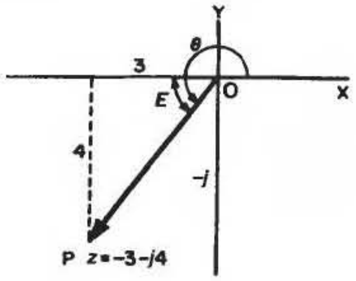
So if $z = 2 + j5$, then $\arg z = \ldots\ldots\ldots\ldots$

47

$\boxed{\arg z = 68°12'}$

$z = 2 + j5$. Then $\arg z = \theta = \tan^{-1}\dfrac{5}{2} = 68°12'$

Warning: In finding θ, there are of course two angles between 0° and 360°, the tangent of which has the value $\dfrac{b}{a}$. We must be careful to use the angle in the correct quadrant. *Always* draw a sketch of the vector to ensure you have the right one.



e.g. Find arg z when $z = -3 - j4$

θ is measured from OX to OP. We first find E, the equivalent acute angle from the triangle shown:

$\tan E = \dfrac{4}{3} = 1{\cdot}333 \qquad \therefore\ E = 53°8'$

Then in this case:

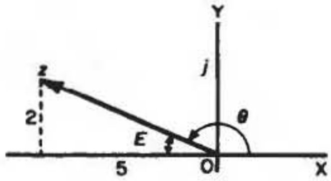
$\theta = 180° + E = 233°8' \quad \arg z = 233°8'$

Now you find $\arg(-5 + j2)$

Move on when you have finished

48

$$\boxed{\arg z = 158^\circ 12'}$$



$$z = -5 + j2$$

$$\tan E = \frac{2}{5} = 0{\cdot}4 \qquad \therefore\ E = 21^\circ 48'$$

In this particular case, $\theta = 180^\circ - E$

$$\therefore\ \theta = 158^\circ 12'$$

Complex numbers in polar form are always of the same shape and differ only in the actual values of r and θ. We often use the shorthand version $r\underline{|\theta}$ to denote the polar form.

e.g. If $z = -5 + j2$, $r = \sqrt{25+4} = \sqrt{29} = 5{\cdot}385$ and from above $\theta = 158^\circ 12'$

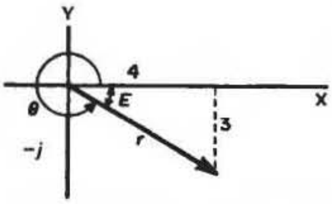
$\therefore$ The full polar form is $z = 5{\cdot}385(\cos 158^\circ 12' + j\sin 158^\circ 12')$ and this can be shortened to $z = 5{\cdot}385\underline{|158^\circ 12'}$

Express in shortened form, the polar form of $(4 - j3)$

Do not forget to draw a sketch diagram first.

49

$$\boxed{z = 5\underline{|323^\circ 8'}}$$



$$r = \sqrt{4^2 + 3^2} \qquad r = 5$$

$$\tan E = 0{\cdot}75 \qquad \therefore\ E = 36^\circ 52'$$

$$\therefore\ \theta = 360^\circ - E = 323^\circ 8'$$

$$\therefore\ z = 5(\cos 323^\circ 8' + j\sin 323^\circ 8')$$

$$= 5\underline{|323^\circ 8'}$$

Of course, given a complex number in polar form, you can convert it into basic form $a + jb$ simply by evaluating the cosine and the sine and multiplying by the value of r.

e.g. $z = 5(\cos 35^\circ + j\sin 35^\circ) = 5(0{\cdot}8192 + j0{\cdot}5736)$

$z = 4{\cdot}0960 + j2{\cdot}8680$

Now you do this one.

Express in the form $a + jb$, $4(\cos 65^\circ + j\sin 65^\circ)$

50

$$\boxed{z = 1{\cdot}6905 + j3{\cdot}6252}$$

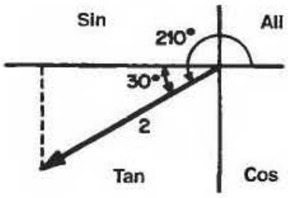
Because

$$z = 4(\cos 65^\circ + j\sin 65^\circ) = 4(0{\cdot}4226 + j0{\cdot}9063) = 1{\cdot}6905 + j3{\cdot}6252$$

If the argument is greater than 90°, care must be taken in evaluating the cosine and sine to include the appropriate signs.

▶

e.g. If $z = 2(\cos 210° + j\sin 210°)$ the vector lies in the third quadrant.



$\cos 210° = -\cos 30°$
$\sin 210° = -\sin 30°$
Then $z = 2(-\cos 30° - j\sin 30°)$
$= 2(-0{\cdot}8660 - j0{\cdot}5)$
$= -1{\cdot}732 - j$

Here you are. What about this one?

Express $z = 5(\cos 140° + j\sin 140°)$ in the form $a + jb$

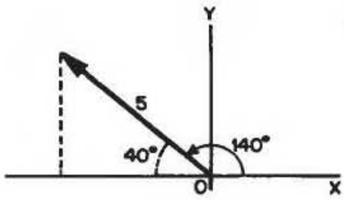
What do you make it?

51

$\boxed{z = -3{\cdot}8300 + j3{\cdot}2140}$

Here are the details:

$\cos 140° = -\cos 40°$
$\sin 140° = \sin 40°$



$z = 5(\cos 140° + j\sin 140°)$
$= 5(-\cos 40° + j\sin 40°)$
$= 5(-0{\cdot}7660 + j0{\cdot}6428)$
$= -3{\cdot}830 + j3{\cdot}214$

Fine. Now by way of revision, work out the following:

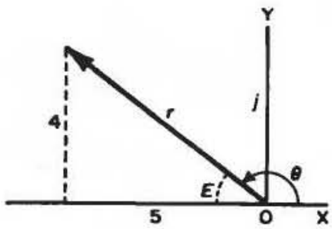
(a) Express $-5 + j4$ in polar form
(b) Express $3\underline{|300°}$ in the form $a + jb$

When you have finished both of them, check your results with those in Frame 52

52

Here is the working:

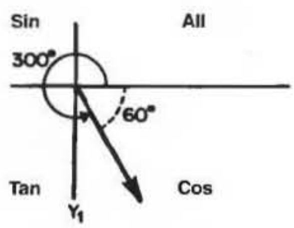
(a) $r^2 = 4^2 + 5^2 = 16 + 25 = 41$
$\therefore\ r = 6{\cdot}403$
$\tan E = 0{\cdot}8 \quad \therefore\ E = 38°40'$
$\therefore\ \theta = 141°20'$



$-5 + j4 = 6{\cdot}403(\cos 141°20' + j\sin 141°20') = \boxed{6{\cdot}403\underline{|141°20'}}$

▶

(b) $3\angle 300^\circ = 3(\cos 300^\circ + j\sin 300^\circ)$
$\cos 300^\circ = \cos 60^\circ$
$\sin 300^\circ = -\sin 60^\circ$
$3\angle 300^\circ = 3(\cos 60^\circ - j\sin 60^\circ)$
$= 3(0{\cdot}500 - j0{\cdot}866)$
$= \boxed{1{\cdot}500 - j2{\cdot}598}$



Move to Frame 53

53

We see then that there are two ways of expressing a complex number:

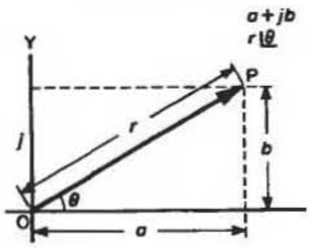
(a) in standard form $z = a + jb$

(b) in polar form $z = r(\cos\theta + j\sin\theta)$

where $r = \sqrt{a^2 + b^2}$

and $\theta = \tan^{-1}\dfrac{b}{a}$

If we remember the simple diagram, we can easily convert from one system to the other:



So on now to Frame 54

Exponential form of a complex number

54

There is still another way of expressing a complex number which we must deal with, for it too has its uses. We shall arrive at it this way:

Many functions can be expressed as series. for example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\ \dots\ \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots\ \dots\ \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\ \dots\ \dots$$

You no doubt have hazy recollections of these series. You had better make a note of them since they have turned up here.

55

If we now take the series for e^x and write $j\theta$ in place of x, we get:

$$\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \dots\dots\dots \\
&= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \dots\dots\dots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \dots\dots\dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\dots\dots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\dots\dots\right) \\
&= \cos\theta + j\sin\theta
\end{aligned}$$

Therefore, $r(\cos\theta + j\sin\theta)$ can now be written as $re^{j\theta}$. This is called the *exponential form* of the complex number. It can be obtained from the polar quite easily since the r value is the same and the angle θ is the same in both. It is important to note, however, that in the exponential form, the angle must be in *radians*.

Move on to the next frame

56

The three ways of expressing a complex number are therefore:

(a) $z = a + jb$
(b) $z = r(\cos\theta + j\sin\theta)$ Polar form
(c) $z = r.e^{j\theta}$ Exponential form

Remember that the exponential form is obtained from the polar form:

(a) the r value is the same in each case
(b) the angle is also the same in each case, but in the exponential form the angle must be in radians.

So, knowing that, change the polar form $5(\cos 60^\circ + j\sin 60^\circ)$ into the exponential form.

Then turn to Frame 57

57

$$\boxed{5e^{j\frac{\pi}{3}}}$$

Because we have $5(\cos 60^\circ + j\sin 60^\circ)$ $\quad r = 5$

$$\theta = 60^\circ = \frac{\pi}{3} \text{ radians}$$

$\therefore$ Exponential form is $5e^{j\frac{\pi}{3}}$

And now a word about negative angles:

We know $e^{j\theta} = \cos\theta + j\sin\theta$

If we replace θ by $-\theta$ in this result, we get

$$\begin{aligned}
e^{-j\theta} &= \cos(-\theta) + j\sin(-\theta) \\
&= \cos\theta - j\sin\theta
\end{aligned}$$

▶

So we have

$$\left.\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned}\right\}$$

Make a note of these

58

There is one operation that we have been unable to carry out with complex numbers before this. That is to find the logarithm of a complex number. The exponential form now makes this possible, since the exponential form consists only of products and powers.

For, if we have:

$$z = re^{j\theta}$$

then we can say:

$$\ln z = \ln r + j\theta$$

e.g. If $z = 6{\cdot}42e^{j1{\cdot}57}$ then

$$\begin{aligned} \ln z &= \ln 6{\cdot}42 + j1{\cdot}57 \\ &= 1{\cdot}8594 + j1{\cdot}57 \end{aligned}$$

and the result is once again a complex number.

And if $z = 3{\cdot}8e^{-j0{\cdot}236}$, then $\ln z =$ ....

59

$$\ln z = \ln 3{\cdot}8 - j0{\cdot}236 = \boxed{1{\cdot}3350 - j0{\cdot}236}$$

Finally, here is an example of a rather different kind. Once you have seen it done, you will be able to deal with others of this kind. Here it is.

Express $e^{1-j\pi/4}$ in the form $a + jb$

Well now, we can write:

$$\begin{aligned} e^{1-j\pi/4} \text{ as } e^1 e^{-j\pi/4} & \\ &= e(\cos\pi/4 - j\sin\pi/4) \\ &= e\left\{\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right\} \\ &= \frac{e}{\sqrt{2}}(1-j) \end{aligned}$$

This brings us to the end of this Programme, except for the **Can You?** checklist and the **Test exercise**. Before you do them, read down the **Revision summary** that follows in the next frame and revise any points on which you are not completely sure.

Move to Frame 60

Revision summary

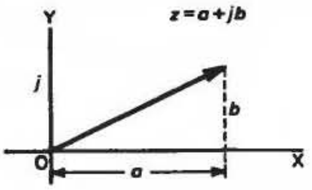
60

1 *Powers of j*

$$j = \sqrt{-1}, \quad j^2 = -1, \quad j^3 = -j, \quad j^4 = 1.$$

A factor j turns a vector through 90° in the positive direction.

2 *Complex numbers*



a =real part
b =imaginary part

3 *Conjugate complex numbers* $(a + jb)$ and $(a - jb)$
The product of two conjugate numbers is always real:

$$(a + jb)(a - jb) = a^2 + b^2$$

4 *Equal complex numbers*

If $a + jb = c + jd$, then $a = c$ and $b = d$.

5 *Polar form of a complex number*

$$\begin{aligned} z &= a + jb \\ &= r(\cos\theta + j\sin\theta) \\ &= r\underline{|\theta} \end{aligned}$$

$$r = \sqrt{a^2 + b^2}; \quad \theta = \tan^{-1}\left\{\frac{b}{a}\right\}$$

also $\quad a = r\cos\theta; \; b = r\sin\theta$

$r =$ the modulus of z written 'mod z' or $|z|$

$\theta =$ the argument of z, written 'arg z'

6 *Exponential form of a complex number*

$$\left.\begin{aligned} z &= r(\cos\theta + j\sin\theta) = re^{j\theta} \\ \text{and} \quad & r(\cos\theta - j\sin\theta) = re^{-j\theta} \end{aligned}\right\} \; \theta \text{ in radians}$$

7 *Logarithm of a complex number*

$$z = re^{j\theta} \quad \therefore \; \ln z = \ln r + j\theta$$

$$\text{or if } \; z = re^{-j\theta} \quad \therefore \; \ln z = \ln r - j\theta$$

Can You?

61 Checklist 1

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Recognize j as standing for $\sqrt{-1}$ and be able to reduce the powers of j to $\pm j$ or ± 1? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 7
• Recognize that all complex numbers are in the form (real part) $+ j$(imaginary part)? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	8
• Add, subtract and multiply complex numbers? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	9 to 15
• Find the complex conjugate of a complex number? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	16 to 20
• Divide complex numbers? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	23 to 27
• State the conditions for the equality of two complex numbers? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	28 to 32
• Draw complex numbers and recognize the parallel law of addition? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	33 to 43
• Convert a complex number from Cartesian to polar form and vice versa? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	44 to 53
• Write a complex number in its exponential form? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	54 to 57
• Obtain the logarithm of a complex number? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	58

Test exercise 1

62

You will find the questions quite straightforward and easy.

1 Simplify: (a) j^3 (b) j^5 (c) j^{12} (d) j^{14}.

2 Express in the form $a + jb$:

(a) $(4 - j7)(2 + j3)$ (b) $(-1 + j)^2$

(c) $(5 + j2)(4 - j5)(2 + j3)$ (d) $\dfrac{4 + j3}{2 - j}$

3 Find the values of x and y that satisfy the equation:
$(x + y) + j(x - y) = 14{\cdot}8 + j6{\cdot}2$

4 Express in polar form:
(a) $3 + j5$ (b) $-6 + j3$ (c) $-4 - j5$

5 Express in the form $a + jb$:
(a) $5(\cos 225^\circ + j \sin 225^\circ)$ (b) $4\angle 330^\circ$

6 Express in exponential form:
(a) $z_1 = 10\angle 37^\circ 15'$ and (b) $z_2 = 10\angle 322^\circ 45'$
Hence find $\ln z_1$ and $\ln z_2$.

7 Express $z = e^{1+j\pi/2}$ in the form $a + jb$.

Now are you ready to start Part 2 of the work on complex numbers

Further problems 1

63

1 Simplify:

(a) $(5 + j4)(3 + j7)(2 - j3)$ (b) $\dfrac{(2 - j3)(3 + j2)}{(4 - j3)}$ (c) $\dfrac{\cos 3x + j \sin 3x}{\cos x + j \sin x}$

2 Express $\dfrac{2 + j3}{j(4 - j5)} + \dfrac{2}{j}$ in the form $a + jb$.

3 If $z = \dfrac{1}{2 + j3} + \dfrac{1}{1 - j2}$, express z in the form $a + jb$.

4 If $z = \dfrac{2 + j}{1 - j}$, find the real and imaginary parts of the complex number $z + \dfrac{1}{z}$.

5 Simplify $(2 + j5)^2 + \dfrac{5(7 + j2)}{3 - j4} - j(4 - j6)$, expressing the result in the form $a + jb$.

▶

6 If $z_1 = 2 + j$, $z_2 = -2 + j4$ and $\dfrac{1}{z_3} = \dfrac{1}{z_1} + \dfrac{1}{z_2}$, evaluate z_3 in the form $a + jb$. If z_1, z_2, z_3 are represented on an Argand diagram by the points P, Q, R, respectively, prove that R is the foot of the perpendicular from the origin on to the line PQ.

7 Points A, B, C, D, on an Argand diagram, represent the complex numbers $9 + j$, $4 + j13$, $-8 + j8$, $-3 - j4$ respectively. Prove that ABCD is a square.

8 If $(2 + j3)(3 - j4) = x + jy$, evaluate x and y.

9 If $(a + b) + j(a - b) = (2 + j5)^2 + j(2 - j3)$, find the values of a and b.

10 If x and y are real, solve the equation:

$$\frac{jx}{1 + jy} = \frac{3x + j4}{x + 3y}$$

11 If $z = \dfrac{a + jb}{c + jd}$, where a, b, c and d are real quantities, show that (a) if z is real then $\dfrac{a}{b} = \dfrac{c}{d}$ and (b) if z is entirely imaginary then $\dfrac{a}{b} = -\dfrac{d}{c}$.

12 Given that $(a + b) + j(a - b) = (1 + j)^2 + j(2 + j)$, obtain the values of a and b.

13 Express $(-1 + j)$ in the form $re^{j\theta}$ where r is positive and $-\pi < \theta < \pi$.

14 Find the modulus of $z = (2 - j)(5 + j12)/(1 + j2)^3$.

15 If x is real, show that $(2 + j)e^{(1+j3)x} + (2 - j)e^{(1-j3)x}$ is also real.

16 Given that $z_1 = R_1 + R + j\omega L$; $z_2 = R_2$; $z_3 = \dfrac{1}{j\omega C_3}$; and $z_4 = R_4 + \dfrac{1}{j\omega C_4}$; and also that $z_1 z_3 = z_2 z_4$, express R and L in terms of the real constants R_1, R_2, R_4, C_3 and C_4.

17 If $z = x + jy$, where x and y are real, and if the real part of $(z + 1)/(z + j)$ is equal to 1, show that the point z lies on a straight line in the Argand diagram.

18 When $z_1 = 2 + j3$, $z_2 = 3 - j4$, $z_3 = -5 + j12$, then $z = z_1 + \dfrac{z_2 z_3}{z_2 + z_3}$. If $E = Iz$, find E when $I = 5 + j6$.

19 If $\dfrac{R_1 + j\omega L}{R_3} = \dfrac{R_2}{R_4 - j\frac{1}{\omega C}}$, where R_1, R_2, R_3, R_4, ω, L and C are real, show that

$$L = \frac{CR_2R_3}{\omega^2 C^2 R_4^2 + 1}$$

20 If z and $\bar{z}$ are conjugate complex numbers, find two complex numbers, $z = z_1$ and $z = z_2$, that satisfy the equation:

$$3z\bar{z} + 2(z - \bar{z}) = 39 + j12$$

On an Argand diagram, these two numbers are represented by the points P and Q. If R represents the number $j1$, show that the angle PRQ is a right angle.

Programme 2

Complex numbers 2

Frames 1 to 60

Learning outcomes

When you have completed this Programme you will be able to:

- Use the shorthand form for a complex number in polar form
- Write complex numbers in polar form using negative angles
- Multiply and divide complex numbers in polar form
- Use DeMoivre's theorem
- Find the roots of a complex number
- Demonstrate trigonometric identities of multiple angles using complex numbers
- Solve loci problems using complex numbers

Introduction

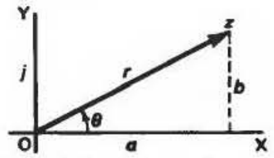
1

In Part 1 of this programme on complex numbers, we discovered how to manipulate them in adding, subtracting, multiplying and dividing. We also finished Part 1 by seeing that a complex number $a + jb$ can also be expressed in polar form, which is always of the form $r(\cos\theta + j\sin\theta)$.

You will remember that values of r and θ can easily be found from the diagram of the given vector:

$$r^2 = a^2 + b^2 \quad \therefore \quad r = \sqrt{a^2 + b^2}$$

$$\text{and } \tan\theta = \frac{b}{a} \quad \therefore \quad \theta = \tan^{-1}\frac{b}{a}$$



To be sure that you have taken the correct value of θ, always *draw a sketch diagram* to see which quadrant the vector is in.

Remember that θ is always measured from ....

2

$\boxed{\text{OX}}$ i.e. the positive axis OX.

Right. As a warming-up exercise, do the following:

Express $z = 12 - j5$ in polar form

Do not forget the sketch diagram. It ensures that you get the correct value for θ.

When you have finished, and not before, move on to Frame 3 to check your result

3

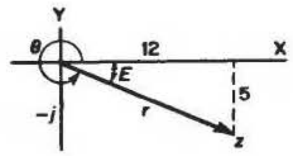
$\boxed{13(\cos 337^\circ 23' + j\sin 337^\circ 23'}$

Here it is, worked out in full:

$$r^2 = 12^2 + 5^2 = 144 + 25 = 169$$

$$\therefore \; r = 13$$

$$\tan E = \frac{5}{12} = 0{\cdot}4167 \quad \therefore \quad E = 22^\circ 37'$$



In this case, $\theta = 360^\circ - E = 360^\circ - 22^\circ 37' \quad \therefore \quad \theta = 337^\circ 23'$

$$z = r(\cos\theta + j\sin\theta) = 13(\cos 337^\circ 23' + j\sin 337^\circ 23')$$

Did you get that right? Here is one more, done in just the same way:

Express $-5 - j4$ in polar form.

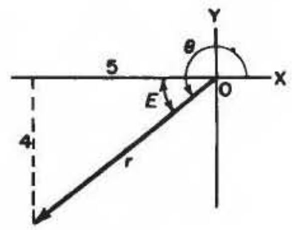
Diagram first of all! Then you cannot go wrong.

When you have the result, on to Frame 4

4

$$\boxed{z = 6{\cdot}403(\cos 218°40' + j\sin 218°40')}$$

Here is the working: check yours.



$r^2 = 5^2 + 4^2 = 25 + 16 = 41$

$\therefore\ r = \sqrt{41} = 6{\cdot}403$

$\tan E = \dfrac{4}{5} = 0{\cdot}8 \ \therefore\ E = 38°40'$

In this case, $\theta = 180° + E = 218°40'$

So $z = -5 - j4 = 6{\cdot}403(\cos 218°40' + j\sin 218°40')$

Since every complex number in polar form is of the same shape, i.e. $r(\cos\theta + j\sin\theta)$ and differs from another complex number simply by the values of r and θ, we have a shorthand method of quoting the result in polar form. Do you remember what it is? The shorthand way of writing the result above, i.e. $6{\cdot}403(\cos 218°40' + j\sin 218°40')$ is ....

5

$$\boxed{6{\cdot}403\lfloor\underline{218°40'}}$$

Correct. Likewise:

$5{\cdot}72(\cos 322°15' + j\sin 322°15')$	is written	$5{\cdot}72\lfloor\underline{322°15'}$
$5(\cos 105° + j\sin 105°)$	is written	$5\lfloor\underline{105°}$
$3{\cdot}4\left(\cos\dfrac{\pi}{6} + j\sin\dfrac{\pi}{6}\right)$	is written	$3{\cdot}4\lfloor\underline{\dfrac{\pi}{6}}$

They are all complex numbers in polar form. They are all the same shape and differ one from another simply by the values of .... and ....

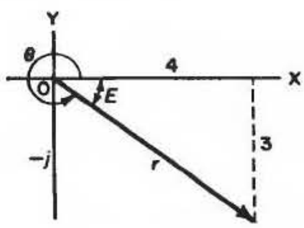
6

$\boxed{r}$ and $\boxed{\theta}$

Now let us consider the following example.

Express $z = 4 - j3$ in polar form.

First the diagram:



From this:

$r = 5$

$\tan E = \dfrac{3}{4} = 0{\cdot}75 \ \therefore\ E = 36°52'$

$\theta = 360° - 36°52' = 323°8'$

$z = 4 - j3 = 5(\cos 323°8' + j\sin 323°8')$

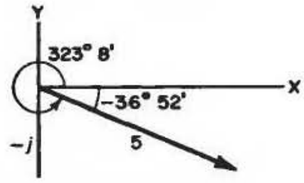
or in shortened form, $z =$ ....

7

$$z = 5\underline{|323^\circ 8'}$$

In the previous example, we have:

$z = 5(\cos 323^\circ 8' + j \sin 323^\circ 8')$



But the direction of the vector, measured from OX, could be given as $-36^\circ 52'$, the minus sign showing that we are measuring the angle in the opposite sense from the usual positive direction.

We could write $z = 5(\cos[-36^\circ 52'] + j \sin[-36^\circ 52'])$. But you alreay know that $\cos[-\theta] = \cos\theta$ and $\sin[-\theta] = -\sin\theta$.

$z = 5(\cos 36^\circ 52' - j \sin 36^\circ 52')$

i.e. very much like the polar form but with a minus sign in the middle. This comes about whenever we use negative angles. In the same way:

$$\begin{aligned} z &= 4(\cos 250^\circ + j \sin 250^\circ) \\ &= 4(\cos[-110^\circ] + j \sin[-110^\circ]) \\ &= 4(\ldots\ldots\ldots\ldots) \end{aligned}$$

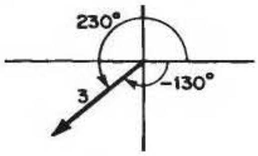
8

$$z = 4(\cos 110^\circ - j \sin 110^\circ)$$

since $\cos(-110^\circ) = \cos 110^\circ$
and $\sin(-110^\circ) = -\sin 110^\circ$

It is sometimes convenient to use this form when the value of θ is greater than 180°, i.e. in the 3rd and 4th quadrants.

Here are some examples:



$$\begin{aligned} z &= 3(\cos 230^\circ + j \sin 230^\circ) \\ &= 3(\cos 130^\circ - j \sin 130^\circ) \end{aligned}$$

Similarly: $z = 3(\cos 300^\circ + j \sin 300^\circ) = 3(\cos 60^\circ - j \sin 60^\circ)$

$z = 4(\cos 290^\circ + j \sin 290^\circ) = 4(\cos 70^\circ - j \sin 70^\circ)$

$z = 2(\cos 215^\circ + j \sin 215^\circ) = 2(\cos 145^\circ - j \sin 145^\circ)$

and $z = 6(\cos 310^\circ + j \sin 310^\circ) = \ldots\ldots\ldots\ldots$

9

$$\boxed{z = 6(\cos 50^\circ - j\sin 50^\circ)}$$

Because $\cos 310^\circ = \cos 50^\circ$
and $\sin 310^\circ = -\sin 50^\circ$

A moment ago we agreed that the minus sign comes about by the use of negative angles. To convert a complex number given in this way back into proper polar form, i.e. with a '+' in the middle, we simply work back the way we came. A complex number with a negative sign in the middle is equivalent to the same complex number with a positive sign, but with the angles made negative.

e.g. $z = 4(\cos 30^\circ - j\sin 30^\circ)$
$= 4(\cos[-30^\circ] + j\sin[-30^\circ])$
$= 4(\cos 330^\circ + j\sin 330^\circ)$ and we are back in proper polar form.

You do this one: Convert $z = 5(\cos 40^\circ - j\sin 40^\circ)$ into proper polar form.

Then on to Frame 10

10

$$\boxed{z = 5(\cos 320^\circ + j\sin 320^\circ)}$$

Because

$$z = 5(\cos 40^\circ - j\sin 40^\circ) = 5(\cos[-40^\circ] + j\sin[-40^\circ])$$
$$= 5(\cos 320^\circ + j\sin 320^\circ)$$

Here is another for you to do.

Express $z = 4(\cos 100^\circ - j\sin 100^\circ)$ in proper polar form.

Do not forget, it all depends on the use of negative angles.

11

$$\boxed{z = 4(\cos 260^\circ + j\sin 260^\circ)}$$

Because

$$z = 4(\cos 100^\circ - j\sin 100^\circ) = 4(\cos[-100^\circ] + j\sin[-100^\circ])$$
$$= 4(\cos 260^\circ + j\sin 260^\circ)$$

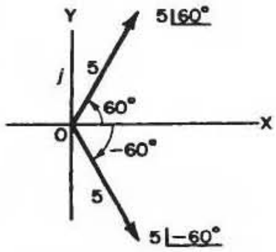
We ought to see how this modified polar form affects our shorthand notation.

▶

Remember, $5(\cos 60^\circ + j\sin 60^\circ)$ is written $5\underline{|60^\circ}$

How then shall we write $5(\cos 60^\circ - j\sin 60^\circ)$?

We know that this really stands for $5(\cos[-60^\circ] + j\sin[-60^\circ])$ so we could write $5\underline{|-60^\circ}$. But instead of using the negative angle we use a different symbol, i.e. $5\underline{|-60^\circ}$ becomes $5\overline{|60^\circ}$.



Similarly, $3(\cos 45^\circ - j\sin 45^\circ) = 3\underline{|-45^\circ} = \ldots\ldots\ldots\ldots$

12

$$\boxed{3\overline{|45^\circ}}$$

This is easy to remember,

for the sign $\underline{|}$ resembles the first quadrant and indicates measuring angles ↺ i.e. in the positive direction,

while the sign $\overline{|}$ resembles the fourth quadrant and indicates measuring angles ↻ i.e. in the negative direction.

e.g. $(\cos 15^\circ + j\sin 15^\circ)$ is written $\underline{|15^\circ}$

but $(\cos 15^\circ - j\sin 15^\circ)$, which is really $(\cos[-15^\circ] + j\sin[-15^\circ])$ is written $\overline{|15^\circ}$

So how do we write (a) $(\cos 120^\circ + j\sin 120^\circ)$
and (b) $(\cos 135^\circ - j\sin 135^\circ)$
in the shorthand way?

13

$$\boxed{\text{(a) } \underline{|120^\circ} \qquad \text{(b) } \overline{|135^\circ}}$$

The polar form at first sight seems to be a complicated way of representing a complex number. However it is very useful, as we shall see. Suppose we multiply together two complex numbers in this form:

Let $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$

Then $z_1z_2 = r_1(\cos\theta_1 + j\sin\theta_1)r_2(\cos\theta_2 + j\sin\theta_2)$

$$= r_1r_2(\cos\theta_1\cos\theta_2 + j\sin\theta_1\cos\theta_2 + j\cos\theta_1\sin\theta_2 + j^2\sin\theta_1\sin\theta_2)$$

Rearranging the terms and remembering that $j^2 = -1$, we get

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

Now the brackets $(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$
and $(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$
ought to ring a bell. What are they?

14

$$\boxed{\begin{aligned}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 &= \cos(\theta_1+\theta_2)\\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 &= \sin(\theta_1+\theta_2)\end{aligned}}$$

In that case, $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)]$.
Note this important result. We have just shown that

$$r_1(\cos\theta_1 + j\sin\theta_1).r_2(\cos\theta_2 + j\sin\theta_2) = r_1 r_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)]$$

i.e. To multiply together two complex numbers in polar form,

(a) multiply the r's together,

(b) add the angles, θ, together.

It is just as easy as that!

$$\begin{aligned}\text{e.g. } & 2(\cos 30^\circ + j\sin 30^\circ) \times 3(\cos 40^\circ + j\sin 40^\circ)\\ &= 2\times 3(\cos[30^\circ+40^\circ] + j\sin[30^\circ+40^\circ])\\ &= 6(\cos 70^\circ + j\sin 70^\circ)\end{aligned}$$

So if we multiply together $5(\cos 50^\circ + j\sin 50^\circ)$ and $2(\cos 65^\circ + j\sin 65^\circ)$ we get ....

15

$$\boxed{10(\cos 115^\circ + j\sin 115^\circ)}$$

Remember, multiply the r's; add the θ's.

Here you are then; all done the same way:

(a) $2(\cos 120^\circ + j\sin 120^\circ) \times 4(\cos 20^\circ + j\sin 20^\circ)$
$\quad = 8(\cos 140^\circ + j\sin 140^\circ)$

(b) $a(\cos\theta + j\sin\theta) \times b(\cos\phi + j\sin\phi)$
$\quad = ab(\cos[\theta+\phi] + j\sin[\theta+\phi])$

(c) $6(\cos 210^\circ + j\sin 210^\circ) \times 3(\cos 80^\circ + j\sin 80^\circ)$
$\quad = 18(\cos 290^\circ + j\sin 290^\circ)$

(d) $5(\cos 50^\circ + j\sin 50^\circ) \times 3(\cos[-20^\circ] + j\sin[-20^\circ])$
$\quad = 15(\cos 30^\circ + j\sin 30^\circ)$

Have you got it? No matter what the angles are, all we do is:

(a) multiply the moduli, (b) add the arguments.

So therefore, $4(\cos 35^\circ + j\sin 35^\circ) \times 3(\cos 20^\circ + j\sin 20^\circ) =$ ....

16

$$\boxed{12(\cos 55^\circ + j\sin 55^\circ)}$$

Now let us see if we can discover a similar set of rules for division.

We already know that to simplify $\dfrac{5+j6}{3+j4}$ we first obtain a denominator that is entirely real by multiplying top and bottom by ....

17

the conjugate of the denominator, i.e. $3 - j4$

Right. Then let us do the same with $\dfrac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)}$

$$\frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} = \frac{r_1(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2)}{r_2(\cos\theta_2 + j\sin\theta_2)(\cos\theta_2 - j\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + j\sin\theta_1\cos\theta_2 - j\cos\theta_1\sin\theta_2 + \sin\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{1}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$$

So, for division, the rule is ....

18

divide the r's and subtract the angle

That is correct.

$$\text{e.g.}\quad \frac{6(\cos 72^\circ + j\sin 72^\circ)}{2(\cos 41^\circ + j\sin 41^\circ)} = 3(\cos 31^\circ + j\sin 31^\circ)$$

So we now have two important rules:

If $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$

then (a) $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$

and (b) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$

The results are still, of course, in polar form.

Now here is one for you to think about.

If $z_1 = 8(\cos 65^\circ + j\sin 65^\circ)$ and $z_2 = 4(\cos 23^\circ + j\sin 23^\circ)$

then (a) $z_1z_2 =$ .... and (b) $\dfrac{z_1}{z_2} =$ ....

19

(a) $z_1z_2 = 32(\cos 88^\circ + j\sin 88^\circ)$

(b) $\dfrac{z_1}{z_2} = 2(\cos 42^\circ + j\sin 42^\circ)$

Of course, we can combine the rules in a single example:

e.g. $\dfrac{5(\cos 60^\circ + j\sin 60^\circ) \times 4(\cos 30^\circ + j\sin 30^\circ)}{2(\cos 50^\circ + j\sin 50^\circ)}$

$= \dfrac{20(\cos 90^\circ + j\sin 90^\circ)}{2(\cos 50^\circ + j\sin 50^\circ)}$

$= 10(\cos 40^\circ + j\sin 40^\circ)$

What does the following product become?

$$4(\cos 20^\circ + j\sin 20^\circ) \times 3(\cos 30^\circ + j\sin 30^\circ) \times 2(\cos 40^\circ + j\sin 40^\circ)$$

Result in next frame

20

$$\boxed{24(\cos 90^\circ + j\sin 90^\circ)}$$

i.e. $(4 \times 3 \times 2)[\cos(20^\circ + 30^\circ + 40^\circ) + j\sin(20^\circ + 30^\circ + 40^\circ)]$

$= 24(\cos 90^\circ + j\sin 90^\circ)$

Now what about a few revision examples on the work we have done so far?

Move to the next frame

21

Revision exercise

Work all these questions and then turn on to Frame 22 and check your results.

1 Express in polar form, $z = -4 + j2$.

2 Express in true polar form, $z = 5(\cos 55^\circ - j\sin 55^\circ)$.

3 Simplify the following, giving the results in polar form:

(a) $3(\cos 143^\circ + j\sin 143^\circ) \times 4(\cos 57^\circ + j\sin 57^\circ)$

(b) $\dfrac{10(\cos 126^\circ + j\sin 126^\circ)}{2(\cos 72^\circ + j\sin 72^\circ)}$

4 Express in the form $a + jb$:

(a) $2(\cos 30^\circ + j\sin 30^\circ)$

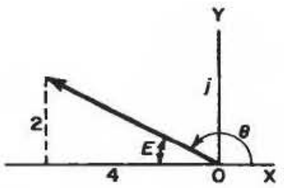
(b) $5(\cos 57^\circ - j\sin 57^\circ)$

Solutions are in Frame 22. Move on and see how you have fared

---

22

1



$r^2 = 2^2 + 4^2 = 4 + 16 = 20$

$\therefore\ r = 4{\cdot}472$

$\tan E = 0{\cdot}5 \quad \therefore\ E = 26^\circ 34'$

$\therefore\ \theta = 153^\circ 26'$

$z = -4 + j2 = 4{\cdot}472(\cos 153^\circ 26' + j\sin 153^\circ 26')$

2 $z = 5(\cos 55^\circ - j\sin 55^\circ) = 5[\cos(-55^\circ) + j\sin(-55^\circ)]$

$= 5(\cos 305^\circ + j\sin 305^\circ)$

3 (a) $3(\cos 143° + j\sin 143°) \times 4(\cos 57° + j\sin 57°)$
$= 3 \times 4[\cos(143° + 57°) + j\sin(143° + 57°)]$
$= 12(\cos 200° + j\sin 200°)$

(b) $\dfrac{10(\cos 126° + j\sin 126°)}{2(\cos 72° + j\sin 72°)}$
$= \dfrac{10}{2}[\cos(126° - 72°) + j\sin(126° - 72°)]$
$= 5(\cos 54° + j\sin 54°)$

4 (a) $2(\cos 30° + j\sin 30°)$
$= 2(0{\cdot}866 + j0{\cdot}5) = 1{\cdot}732 + j$

(b) $5(\cos 57° - j\sin 57°)$
$= 5(0{\cdot}5446 - j0{\cdot}8387)$
$= 2{\cdot}723 - j4{\cdot}193$

Now continue the Programme in Frame 23

23

Now we are ready to go on to a very important section which follows from our work on multiplication of complex numbers in polar form.

We have already established that:

if $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$

then $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$

So if $z_3 = r_3(\cos\theta_3 + j\sin\theta_3)$ then we have

$$z_1z_2z_3 = r_1r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]r_3(\cos\theta_3 + j\sin\theta_3)$$
$$= \ldots\ldots\ldots\ldots$$

24

$$z_1z_2z_3 = r_1r_2r_3[\cos(\theta_1 + \theta_2 + \theta_3) + j\sin(\theta_1 + \theta_2 + \theta_3)]$$

because in multiplication, we multiply the moduli and add the arguments

Now suppose that z_1, z_2, z_3 are all alike and that each is equal to $z = r(\cos\theta + j\sin\theta)$. Then the result above becomes:

$$z_1z_2z_3 = z^3 = r.r.r[\cos(\theta + \theta + \theta) + j\sin(\theta + \theta + \theta)]$$
$$= r^3(\cos 3\theta + j\sin 3\theta)$$

or $$z^3 = [r(\cos\theta + j\sin\theta)]^3 = r^3(\cos\theta + j\sin\theta)^3$$
$$= r^3(\cos 3\theta + j\sin 3\theta)$$

That is, if we wish to cube a complex number in polar form, we just cube the modulus (r value) and multiply the argument (θ) by 3.

Similarly, to square a complex number in polar form, we square the modulus (r value) and multiply the argument (θ) by ....

25

$\boxed{2}$ i.e. $[r(\cos\theta + j\sin\theta)]^2 = r^2(\cos 2\theta + j\sin 2\theta)$

Let us take another look at these results:

$$[r(\cos\theta + j\sin\theta)]^2 = r^2(\cos 2\theta + j\sin 2\theta)$$
$$[r(\cos\theta + j\sin\theta)]^3 = r^3(\cos 3\theta + j\sin 3\theta)$$

Similarly:

$$[r(\cos\theta + j\sin\theta)]^4 = r^4(\cos 4\theta + j\sin 4\theta)$$
$$[r(\cos\theta + j\sin\theta)]^5 = r^5(\cos 5\theta + j\sin 5\theta) \quad \text{and so on}$$

In general, then, we can say:

$[r(\cos\theta + j\sin\theta)]^n = \dots\dots\dots\dots$

26

$$[r(\cos\theta + j\sin\theta)]^n = \boxed{r^n(\cos n\theta + j\sin n\theta)}$$

This general result is very important and is called *DeMoivre's theorem.* It says that to raise a complex number in polar form to any power n, we raise the r to the power n and multiply the angle by n:

e.g. $[4(\cos 50^\circ + j\sin 50^\circ)]^2 = 4^2[\cos(2 \times 50^\circ) + j\sin(2 \times 50^\circ)]$
$= 16(\cos 100^\circ + j\sin 100^\circ)$

and $[3(\cos 110^\circ + j\sin 110^\circ)]^3 = 27(\cos 330^\circ + j\sin 330^\circ)$

and in the same way:

$[2(\cos 37^\circ + j\sin 37^\circ)]^4 = \dots\dots\dots\dots$

27

$$\boxed{16(\cos 148^\circ + j\sin 148^\circ)}$$

This is where the polar form really comes into its own! For DeMoivre's theorem also applies when we are raising the complex number to a fractional power, i.e. when we are finding the roots of a complex number.

e.g. To find the square root of $z = 4(\cos 70^\circ + j\sin 70^\circ)$

We have $\sqrt{z} = z^{\frac{1}{2}} = [4(\cos 70^\circ + j\sin 70^\circ)]^{\frac{1}{2}}$ i.e. $n = \dfrac{1}{2}$

$$= 4^{\frac{1}{2}}\left(\cos\frac{70^\circ}{2} + j\sin\frac{70^\circ}{2}\right)$$
$$= 2(\cos 35^\circ + j\sin 35^\circ)$$

It works every time, no matter whether the power is positive, negative, whole number or fraction. In fact, DeMoivre's theorem is so important, let us write it down again. Here goes:

If $z = r(\cos\theta + j\sin\theta)$, then $z^n = \dots\dots\dots\dots$

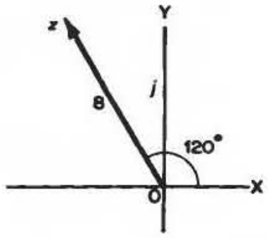
28

$$z = r(\cos\theta + j\sin\theta), \text{ then } \boxed{z^n = r^n(\cos n\theta + j\sin n\theta)}$$

for any value of n.

Look again at finding a root of a complex number. Let us find the cube root of $z = 8(\cos 120^\circ + j\sin 120^\circ)$. Here is the given complex number shown on an Argand diagram:

$z = 8\underline{|120^\circ}$



Of course, we could say that θ was '1 revolution + 120°': the vector would still be in the same position, or, for that matter (2 revs + 120°), (3 revs + 120°) etc. i.e. $z = 8\underline{|120^\circ}$ or $8\underline{|480^\circ}$ or $8\underline{|840^\circ}$ or $8\underline{|1200^\circ}$ etc. and if we now apply DeMoivre's theorem to each of these, we get:

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\underline{\left|\frac{120^\circ}{3}\right.} \text{ or } 8^{\frac{1}{3}}\underline{\left|\frac{480^\circ}{3}\right.} \text{ or } \ldots\ldots\ldots\ldots \text{ or } \ldots\ldots\ldots\ldots \text{ etc.}$$

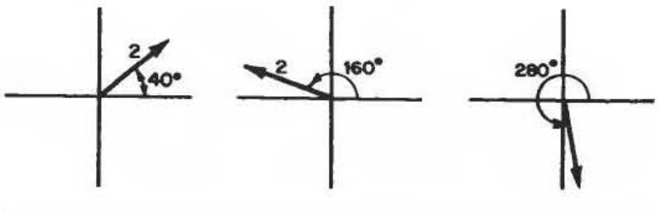
29

$$\boxed{z^{\frac{1}{3}} = 8^{\frac{1}{3}}\underline{\left|\frac{120^\circ}{3}\right.} \text{ or } 8^{\frac{1}{3}}\underline{\left|\frac{480^\circ}{3}\right.} \text{ or } 8^{\frac{1}{3}}\underline{\left|\frac{840^\circ}{3}\right.} \text{ or } 8^{\frac{1}{3}}\underline{\left|\frac{1200^\circ}{3}\right.}}$$

If we simplify these, we get:

$z^{\frac{1}{3}} = 2\underline{|40^\circ}$ or $2\underline{|160^\circ}$ or $2\underline{|280^\circ}$ or $2\underline{|400^\circ}$etc.

If we put each of these on an Argand diagram, as follows:

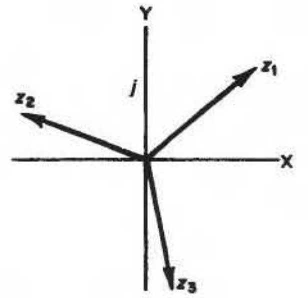


we see we have three quite different results for the cube root of z and also that the fourth diagram would be a repetition of the first. Any subsequent calculations merely repeat these three positions.

Make a sketch of the first three vectors on a single Argand diagram

30

Here they are. The cube roots of $z = 8(\cos 120^\circ + j\sin 120^\circ)$:



$z_1 = 2\underline{|40^\circ}$
$z_2 = 2\underline{|160^\circ}$
$z_3 = 2\underline{|280^\circ}$

We see, therefore, that there are 3 cube roots of a complex number. Also, if you consider the angles, you see that the 3 roots are equally spaced round the diagram, any two adjacent vectors being separated by .... degrees.

31

$\boxed{120^\circ}$

That is right. Therefore all we need to do in practice is to find the first of the roots and simply add 120° on to get the next – and so on.

Notice that the three cube roots of a complex number are equal in modulus (or size) and equally spaced at intervals of $\dfrac{360^\circ}{3}$ i.e. 120°.

Now let us take another example. On to the next frame

32

To find the three cube roots of $z = 5(\cos 225^\circ + j\sin 225^\circ)$

The first root is given by

$$z_1 = z^{\frac{1}{3}} = 5^{\frac{1}{3}}\left(\cos\frac{225^\circ}{3} + j\sin\frac{225^\circ}{3}\right)$$
$$= 1{\cdot}71(\cos 75^\circ + j\sin 75^\circ)$$
$$z_1 = 1{\cdot}71\underline{|75^\circ}$$

We know that the other cube roots are the same length (modulus), i.e. 1·71, and separated at intervals of $\dfrac{360^\circ}{3}$, i.e. 120°.

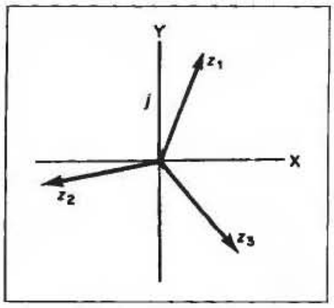
So the three cube roots are:

$z_1 = 1{\cdot}71\underline{|75^\circ}$
$z_2 = 1{\cdot}71\underline{|195^\circ}$
$z_3 = 1{\cdot}71\underline{|315^\circ}$

It helps to see them on an Argand diagram, so sketch them on a combined diagram.

33

Here they are:



We find any roots of a complex number in the same way:

(a) Apply DeMoivre's theorem to find the first of the n roots.

(b) The other roots will then be distributed round the diagram at regular intervals of $\dfrac{360^\circ}{n}$.

A complex number, therefore, has:

2 square roots, separated by $\dfrac{360^\circ}{2}$ i.e. 180°

3 cube roots, separated by $\dfrac{360^\circ}{3}$ i.e. 120°

4 fourth roots, separated by $\dfrac{360^\circ}{4}$ i.e. 90°

5 fifth roots, separated by ....etc.

34

There would be 5 fifth roots separated by $\boxed{\dfrac{360^\circ}{5} \text{ i.e. } 72^\circ}$

And now: To find the 5 fifth roots of $12\underline{|300^\circ}$

$z = 12\underline{|300^\circ} \quad \therefore \; z_1 = 12^{\frac{1}{5}}\underline{\left|\dfrac{300^\circ}{5}\right.} = 12^{\frac{1}{5}}\underline{|60^\circ}$

We now have to find the value of $12^{\frac{1}{5}}$. Do it by logs.

Let $A = 12^{\frac{1}{5}}$. Then $\log A = \dfrac{1}{5}\log 12 = \dfrac{1}{5}(1{\cdot}0792) = 0{\cdot}2158$

So: $\qquad A = 1{\cdot}644$

The first of the 5 fifth roots is therefore $z_1 = 1{\cdot}644\underline{|60^\circ}$

The others will be of the same modulus, i.e. 1·644, and equally separated at intervals of $\dfrac{360^\circ}{5}$ i.e. 72°.

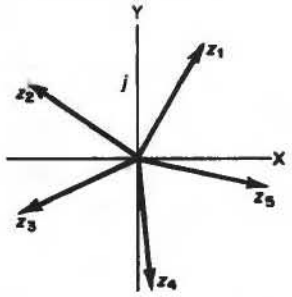
So the required 5 fifth roots of $12\underline{|300^\circ}$ are:

$z_1 = 1{\cdot}644\underline{|60^\circ} \qquad z_2 = 1{\cdot}644\underline{|132^\circ} \qquad z_3 = 1{\cdot}644\underline{|204^\circ}$

$z_4 = 1{\cdot}644\underline{|276^\circ} \qquad z_5 = 1{\cdot}644\underline{|348^\circ}$

Sketch them on an Argand diagram, as before.

35



$$z_1 = 1{\cdot}644\underline{|\ 60^\circ}$$
$$z_2 = 1{\cdot}644\underline{|132^\circ}$$
$$z_3 = 1{\cdot}644\underline{|204^\circ}$$
$$z_4 = 1{\cdot}644\underline{|276^\circ}$$
$$z_5 = 1{\cdot}644\underline{|348^\circ}$$

Although there are 5 fifth roots of a complex number, we are sometimes asked to find the *principal root*. This is always the root whose vector is nearest to the positive OX axis.

In some cases, it may be the first root. In others, it may be the last root. The only test is to see which root is nearest to the positive OX axis. If the first and last root are equidistant from the x-axis, the principal root is taken to be the first root.

In the example above, the *principal root* is therefore ....

36

$$\boxed{z_5 = 1{\cdot}644\underline{|348^\circ}}$$

Good. Now here is another example worked in detail. Follow it.

We have to find the 4 fourth roots of $z = 7(\cos 80^\circ + j\sin 80^\circ)$

The first root $z_1 = 7^{\frac{1}{4}}\underline{\left|\dfrac{80^\circ}{4}\right.} = 7^{\frac{1}{4}}\underline{|20^\circ}$

Now find $7^{\frac{1}{4}}$ by logs. Let $A = 7^{\frac{1}{4}}$

Then $\log A = \dfrac{1}{4}\log 7 = \dfrac{1}{4}(0{\cdot}8451) = 0{\cdot}2113$ and $A = 1{\cdot}627$

$\therefore$ $z_1 = 1{\cdot}627\underline{|20^\circ}$

The other roots will be separated by intervals of $\dfrac{360^\circ}{4} = 90^\circ$

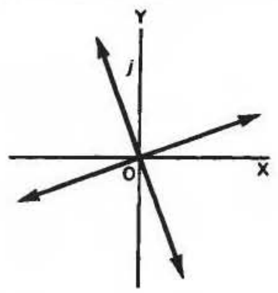
Therefore the 4 fourth roots are:

$z_1 = 1{\cdot}627\underline{|20^\circ}$ $\quad$ $z_2 = 1{\cdot}627\underline{|110^\circ}$

$z_3 = 1{\cdot}627\underline{|200^\circ}$ $\quad$ $z_4 = 1{\cdot}627\underline{|290^\circ}$

And once again, draw an Argand diagram to illustrate these roots.

37



$z_1 = 1{\cdot}627\underline{|\ 20^\circ}$
$z_2 = 1{\cdot}627\underline{|110^\circ}$
$z_3 = 1{\cdot}627\underline{|200^\circ}$
$z_4 = 1{\cdot}627\underline{|290^\circ}$

And in this example the principal fourth root is ....

38

$$\boxed{z_1 = 1{\cdot}627\underline{|20^\circ}}$$

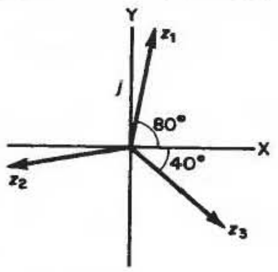
since it is the root nearest to the positive OX axis.

Now you can do one entirely on your own. Here it is.

Find the three cube roots of $6(\cos 240^\circ + j\sin 240^\circ)$. Represent them on an Argand diagram and indicate which is the principal cube root.

When you have finished it, move on to Frame 39 and check your results

39



$z_1 = 1{\cdot}817\underline{|\ 80^\circ}$
$z_2 = 1{\cdot}817\underline{|200^\circ}$
$z_3 = 1{\cdot}817\underline{|320^\circ}$
Principal root: $z_3 = 1{\cdot}817\underline{|320^\circ}$

Here is the working:

$$z = 6\underline{|240^\circ} \qquad z_1 = 6^{\frac{1}{3}}\underline{\left|\frac{240^\circ}{3}\right.} = 1{\cdot}817\underline{|80^\circ}$$

Interval between roots $= \dfrac{360^\circ}{3} = 120^\circ$

Therefore the roots are:

$$z_1 = 1{\cdot}817\underline{|80^\circ} \qquad z_2 = 1{\cdot}817\underline{|200^\circ} \qquad z_3 = 1{\cdot}817\underline{|320^\circ}$$

The principal root is the root nearest to the positive OX axis. In this case, then, the principal root is $z_3 = 1{\cdot}817\underline{|320^\circ}$

On to the next frame

40

Expansions of sin $n\theta$ and cos $n\theta$, where n is a positive integer

By DeMoivre's theorem, we know that:

$$\cos n\theta + j\sin n\theta = (\cos\theta + j\sin\theta)^n$$

The method is simply to expand the right-hand side as a binomial series, after which we can equate real and imaginary parts.

An example will soon show you how it is done:

To find expansions for $\cos 3\theta$ and $\sin 3\theta$.

We have:

$$\cos 3\theta + j\sin 3\theta = (\cos\theta + j\sin\theta)^3$$

$$= (c + js)^3 \qquad \text{where } c \equiv \cos\theta, \quad s \equiv \sin\theta$$

Now expand this by the binomial series – like $(a+b)^3$ so that

$\cos 3\theta + j\sin 3\theta = \dots\dots\dots\dots$

41

$$\boxed{c^3 + j3c^2s - 3cs^2 - js^3}$$

Because

$$\begin{aligned}\cos 3\theta + j\sin 3\theta &= c^3 + 3c^2(js) + 3c(js)^2 + (js)^3\\ &= c^3 + j3c^2s - 3cs^2 - js^3 && \text{since } j^2 = -1\\ &= (c^3 - 3cs^2) + j(3c^2s - s^3) && j^3 = -j\end{aligned}$$

Now, equating real parts and imaginary parts, we get

$\cos 3\theta = \dots\dots\dots\dots$

and $\sin 3\theta = \dots\dots\dots\dots$

42

$$\boxed{\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\end{aligned}}$$

If we wish, we can replace $\sin^2\theta$ by $(1 - \cos^2\theta)$
and $\cos^2\theta$ by $(1 - \sin^2\theta)$

so that we could write the results above as:

$\cos 3\theta = \dots\dots\dots$ (all in terms of $\cos\theta$)

$\sin 3\theta = \dots\dots\dots$ (all in terms of $\sin\theta$)

43

$$\boxed{\begin{aligned}\cos 3\theta &= 4\cos^3\theta - 3\cos\theta\\ \sin 3\theta &= 3\sin\theta - 4\sin^3\theta\end{aligned}}$$

Because

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

and

$$\begin{aligned}\sin 3\theta &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

While these results are useful, it is really the method that counts.

So now do this one in just the same way:

Obtain an expression for $\cos 4\theta$ in terms of $\cos\theta$.

When you have finished, check your result with the next frame

44

$$\boxed{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1}$$

Working: $$\begin{aligned}\cos 4\theta + j\sin 4\theta &= (\cos\theta + j\sin\theta)^4\\ &= (c+js)^4\\ &= c^4 + 4c^3(js) + 6c^2(js)^2 + 4c(js)^3 + (js)^4\\ &= c^4 + j4c^3s - 6c^2s^2 - j4cs^3 + s^4\\ &= (c^4 - 6c^2s^2 + s^4) + j(4c^3s - 4cs^3)\end{aligned}$$

Equating real parts: $$\begin{aligned}\cos 4\theta &= c^4 - 6c^2s^2 + s^4\\ &= c^4 - 6c^2(1-c^2) + (1-c^2)^2\\ &= c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4\\ &= 8c^4 - 8c^2 + 1\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

Now for a different problem.

On to the next frame

45

Expansions for $\cos^n\theta$ and $\sin^n\theta$ in terms of sines and cosines of multiples of θ

Let $\qquad z = \cos\theta + j\sin\theta$

then $\qquad \dfrac{1}{z} = z^{-1} = \cos\theta - j\sin\theta$

$$\therefore\ z + \frac{1}{z} = 2\cos\theta \text{ and } z - \frac{1}{z} = j2\sin\theta$$

Also, by DeMoivre's theorem:

$$z^n = \cos n\theta + j\sin n\theta$$

and $\qquad \dfrac{1}{z^n} = z^{-n} = \cos n\theta - j\sin n\theta$

$$\therefore\ z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = j2\sin n\theta$$

Let us collect these four results together: $z = \cos\theta + j\sin\theta$

$z + \dfrac{1}{z} = 2\cos\theta$	$z - \dfrac{1}{z} = j2\sin\theta$
$z^n + \dfrac{1}{z^n} = 2\cos n\theta$	$z^n - \dfrac{1}{z^n} = j2\sin n\theta$

Make a note of these results in your record book. Then move on and we will see how we use them

46

We shall expand $\cos^3\theta$ as an example.

From our results: $z + \dfrac{1}{z} = 2\cos\theta$

$$\therefore\ (2\cos\theta)^3 = \left(z + \frac{1}{z}\right)^3$$
$$= z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z^2}\right) + \frac{1}{z^3}$$
$$= z^3 + 3z + 3\frac{1}{z} + \frac{1}{z^3}$$

Now here is the trick: we rewrite this, collecting the terms up in pairs from the two extreme ends, thus:

$$(2\cos\theta)^3 = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

And, from the four results that we noted:

$$z + \frac{1}{z} = \ldots\ldots\ldots\ldots$$

and $\qquad z^3 + \dfrac{1}{z^3} = \ldots\ldots\ldots\ldots$

47

$$\boxed{z + \frac{1}{z} = 2\cos\theta: \;\; z^3 + \frac{1}{z^3} = 2\cos 3\theta}$$

$$\therefore \;\; (2\cos\theta)^3 = 2\cos 3\theta + 3 \times 2\cos\theta$$
$$8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$$
$$4\cos^3\theta = \cos 3\theta + 3\cos\theta$$
$$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

Now one for you:

Find an expression for $\sin^4\theta$.

Work in the same way, but, this time, remember that

$z - \frac{1}{z} = j\,2\sin\theta$ and $z^n - \frac{1}{z^n} = j\,2\sin n\theta$.

When you have obtained a result, check it with the next frame

48

$$\boxed{\sin^4\theta = \frac{1}{8}[\cos 4\theta - 4\cos 2\theta + 3]}$$

Because we have:

$$z - \frac{1}{z} = j\,2\sin\theta; \quad z^n - \frac{1}{z^n} = j\,2\sin n\theta$$

$$\therefore \;\; (j\,2\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$$
$$= z^4 - 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) - 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$$
$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

Now: $\quad z^n + \frac{1}{z^n} = 2\cos n\theta$

$$\therefore \;\; 16\sin^4\theta = 2\cos 4\theta - 4 \times 2\cos 2\theta + 6$$
$$\therefore \;\; \sin^4\theta = \frac{1}{8}[\cos 4\theta - 4\cos 2\theta + 3]$$

They are all done in the same way: once you know the trick, the rest is easy.

Now let us move on to something new

Loci problems

49

We are sometimes required to find the locus of a point which moves in the Argand diagram according to some stated condition. Before we work through one or two examples of this kind, let us just revise a couple of useful points.

You will remember that when we were representing a complex number in polar form, i.e. $z = a + jb = r(\cos\theta + j\sin\theta)$, we said that:

(a) r is called the *modulus* of z and is written 'mod z' or $|z|$ and
(b) θ is called the *argument* of z and is written 'arg z'.

Also $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left\{\dfrac{b}{a}\right\}$

so that $|z| = \sqrt{a^2 + b^2}$ and $\arg z = \tan^{-1}\left\{\dfrac{a}{b}\right\}$

Similarly, if $z = x + jy$ then $|z| = \ldots\ldots\ldots\ldots$

and $\arg z = \ldots\ldots\ldots\ldots$

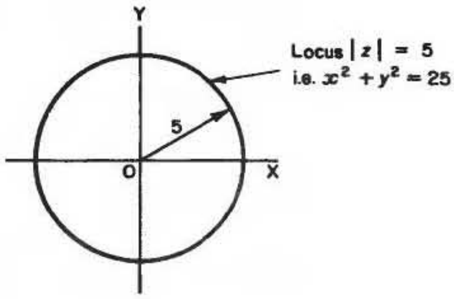
50

$$\boxed{|z| = \sqrt{x^2 + y^2} \text{ and } \arg z = \tan^{-1}\left\{\frac{y}{x}\right\}}$$

Keep those in mind and we are now ready to tackle some examples.

Example 1

If $z = x + jy$, find the locus defined as $|z| = 5$.



Now we know that in this case, $|z| = \sqrt{x^2 + y^2}$

The locus is defined as $\sqrt{x^2 + y^2} = 5$

$\therefore\ x^2 + y^2 = 25$

This is a circle, with centre at the origin and with radius 5.

That was easy enough. Move on for Example 2

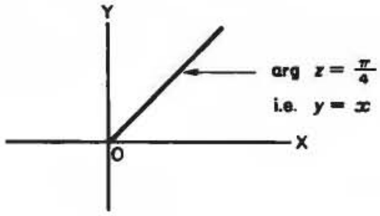
51

Example 2

If $z = x + jy$, find the locus defined as $\arg z = \frac{\pi}{4}$.

In this case: $\arg z = \tan^{-1}\left\{\frac{y}{x}\right\} \quad \therefore \quad \tan^{-1}\left\{\frac{y}{x}\right\} = \frac{\pi}{4}$

$$\therefore \quad \frac{y}{x} = \tan\frac{\pi}{4} = \tan 45^\circ = 1 \quad \therefore \quad \frac{y}{x} = 1 \quad \therefore \quad y = x$$



So the locus $\arg z = \frac{\pi}{4}$ is therefore the straight line $y = x$ and $y > 0$.

All locus problems at this stage are fundamentally of one of these kinds. Of course, the given condition may look a trifle more involved, but the approach is always the same.

Let us look at a more complicated one. Next frame

52

Example 3

If $z = x + jy$, find the equation of the locus $\left|\frac{z+1}{z-1}\right| = 2$.

Since $z = x + jy$:

$$z + 1 = x + jy + 1 = (x+1) + jy = r_1\underline{|\theta_1} = z_1$$
$$z - 1 = x + jy - 1 = (x-1) + jy = r_2\underline{|\theta_2} = z_2$$

$$\therefore \quad \frac{z+1}{z-1} = \frac{r_1\underline{|\theta_1}}{r_2\underline{|\theta_2}} = \frac{r_1}{r_2}\underline{|\theta_1 - \theta_2}$$

$$\therefore \quad \left|\frac{z+1}{z-1}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} = \frac{\sqrt{(x+1)^2 + y^2}}{\sqrt{(x-1)^2 + y^2}}$$

$$\therefore \quad \frac{\sqrt{(x+1)^2 + y^2}}{\sqrt{(x-1)^2 + y^2}} = 2$$

$$\therefore \quad \frac{(x+1)^2 + y^2}{(x-1)^2 + y^2} = 4$$

All that now remains is to multiply across by the denominator and tidy up the result. So finish it off in its simplest form.

53

We had $\dfrac{(x+1)^2+y^2}{(x-1)^2+y^2}=4$

$$\text{So therefore} \quad (x+1)^2+y^2=4\left\{(x-1)^2+y^2\right\}$$
$$x^2+2x+1+y^2=4(x^2-2x+1+y^2)$$
$$=4x^2-8x+4+4y^2$$
$$\therefore\ 3x^2-10x+3+3y^2=0$$

This is the equation of the given locus.

Although this takes longer to write out than either of the first two examples, the basic principle is the same. The given condition must be a function of either the modulus or the argument.

Move on now to Frame 54 for Example 4

54

Example 4

If $z=x+jy$, find the equation of the locus $\arg(z^2)=-\dfrac{\pi}{4}$.

$$z=x+jy=r\underline{|\theta} \quad \therefore\ \arg z=\theta=\tan^{-1}\left\{\frac{y}{x}\right\}$$
$$\therefore\ \tan\theta=\frac{y}{x}$$
$$\therefore\ \text{By DeMoivre's theorem, } z^2=r^2\underline{|2\theta}$$
$$\therefore\ \arg(z^2)=2\theta=-\frac{\pi}{4}$$
$$\therefore\ \tan 2\theta=\tan\left(-\frac{\pi}{4}\right)=-1$$
$$\therefore\ \frac{2\tan\theta}{1-\tan^2\theta}=-1$$
$$\therefore\ 2\tan\theta=\tan^2\theta-1$$
$$\text{But} \quad \tan\theta=\frac{y}{x} \quad \therefore\ \frac{2y}{x}=\frac{y^2}{x^2}-1$$
$$2xy=y^2-x^2 \quad \therefore\ y^2=x^2+2xy$$

In that example, the given condition was a function of the argument. Here is one for you to do:

If $z=x+jy$, find the equation of the locus $\arg(z+1)=\dfrac{\pi}{3}$.

Do it carefully; then check with the next frame

55 Here is the solution set out in detail.

If $z = x + jy$, find the locus $\arg(z+1) = \dfrac{\pi}{3}$.

$$z = x + jy \quad \therefore \; z + 1 = x + jy + 1 = (x+1) + jy$$

$$\arg(z+1) = \tan^{-1}\left\{\frac{y}{x+1}\right\} = \frac{\pi}{3} \quad \therefore \; \frac{y}{x+1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$y = \sqrt{3}(x+1) \text{ for } y > 0$$

And that is all there is to that.

Now do this one. You will have no trouble with it.

If $z = x + jy$, find the equation of the locus $|z - 1| = 5$

When you have finished it, move on to Frame 56

56 Here it is: $z = x + jy$, given locus $|z - 1| = 5$

$$z - 1 = x + jy - 1 = (x - 1) + jy$$

$$\therefore \; |z - 1| = \sqrt{(x-1)^2 + y^2} = 5 \qquad \therefore \; (x-1)^2 + y^2 = 25$$

$$\therefore \; x^2 - 2x + 1 + y^2 = 25 \qquad \therefore \; x^2 - 2x + y^2 = 24$$

Every one is very much the same.

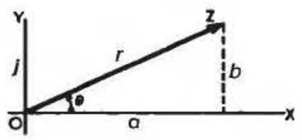
This brings us to the end of this Programme, except for the final **Can You?** checklist and **Test exercise**. Before you work through them, read down the **Revision summary** (Frame 57), just to refresh your memory of what we have covered in this Programme.

So on now to Frame 57

57

Revision summary

1



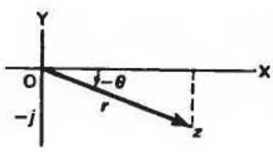
Polar form of a complex number

$$z = a + jb = r(\cos\theta + j\sin\theta) = r\underline{|\theta}$$

$$r = \text{mod } z = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \arg z = \tan^{-1}\left\{\frac{a}{b}\right\}$$

2



Negative angles

$$z = r(\cos[-\theta] + j\sin[-\theta])$$

$$\cos[-\theta] = \cos\theta$$

$$\sin[-\theta] = -\sin\theta$$

$$\therefore \; z = r(\cos\theta - j\sin\theta) = r\overline{\theta|}$$

3 *Multiplication and division in polar form*

If $z_1 = r_1\underline{|\theta_1}$; $z_2 = r_2\underline{|\theta_2}$

then $z_1 z_2 = r_1 r_2\underline{|\theta_1 + \theta_2}$

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\underline{|\theta_1 - \theta_2}$

4 *DeMoivre's theorem*

If $z = r(\cos\theta + j\sin\theta)$, then $z^n = r^n(\cos n\theta + j\sin n\theta)$

5 *Exponential form of a complex number*

$z = a + jb$ standard form

$= r(\cos\theta + j\sin\theta)$ polar form

$= re^{j\theta}$ [θ in radians] exponential form

Also $e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

6 *Logarithm of a complex number*

$z = re^{j\theta}$ $\therefore$ $\ln z = \ln r + j\theta$

7 *Loci problems*

If $z = x + jy$, $|z| = \sqrt{x^2 + y^2}$

$\arg z = \tan^{-1}\left\{\dfrac{y}{x}\right\}$

That's it! Now you are ready for the ***Can You?*** *checklist in Frame 58 and the* ***Test exercise*** *in Frame 59*

Can You?

Checklist 2

58

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Use the shorthand form for a complex number in polar form? 1 to 6
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Write complex numbers in polar form using negative angles? 7 to 12
  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

Frames

- Multiply and divide complex numbers in polar form? 13 to 20

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use DeMoivre's theorem? 23 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the roots of a complex number? 27 to 39

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Demonstrate trigonometric identities of multiple angles using complex numbers? 40 to 48

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve loci problems using complex numbers? 49 to 56

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 2

59

1. Express in polar form, $z = -5 - j3$.

2. Express in the form $a + jb$:

   (a) $2\angle 156^\circ$

   (b) $5\overline{|37^\circ}$

3. If $z_1 = 12(\cos 125^\circ + j\sin 125^\circ)$ and $z_2 = 3(\cos 72^\circ + j\sin 72^\circ)$, find (a) z_1z_2 and (b) $\dfrac{z_1}{z_2}$ giving the results in polar form.

4. If $z = 2(\cos 25^\circ + j\sin 25^\circ)$, find z^3 in polar form.

5. Find the three cube roots of $8(\cos 264^\circ + j\sin 264^\circ)$ and state which of them is the principal cube root. Show all three roots on an Argand diagram.

6. Expand $\sin 4\theta$ in powers of $\sin\theta$ and $\cos\theta$.

7. Express $\cos^4\theta$ in terms of cosines of multiples of θ.

8. If $z = x + jy$, find the equations of the two loci defined by:

   (a) $|z - 4| = 3$ (b) $\arg(z + 2) = \dfrac{\pi}{6}$

Further problems 2

60

1 If $z = x + jy$, where x and y are real, find the values of x and y when $\frac{3z}{1-j} + \frac{3z}{j} = \frac{4}{3-j}$.

2 In the Argand diagram, the origin is the centre of an equilateral triangle and one vertex of the triangle is the point $3 + j\sqrt{3}$. Find the complex numbers representing the other vertices.

3 Express $2 + j3$ and $1 - j2$ in polar form and apply DeMoivre's theorem to evaluate $\frac{(2+j3)^4}{1-j2}$. Express the result in the form $a + jb$ and in exponential form.

4 Find the fifth roots of $-3 + j3$ in polar form and in exponential form.

5 Express $5 + j12$ in polar form and hence evaluate the principal value of $\sqrt[3]{(5 + j12)}$, giving the results in the form $a + jb$ and in the form $re^{j\theta}$.

6 Determine the fourth roots of -16, giving the results in the form $a + jb$.

7 Find the fifth roots of -1, giving the results in polar form. Express the principal root in the form $re^{j\theta}$.

8 Determine the roots of the equation $x^3 + 64 = 0$ in the form $a + jb$, where a and b are real.

9 Determine the three cube roots of $\frac{2-j}{2+j}$ giving the result in modulus/argument form. Express the principal root in the form $a + jb$.

10 Show that the equation $z^3 = 1$ has one real root and two other roots which are not real, and that, if one of the non-real roots is denoted by ω, the other is then ω^2. Mark on the Argand diagram the points which represent the three roots and show that they are the vertices of an equilateral triangle.

11 Determine the fifth roots of $(2 - j5)$, giving the results in modulus/argument form. Express the principal root in the form $a + jb$ and in the form $re^{j\theta}$.

12 Solve the equation $z^2 + 2(1 + j)z + 2 = 0$, giving each result in the form $a + jb$, with a and b correct to 2 places of decimals.

13 Express $e^{1-j\pi/2}$ in the form $a + jb$.

14 Obtain the expansion of $\sin 7\theta$ in powers of $\sin\theta$.

15 Express $\sin^6 x$ as a series of terms which are cosines of angles that are multiples of x.

▶

16 If $z = x + jy$, where x and y are real, show that the locus $\left|\dfrac{z-2}{z+2}\right| = 2$ is a circle and determine its centre and radius.

17 If $z = x + jy$, show that the locus $\arg\left\{\dfrac{z-1}{z-j}\right\} = \dfrac{\pi}{6}$ is a circle. Find its centre and radius.

18 If $z = x + jy$, determine the Cartesian equation of the locus of the point z which moves in the Argand diagram so that

$$|z + j2|^2 + |z - j2|^2 = 40.$$

19 If $z = x + jy$, determine the equations of the two loci:

(a) $\left|\dfrac{z+2}{z}\right| = 3$ and (b) $\arg\left\{\dfrac{z+2}{z}\right\} = \dfrac{\pi}{4}$

20 If $z = x + jy$, determine the equations of the loci in the Argand diagram, defined by:

(a) $\left|\dfrac{z+2}{z-1}\right| = 2$ and (b) $\arg\left\{\dfrac{z-1}{z+2}\right\} = \dfrac{\pi}{2}$

21 Prove that:

(a) if $|z_1 + z_2| = |z_1 - z_2|$, the difference of the arguments of z_1 and z_2 is $\dfrac{\pi}{2}$

(b) if $\arg\left\{\dfrac{z_1 + z_2}{z_1 - z_2}\right\} = \dfrac{\pi}{2}$, then $|z_1| = |z_2|$

22 If $z = x + jy$, determine the loci in the Argand diagram, defined by:

(a) $|z + j2|^2 - |z - j2|^2 = 24$

(b) $|z + jk|^2 + |z - jk|^2 = 10k^2 \quad (k > 0)$

Hyperbolic functions

Frames

1 to 54

Learning outcomes

When you have completed this Programme you will be able to:

- Define the hyperbolic functions in terms of the exponential function
- Express the hyperbolic functions as power series
- Recognize the graphs of the hyperbolic functions
- Evaluate hyperbolic functions and their inverses
- Determine the logarithmic form of the inverse hyperbolic functions
- Prove hyperbolic trigonometric identities
- Understand the relationship between the circular and the hyperbolic trigonometric functions

Introduction

1

The cosine of an angle was first defined as the ratio of two sides of a right-angled triangle – adjacent over hypotenuse. In Programme F.12 of Part I you learnt how to extend the definition of a cosine to *any* angle, positive or negative. You might just check that out to refresh your memory by re-reading Frame 31 of Programme F.12.

Now, in Frames 55 to 57 of Programme 1 in Part II you learnt how a complex number of unit length could be written in either polar or exponential form, giving rise to the equations:

$$\cos\theta + j\sin\theta = e^{j\theta}$$

$$\cos\theta - j\sin\theta = e^{-j\theta}$$

If these two equations are added you find that:

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \text{ so that } \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

If θ is replaced by jx in this last equation you find that:

$$\cos jx = \frac{e^{jjx} + e^{-jjx}}{2} = \frac{e^{-x} + e^{x}}{2}$$

where the right-hand side is *entirely real*. In fact, you have seen this before in Frame 75 of Programme F.12, it is the even part of the exponential function which is called the *hyperbolic cosine*:

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \text{ so that } \cos jx = \cosh x$$

The graph of $y = \cosh x$ is called a *catenary* from the Latin word *catena* meaning chain because the shape of the graph is the shape of a hanging chain.

Move on to Frame 2 and start the Programme

2

You may remember that of the many functions that can be expressed as a series of powers of x, a common one is e^x:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

If we replace x by $-x$, we get:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

and these two functions e^x and e^{-x} are the foundations of the definitions we are going to use.

▶

(a) If we take the value of e^x, subtract e^{-x}, and divide by 2, we form what is defined as the hyperbolic sine of x:

$$\frac{e^x - e^{-x}}{2} = \textit{hyperbolic sine} \text{ of } x$$

This is a lot to write every time we wish to refer to it, so we shorten it to $\sinh x$, the h indicating its connection with the hyperbola. We pronounce it 'shine x'.

$$\frac{e^x - e^{-x}}{2} = \sinh x$$

So, in the same way, $\dfrac{e^y - e^{-y}}{2}$ would be written as ....

3

$$\boxed{\sinh y}$$

In much the same way, we have two other definitions:

(b) $\dfrac{e^x + e^{-x}}{2} = \textit{hyperbolic cosine}$ of x

$= \cosh x$ [pronounced 'cosh x']

(c) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \textit{hyperbolic tangent}$ of x

$= \tanh x$ [pronounced 'than x']

We must start off by learning these definitions, for all the subsequent developments depend on them.

So now then; what was the definition of $\sinh x$?

$\sinh x =$ ....

4

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

Here they are together so that you can compare them:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Make a copy of these in your record book for future reference when necessary.

5

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}; \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

We started the programme by referring to e^x and e^{-x} as series of powers of x. It should not be difficult therefore to find series at least for $\sinh x$ and for $\cosh x$. Let us try.

(a) *Series for sinh x*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

If we subtract, we get:

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} \dots$$

Divide by 2:

$$\frac{e^x - e^{-x}}{2} = \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

(b) If we add the series for e^x and e^{-x}, we get a similar result. What is it?

When you have decided, move on to Frame 6

6

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

Because we have:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$
$$\therefore\ e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots$$
$$\therefore\ \frac{e^x + e^{-x}}{2} = \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Move on to Frame 7

7

So we have:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

Note: All terms positive: $\sinh x$ has all the odd powers
$\cosh x$ has all the even powers

▶

We cannot easily get a series for $\tanh x$ by this process, so we will leave that one to some other time.

Make a note of these two series in your record book. Then, cover up what you have done so far and see if you can write down the definitions of:

(a) $\sinh x =$ ....

(b) $\cosh x =$ ....

(c) $\tanh x =$ .... No looking!

8

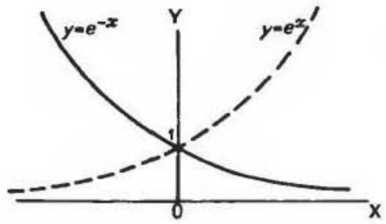
$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}; \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

All correct? Right.

Graphs of hyperbolic functions

9

We shall get to know quite a lot about these hyperbolic functions if we sketch the graphs of these functions. Since they depend on the values of e^x and e^{-x}, we had better just refresh our memories of what these graphs look like.



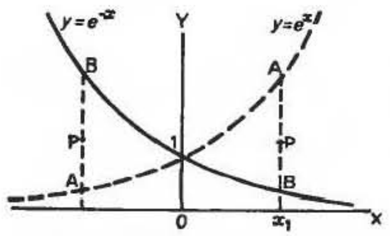
$y = e^x$ and $y = e^{-x}$ cross the y-axis at the point $y = 1$ $(e^0 = 1)$. Each graph then approaches the x-axis as an asymptote, getting nearer and nearer to it as it goes away to infinity in each direction, without actually crossing it,.

So, for what range of values of x are e^x and e^{-x} positive?

10

e^x and e^{-x} are positive for all values of x

Correct, since the graphs are always above the x-axis.



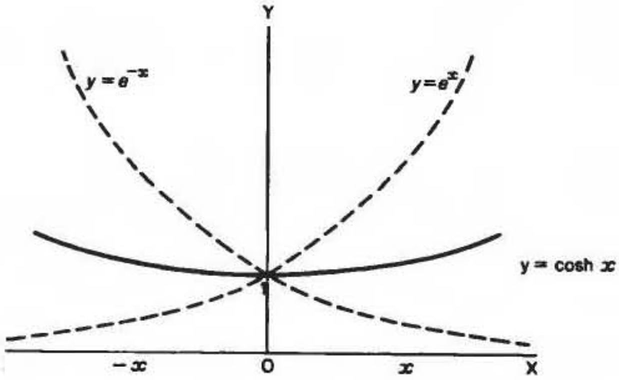
At any value of x, e.g. $x = x_1$, $\cosh x = \frac{e^x + e^{-x}}{2}$, i.e. the value of $\cosh x$ is the average of the values of e^x and e^{-x} at that value of x. This is given by P, the mid-point of AB.

If we can imagine a number of ordinates (or verticals) like AB and we plot their mid-points, we shall obtain the graph of $y = \cosh x$.

Can you sketch in what the graph will look like?

11

Here it is:



We see from the graph of $y=\cosh x$ that:

(a) $\cosh 0=1$

(b) the value of $\cosh x$ is never less than 1

(c) the curve is symmetrical about the y-axis, i.e.

$$\cosh(-x)=\cosh x$$

(d) for any given value of $\cosh x$, there are two values of x, equally spaced about the origin, i.e. $x=\pm a$.

Now let us see about the graph of $y=\sinh x$ in the same sort of way.

12

$$\sinh x=\frac{e^x-e^{-x}}{2}$$

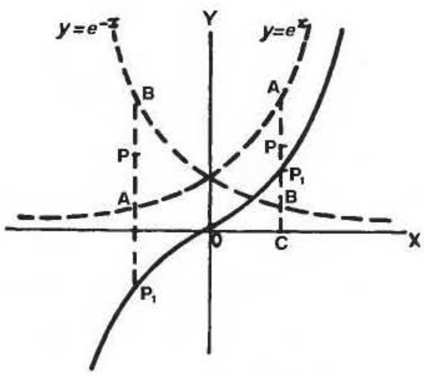
On the diagram:

$CA=e^x$

$CB=e^{-x}$

$BA=e^x-e^{-x}$

$$BP=\frac{e^x-e^{-x}}{2}$$

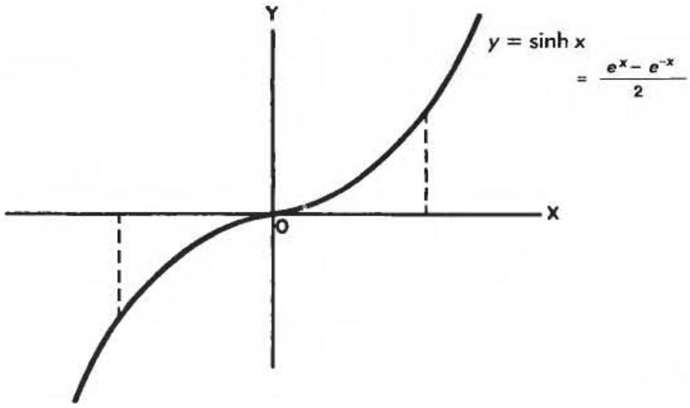


The corresponding point on the graph of $y=\sinh x$ is thus obtained by standing the ordinate BP on the x-axis at C, i.e. P_1.

Note that on the left of the origin, BP is negative and is therefore placed below the x-axis.

So what can we say about $y=\sinh x$?

13



From the graph of $y = \sinh x$, we see:

(a) $\sinh 0 = 0$

(b) $\sinh x$ can have all values from $-\infty$ to $+\infty$

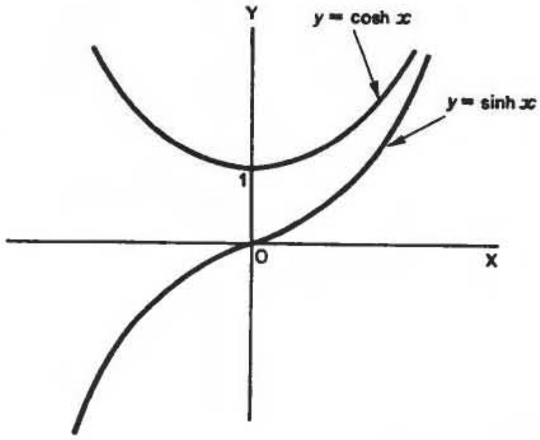
(c) the curve is symmetrical about the origin, i.e.

$$\sinh(-x) = -\sinh x$$

(d) for a given value of $\sinh x$, there is only one real value of x.

If we draw $y = \sinh x$ and $y = \cosh x$ on the same graph, what do we get?

14



Note that $y = \sinh x$ is always outside $y = \cosh x$, but gets nearer to it as x increases:

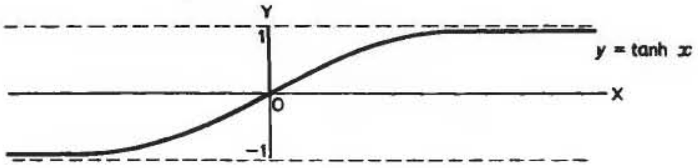
i.e. as $x \to \infty$, $\sinh x \to \cosh x$

And now let us consider the graph of $y = \tanh x$.

Move on

15

It is not easy to build $y = \tanh x$ directly from the graphs of $y = e^x$ and $y = e^{-x}$. If, however, we take values of e^x and e^{-x} and then calculate $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ and plot points, we get a graph as shown:



We see:

(a) $\tanh 0 = 0$

(b) $\tanh x$ always lies between $y = -1$ and $y = 1$

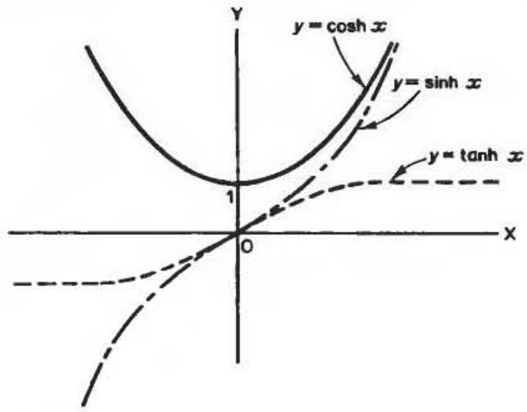
(c) $\tanh(-x) = -\tanh x$

(d) as $x \to \infty$, $\tanh x \to 1$
as $x \to -\infty$, $\tanh x \to -1$

Finally, let us now sketch all three graphs on one diagram so that we can compare them and distinguish between them.

16

Here they are:



One further point to note:

At the origin, $y = \sinh x$ and $y = \tanh x$ have the same slope. The two graphs therefore slide into each other and out again. They do not cross each other at three distinct points (as some people think).

It is worth while to remember this combined diagram: sketch it in your record book for reference.

17

Revision exercise

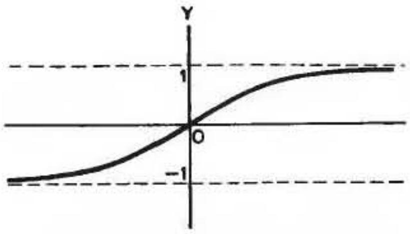
Fill in the following:

(a) $\dfrac{e^x + e^{-x}}{2} = \ldots\ldots\ldots\ldots$

(b) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \ldots\ldots\ldots\ldots$

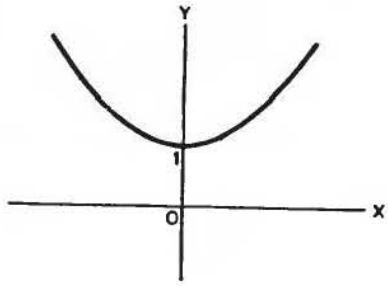
(c) $\dfrac{e^x - e^{-x}}{2} = \ldots\ldots\ldots\ldots$

(d)



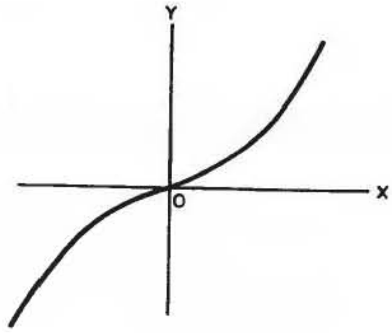
$y = \ldots\ldots\ldots\ldots$

(e)



$y = \ldots\ldots\ldots\ldots$

(f)



$y = \ldots\ldots\ldots\ldots$

Results in the next frame. Check your answers carefully

18

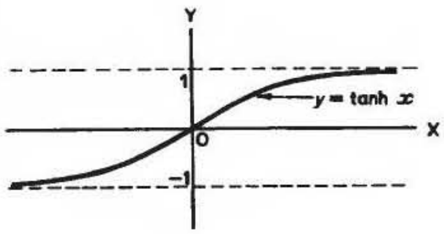
Here are the results: check yours.

(a) $\dfrac{e^x + e^{-x}}{2} = \cosh x$

(b) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$

(c) $\dfrac{e^x - e^{-x}}{2} = \sinh x$

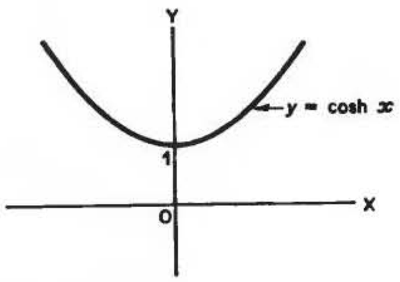
(d)



$y = \tanh x$

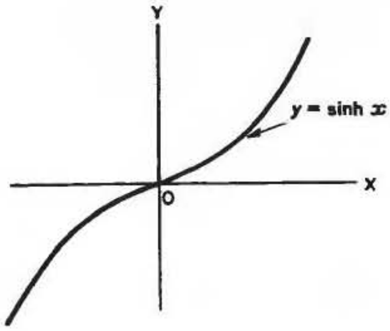
▶

(e)



$y = \cosh x$

(f)



$y = \sinh x$

Now we can continue with the next piece of work

Evaluation of hyperbolic functions

19

The values of $\sinh x$, $\cosh x$ and $\tanh x$ can be found using a calculator in just the same manner as the values of the circular trigonometric expressions were found. However, if your calculator does not possess the facility to work out hyperbolic expressions then their values can still be found by using the exponential key instead.

Example 1

To evaluate sinh 1·275

$$\text{Now } \sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \therefore \quad \sinh 1{\cdot}275 = \frac{1}{2}(e^{1\cdot 275} - e^{-1\cdot 275}).$$

We now have to evaluate $e^{1\cdot 275}$ and $e^{-1\cdot 275}$.

Using your calculator, you will find that:

$$e^{1\cdot 275} = 3{\cdot}579 \text{ and } e^{-1\cdot 275} = \frac{1}{3{\cdot}579} = 0{\cdot}2794$$

$$\therefore \quad \sinh 1{\cdot}275 = \frac{1}{2}(3{\cdot}579 - 0{\cdot}279)$$

$$= \frac{1}{2}(3{\cdot}300) = 1{\cdot}65$$

$$\therefore \quad \sinh 1{\cdot}275 = 1{\cdot}65$$

In the same way, you now find the value of cosh 2·156.

When you have finished, move on to Frame 20

20

$$\boxed{\cosh 2{\cdot}156 = 4{\cdot}377}$$

Here is the working:

Example 2

$$\cosh 2{\cdot}156 = \frac{1}{2}(e^{2{\cdot}156} + e^{-2{\cdot}156})$$
$$\therefore \quad \cosh 2{\cdot}156 = \frac{1}{2}(8{\cdot}637 + 0{\cdot}116)$$
$$= \frac{1}{2}(8{\cdot}753) = 4{\cdot}377$$
$$\therefore \quad \cosh 2{\cdot}156 = 4{\cdot}377$$

Right, one more. Find the value of $\tanh 1{\cdot}27$.

When you have finished, move on to Frame 21

21

$$\boxed{\tanh 1{\cdot}27 = 0{\cdot}8538}$$

Here is the working.

Example 3

$$\tanh 1{\cdot}27 = \frac{e^{1{\cdot}27} - e^{-1{\cdot}27}}{e^{1{\cdot}27} + e^{-1{\cdot}27}}$$
$$\therefore \quad \tanh 1{\cdot}27 = \frac{3{\cdot}561 - 0{\cdot}281}{3{\cdot}561 + 0{\cdot}281} = \frac{3{\cdot}280}{3{\cdot}842}$$
$$\tanh 1{\cdot}27 = 0{\cdot}8538$$

So, evaluating sinh, cosh and tanh is easy enough and depends mainly on being able to evaluate e^k, where k is a given number.

And now let us look at the reverse process. So on to Frame 22

Inverse hyperbolic functions

22

Example 1

To find $\sinh^{-1} 1{\cdot}475$, i.e. to find the value of x such that $\sinh x = 1{\cdot}475$.

Here it is: $\sinh x = 1{\cdot}475 \quad \therefore \quad \frac{1}{2}(e^x - e^{-x}) = 1{\cdot}475$

$$\therefore \; e^x - \frac{1}{e^x} = 2{\cdot}950$$

Multiplying both sides by e^x: $(e^x)^2 - 1 = 2{\cdot}95(e^x)$

$$(e^x)^2 - 2{\cdot}95(e^x) - 1 = 0$$

This is a quadratic equation and can be solved as usual, giving:

$$e^x = \frac{2{\cdot}95 \pm \sqrt{2{\cdot}95^2 + 4}}{2} = \frac{2{\cdot}95 \pm \sqrt{8{\cdot}703 + 4}}{2}$$

$$= \frac{2{\cdot}95 \pm \sqrt{12{\cdot}703}}{2} = \frac{2{\cdot}95 \pm 3{\cdot}564}{2}$$

$$= \frac{6{\cdot}514}{2} \text{ or } -\frac{0{\cdot}614}{2} = 3{\cdot}257 \text{ or } -0{\cdot}307$$

But e^x is always positive for real values of x. Therefore the only real solution is given by $e^x = 3{\cdot}257$.

$$\therefore \; x = \ln 3{\cdot}257 = 1{\cdot}1808$$

$$\therefore \; x = 1{\cdot}1808$$

Example 2

Now you find $\cosh^{-1} 2{\cdot}364$ in the same way.

23

$$\boxed{\cosh^{-1} 2{\cdot}364 = \pm 1{\cdot}5054 \text{ to 4 dp}}$$

Because

To evaluate $\cosh^{-1} 2{\cdot}364$, let $x = \cosh^{-1} 2{\cdot}364$

$$\therefore \; \cosh x = 2{\cdot}364 \quad \therefore \; \frac{e^x + e^{-x}}{2} = 2{\cdot}364 \quad \therefore \; e^x + \frac{1}{e^x} = 4{\cdot}728$$

$$(e^x)^2 - 4{\cdot}728(e^x) + 1 = 0$$

$$e^x = \frac{4{\cdot}728 \pm \sqrt{4{\cdot}728^2 - 4}}{2}$$

$$= \frac{1}{2}(4{\cdot}728 \pm 4{\cdot}284\ldots)$$

$$= 4{\cdot}5060\ldots \text{ or } 0{\cdot}2219\ldots$$

Therefore:

$$x = \ln 4{\cdot}5060\ldots \text{ or } \ln 0{\cdot}2219\ldots$$

$$= \pm 1{\cdot}5054 \text{ to 4 dp}$$

Before we do the next one, do you remember the exponential definition of $\tanh x$? Well, what is it?

24

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

That being so, we can now evaluate $\tanh^{-1} 0{\cdot}623$.

Let $x = \tanh^{-1} 0{\cdot}623 \;\therefore\; \tanh x = 0{\cdot}623$

$$\therefore \frac{e^x - e^{-x}}{e^x + e^{-x}} = 0{\cdot}623$$

$$\therefore e^x - e^{-x} = 0{\cdot}623(e^x + e^{-x})$$

$$\therefore (1 - 0{\cdot}623)e^x = (1 + 0{\cdot}623)e^{-x}$$

$$0{\cdot}377e^x = 1{\cdot}623e^{-x}$$

$$= \frac{1{\cdot}623}{e^x}$$

$$\therefore (e^x)^2 = \frac{1{\cdot}623}{0{\cdot}377}$$

$$\therefore e^x = 2{\cdot}075$$

$$\therefore x = \ln 2{\cdot}075 = 0{\cdot}7299$$

$$\therefore \tanh^{-1} 0{\cdot}623 = 0{\cdot}730$$

Now one for you to do on your own. Evaluate $\sinh^{-1} 0{\cdot}5$.

25

$$\sinh^{-1} 0{\cdot}5 = 0{\cdot}4812$$

Check your working.

Let $x = \sinh^{-1} 0{\cdot}5 \;\therefore\; \sinh x = 0{\cdot}5$

$$\therefore \frac{e^x - e^{-x}}{2} = 0{\cdot}5 \quad \therefore e^x - \frac{1}{e^x} = 1$$

$$\therefore (e^x)^2 - 1 = e^x$$

$$\therefore (e^x)^2 - (e^x) - 1 = 0$$

$$e^x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$= \frac{3{\cdot}2361}{2} \text{ or } \frac{-1{\cdot}2361}{2}$$

$$= 1{\cdot}6180 \text{ or } -0{\cdot}6180$$

$$\therefore x = \ln 1{\cdot}6180 = 0{\cdot}4812$$

$$\sinh^{-1} 0{\cdot}5 = 0{\cdot}4812$$

$e^x = -0{\cdot}6180$ gives no real value of x

And just one more! Evaluate $\tanh^{-1} 0{\cdot}75$.

26

$$\boxed{\tanh^{-1} 0{\cdot}75 = 0{\cdot}9730}$$

Let $x = \tanh^{-1} 0{\cdot}75 \quad \therefore \quad \tanh x = 0{\cdot}75$

$$\therefore \quad \frac{e^x - e^{-x}}{e^x + e^{-x}} = 0{\cdot}75$$

$$e^x - e^{-x} = 0{\cdot}75(e^x + e^{-x})$$

$$(1 - 0{\cdot}75)e^x = (1 + 0{\cdot}75)e^{-x}$$

$$0{\cdot}25e^x = 1{\cdot}75e^{-x}$$

$$(e^x)^2 = \frac{1{\cdot}75}{0{\cdot}25} = 7$$

$$e^x = \pm\sqrt{7} = \pm 2{\cdot}6458$$

But remember that e^x cannot be negative for real values of x.
Therefore $e^x = 2{\cdot}6458$ is the only real solution.

$$\therefore \quad x = \ln 2{\cdot}6458 = 0{\cdot}9730$$

$$\tanh^{-1} 0{\cdot}75 = 0{\cdot}9730$$

Log form of the inverse hyperbolic functions

27

Let us do the same thing in a general way.
To find $\tanh^{-1} x$ in log form.
As usual, we start off with: Let $y = \tanh^{-1} x \quad \therefore \quad x = \tanh y$

$$\therefore \quad \frac{e^y - e^{-y}}{e^y + e^{-y}} = x \quad \therefore \quad e^y - e^{-y} = x(e^y + e^{-y})$$

$$e^y(1 - x) = e^{-y}(1 + x) = \frac{1}{e^y}(1 + x)$$

$$e^{2y} = \frac{1 + x}{1 - x}$$

$$\therefore \quad 2y = \ln\left\{\frac{1 + x}{1 - x}\right\}$$

$$\therefore \quad y = \tanh^{-1} x = \frac{1}{2}\ln\left\{\frac{1 + x}{1 - x}\right\}$$

So that: $\quad \tanh^{-1} 0{\cdot}5 = \frac{1}{2}\ln\left\{\frac{1{\cdot}5}{0{\cdot}5}\right\}$

$$= \frac{1}{2}\ln 3 = \frac{1}{2}(1{\cdot}0986) = 0{\cdot}5493$$

And similarly: $\quad \tanh^{-1}(-0{\cdot}6) = \ldots\ldots\ldots\ldots$

28

$$\boxed{\tanh^{-1}(-0{\cdot}6) = -0{\cdot}6931}$$

Because

$$\tanh^{-1} x = \frac{1}{2}\ln\left\{\frac{1+x}{1-x}\right\}$$

$$\therefore \ \tanh^{-1}(-0{\cdot}6) = \frac{1}{2}\ln\left\{\frac{1-0{\cdot}6}{1+0{\cdot}6}\right\} = \frac{1}{2}\ln\left\{\frac{0{\cdot}4}{1{\cdot}6}\right\}$$

$$= \frac{1}{2}\ln 0{\cdot}25$$

$$= \frac{1}{2}(-1{\cdot}3863)$$

$$= -0{\cdot}6931$$

Now, in the same way, find an expression for $\sinh^{-1} x$.
Start off by saying: Let $y = \sinh^{-1} x \quad \therefore \ x = \sinh y$

$$\therefore \ \frac{e^y - e^{-y}}{2} = x \quad \therefore \ e^y - e^{-y} = 2x \quad \therefore \ e^y - \frac{1}{e^y} = 2x$$

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Now finish it off – result in Frame 29

29

$$\boxed{\sinh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}}$$

Because

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= x \pm \sqrt{x^2+1}$$

$$e^y = x + \sqrt{x^2+1} \text{ or } e^y = x - \sqrt{x^2+1}$$

At first sight, there appear to be two results, but notice this:

In the second result $\quad \sqrt{x^2+1} > x$

$\therefore \ e^y = x - (\text{something } > x)$, i.e. negative

Therefore we can discard the second result as far as we are concerned since powers of e are always positive. (Remember the graph of e^x.)

The only real solution then is given by $e^y = x + \sqrt{x^2+1}$

$$y = \sinh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}$$

30

Finally, let us find the general expression for $\cosh^{-1} x$.

Let $\quad y = \cosh^{-1} x \qquad \therefore\ x = \cosh y = \dfrac{e^y + e^{-y}}{2}$

$$\therefore\ e^y + \frac{1}{e^y} = 2x \qquad \therefore\ (e^y)^2 - 2x(e^y) + 1 = 0$$

$$\therefore\ e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\therefore\ e^y = x + \sqrt{x^2 - 1} \quad \text{and} \quad e^y = x - \sqrt{x^2 - 1}$$

Both these results are positive, since $\sqrt{x^2 - 1} < x$.

However $\quad \dfrac{1}{x + \sqrt{x^2 - 1}} = \dfrac{1}{x + \sqrt{x^2 - 1}} \cdot \dfrac{x - \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}$

$$= \frac{x - \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)} = x - \sqrt{x^2 - 1}$$

So our results can be written:

$$e^y = x + \sqrt{x^2 - 1} \quad \text{and} \quad e^y = \frac{1}{x + \sqrt{x^2 - 1}}$$

$$e^y = x + \sqrt{x^2 - 1} \text{ or } \left\{x + \sqrt{x^2 - 1}\right\}^{-1}$$

$$\therefore\ y = \ln\left\{x + \sqrt{x^2 - 1}\right\} \text{ or } -\ln\left\{x + \sqrt{x^2 - 1}\right\}$$

$$\therefore \qquad \cosh^{-1} x = \pm\ln\left\{x + \sqrt{x^2 - 1}\right\}$$

Notice that the plus and minus signs give two results which are symmetrical about the y-axis (agreeing with the graph of $y = \cosh x$).

31

Here are the three general results collected together:

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\}$$

$$\cosh^{-1} x = \pm\ln\left\{x + \sqrt{x^2 - 1}\right\}$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left\{\frac{1 + x}{1 - x}\right\}$$

Add these to your list in your record book. They will be useful. Compare the first two carefully, for they are very nearly alike. Note also that:

(a) $\sinh^{-1} x$ has only one value

(b) $\cosh^{-1} x$ has two values.

So what comes next? We shall see in Frame 32

Hyperbolic identities

There is no need to recoil in horror. You will see before long that we have an easy way of doing these. First of all, let us consider one or two relationships based on the basic definitions. 32

(1) The first set are really definitions themselves. Like the trig ratios, we have reciprocal hyperbolic functions:

(a) $\coth x$ (i.e. hyperbolic cotangent) $= \dfrac{1}{\tanh x}$

(b) $\operatorname{sech} x$ (i.e. hyperbolic secant) $= \dfrac{1}{\cosh x}$

(c) $\operatorname{cosech} x$ (i.e. hyperbolic cosecant) $= \dfrac{1}{\sinh x}$

These, by the way, are pronounced (a) coth, (b) sheck and (c) co-sheck respectively.

These remind us, once again, how like trig functions these hyperbolic functions are.

Make a list of these three definitions: then move on to Frame 33

(2) Let us consider $\dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{2} \div \dfrac{e^x + e^{-x}}{2}$ 33

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$$

$\therefore\ \tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad \left\{ \begin{array}{l} \text{Very much like} \\ \tan\theta = \dfrac{\sin\theta}{\cos\theta} \end{array} \right\}$

(3) $\cosh x = \dfrac{1}{2}(e^x + e^{-x})$; $\qquad \sinh x = \dfrac{1}{2}(e^x - e^{-x})$

Add these results: $\qquad \cosh x + \sinh x = e^x$

Subtract: $\qquad \cosh x - \sinh x = e^{-x}$

Multiply these two expressions together:

$(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x.e^{-x}$

$\therefore\ \cosh^2 x - \sinh^2 x = 1$

$\left\{ \begin{array}{l} \text{In trig, we have } \cos^2\theta + \sin^2\theta = 1, \\ \text{so there is a difference in sign here.} \end{array} \right\}$

On to Frame 34

34

(4) We have just established that $\cosh^2 x - \sinh^2 x = 1$.

Divide by $\cosh^2 x$:

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\therefore\ 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\therefore\ \operatorname{sech}^2 x = 1 - \tanh^2 x$$

{Something like $\sec^2\theta = 1 + \tan^2\theta$, isn't it?}

(5) If we start again with $\cosh^2 x - \sinh^2 x = 1$ and divide this time by $\sinh^2 x$, we get:

$$\frac{\cosh^2 x}{\sinh^2 x} - 1 = \frac{1}{\sinh^2 x}$$

$$\therefore\ \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\therefore\ \operatorname{cosech}^2 x = \coth^2 x - 1$$

{In trig, we have $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$, so there is a sign difference here too.}

Move on to Frame 35

35

(6) We have already used the fact that

$\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$

If we square each of these statements, we obtain

(a) ....

(b) ....

36

$$\boxed{\begin{array}{ll}\text{(a)} & \cosh^2 x + 2\sinh x\cosh x + \sinh^2 x = e^{2x}\\ \text{(b)} & \cosh^2 x - 2\sinh x\cosh x + \sinh^2 x = e^{-2x}\end{array}}$$

So if we subtract as they stand, we get:

$$4\sinh x\cosh x = e^{2x} - e^{-2x}$$

$$\therefore\ 2\sinh x\cosh x = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

$$\therefore\ \sinh 2x = 2\sinh x\cosh x$$

If however we add the two lines together, we get ....

37

$2(\cosh^2 x + \sinh^2 x) = e^{2x} + e^{-2x}$

$\therefore\ \cosh^2 x + \sinh^2 x = \dfrac{e^{2x} + e^{-2x}}{2} = \cosh 2x$

$\therefore\ \cosh 2x = \cosh^2 x + \sinh^2 x$

We already know that $\quad \cosh^2 x - \sinh^2 x = 1$

$\therefore\ \cosh^2 x = 1 + \sinh^2 x$

Substituting this in our previous result, we have:

$\cosh 2x = 1 + \sinh^2 x + \sinh^2 x$

$\therefore\ \cosh 2x = 1 + 2\sinh^2 x$

Or we could say $\quad \cosh^2 x - 1 = \sinh^2 x$

$\therefore\ \cosh 2x = \cosh^2 x + (\cosh^2 x - 1)$

$\therefore\ \cosh 2x = 2\cosh^2 x - 1$

Now we will collect all these hyperbolic identities together and compare them with the corresponding trig identities.

These are all listed in the next frame, so move on

38

	Trig identities	*Hyperbolic identities*
(1)	$\cot x = 1/\tan x$	$\coth x = 1/\tanh x$
	$\sec x = 1/\cos x$	$\text{sech}\ x = 1/\cosh x$
	$\text{cosec}\ x = 1/\sin x$	$\text{cosech}\ x = 1/\sinh x$
(2)	$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
	$\sec^2 x = 1 + \tan^2 x$	$\text{sech}^2 x = 1 - \tanh^2 x$
	$\text{cosec}^2 x = 1 + \cot^2 x$	$\text{cosech}^2 x = \coth^2 x - 1$
(3)	$\sin 2x = 2\sin x \cos x$	$\sinh 2x = 2\sinh x \cosh x$
	$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
	$= 1 - 2\sin^2 x$	$= 1 + 2\sinh^2 x$
	$= 2\cos^2 x - 1$	$= 2\cosh^2 x - 1$

If we look at these results, we find that some of the hyperbolic identities follow exactly the trig identities: others have a difference in sign. This change of sign occurs whenever $\sin^2 x$ in the trig results is being converted into $\sinh^2 x$ to form the corresponding hyperbolic identities. This sign change also occurs when $\sin^2 x$ is involved without actually being written as such. For example, $\tan^2 x$ involves $\sin^2 x$ since $\tan^2 x$ could be written as $\dfrac{\sin^2 x}{\cos^2 x}$. The change of sign therefore occurs with

$\tan^2 x$ when it is being converted into $\tanh^2 x$

$\cot^2 x$ when it is being converted into $\coth^2 x$

$\text{cosec}^2\, x$ when it is being converted into $\text{cosech}^2\, x$

▶

The sign change also occurs when we have a product of two sinh terms, e.g. the trig identity $\cos(A+B)=\cos A\cos B-\sin A\sin B$ gives the hyperbolic identity $\cosh(A+B)=\cosh A\cosh B+\sinh A\sinh B$.

Apart from this one change, the hyperbolic identities can be written down from the trig identities which you already know.
For example:

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x} \text{ becomes } \tanh 2x=\frac{2\tanh x}{1+\tanh^2 x}$$

So provided you know your trig identities, you can apply the rule to form the corresponding hyperbolic identities.

Relationship between trigonometric and hyperbolic functions

39

From our previous work on complex numbers, we know that:

$$e^{j\theta}=\cos\theta+j\sin\theta$$

and $$e^{-j\theta}=\cos\theta-j\sin\theta$$

Adding these two results together, we have:

$e^{j\theta}+e^{-j\theta}=\dots\dots\dots\dots$

40

$$\boxed{2\cos\theta}$$

So that: $\cos\theta=\dfrac{e^{j\theta}+e^{-j\theta}}{2}$

which is of the form $\dfrac{e^x+e^{-x}}{2}$, with x replaced by $(j\theta)$

$\therefore\ \cos\theta=\dots\dots\dots\dots$

41

$$\boxed{\cosh j\theta}$$

Here, then, is our first relationship:

$\cos\theta=\cosh j\theta$

Make a note of that for the moment: then on to Frame 42

42

If we return to our two original statements:

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

and this time subtract, we get a similar kind of result

$e^{j\theta} - e^{-j\theta} = \ldots\ldots\ldots\ldots$

43

$\boxed{2j\sin\theta}$

So that: $j\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2}$

$= \ldots\ldots\ldots\ldots$

44

$\boxed{\sinh j\theta}$

So: $\sinh j\theta = j\sin\theta$

Make a note of that also

45

So far, we have two important results:

(a) $\cosh j\theta = \cos\theta$

(b) $\sinh j\theta = j\sin\theta$

Now if we substitute $\theta = jx$ in the first of these results, we have:

$\cos jx = \cosh(j^2x) = \cosh(-x) \;\therefore\; \cos jx = \cosh x \quad [\text{since } \cosh(-x) = \cosh x]$

Writing this in reverse order, gives:

$\cosh x = \cos jx$ Another result to note

Now do exactly the same with the second result above, i.e. put $\theta = jx$ in the relationship $j\sin\theta = \sinh j\theta$ and simplify the result. What do you get?

46

$\boxed{j\sinh x = \sin jx}$

Because we have:

$$\begin{aligned} j\sin\theta &= \sinh j\theta \\ j\sin jx &= \sinh(j^2x) \\ &= \sinh(-x) \\ &= -\sinh x \quad [\text{since } \sinh(-x) = -\sinh x] \end{aligned}$$

Finally, divide both sides by j, and we have

$\sin jx = j\sinh x$

On to the next frame

47

Now let us collect together the results we have established. They are so nearly alike, that we must distinguish between them:

$\sin jx = j\sinh x$	$\sinh jx = j\sin x$
$\cos jx = \cosh x$	$\cosh jx = \cos x$

and, by division, we can also obtain:

$\tan jx = j\tanh x$	$\tanh jx = j\tan x$

Copy the complete table into your record book for future use.

48

Here is an example of an application of these results:
Find an expansion for $\sin(x + jy)$.
Now we know that:

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
$$\therefore \quad \sin(x + jy) = \sin x\cos jy + \cos x\sin jy$$

so using the results we have listed, we can replace

$\cos jy$ by ....
and $\sin jy$ by ....

49

$\cos jy = \cosh y$	$\sin jy = j\sinh y$

So that:

$$\sin(x + jy) = \sin x\cos jy + \cos x\sin jy$$
becomes $$\sin(x + jy) = \sin x\cosh y + j\cos x\sinh y$$

Note: $\sin(x + jy)$ is a function of the angle $(x + jy)$, which is, of course, a complex quantity. In this case, $(x + jy)$ is referred to as a *complex variable* and you will most likely deal with this topic at a later stage of your course.

Meanwhile, here is just one example for you to work through:
Find an expansion for $\cos(x - jy)$.

Then check with Frame 50

50

$$\boxed{\cos(x - jy) = \cos x\cosh y + j\sin x\sinh y}$$

Here is the working:

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$
$$\therefore \quad \cos(x - jy) = \cos x\cos jy + \sin x\sin jy$$
But $\cos jy = \cosh y$
and $\sin jy = j\sinh y$
$$\therefore \quad \cos(x - jy) = \cos x\cosh y + j\sin x\sinh y$$

51

All that now remains is the **Can You?** checklist and the **Test exercise**, but before working through them, look through your notes, or revise any parts of the Programme on which you are not perfectly clear.

When you are ready, move on to the next frame

Can You?

Checklist 3

52

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Define the hyperbolic functions in terms of the exponential function? — 1 to 4
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Express the hyperbolic functions as power series? — 5 to 8
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognize the graphs of the hyperbolic functions? — 9 to 16
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate hyperbolic functions and their inverses? — 19 to 26
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine the logarithmic form of the inverse hyperbolic functions? — 27 to 31
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Prove hyperbolic trigonometric identities? — 32 to 38
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Understand the relationship between the circular and the hyperbolic trigonometric functions? — 39 to 50
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 3

53

1 On the same axes, draw sketch graphs of (a) $y = \sinh x$, (b) $y = \cosh x$, (c) $y = \tanh x$.

2 If $L = 2C\sinh\dfrac{H}{2C}$, find L when $H = 63$ and $C = 50$.

3 If $v^2 = 1{\cdot}8L\tanh\dfrac{6{\cdot}3d}{L}$, find v when $d = 40$ and $L = 315$.

4 Calculate from first principles, the value of:
(a) $\sinh^{-1} 1{\cdot}532$ (b) $\cosh^{-1} 1{\cdot}25$

5 If $\tanh x = \dfrac{1}{3}$, find e^{2x} and hence evaluate x.

6 The curve assumed by a heavy chain or cable is:

$y = C\cosh\dfrac{x}{C}$

If $C = 50$, calculate: (a) the value of y when $x = 109$
(b) the value of x when $y = 75$.

7 Simplify $\dfrac{1 + \sinh 2A + \cosh 2A}{1 - \sinh 2A - \cosh 2A}$.

8 Obtain the expansion of $\sin(x - jy)$ in terms of the trigonometric and hyperbolic functions of x and y.

Further problems 3

54

1 Prove that $\cosh 2x = 1 + 2\sinh^2 x$.

2 Express $\cosh 2x$ and $\sinh 2x$ in exponential form and hence solve for real values of x, the equation:

$2\cosh 2x - \sinh 2x = 2$

3 If $\sinh x = \tan y$, show that $x = \ln(\tan y \pm \sec y)$.

4 If $a = c\cosh x$ and $b = c\sinh x$, prove that

$(a + b)^2 e^{-2x} = a^2 - b^2$

5 Evaluate: (a) $\tanh^{-1} 0{\cdot}75$ and (b) $\cosh^{-1} 2$.

6 Prove that $\tanh^{-1}\left\{\dfrac{x^2 - 1}{x^2 + 1}\right\} = \ln x$.

7 Express (a) $\cosh\frac{1+j}{2}$ and (b) $\sinh\frac{1+j}{2}$ in the form $a+jb$, giving a and b to 4 significant figures.

8 Prove that:

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

(b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Hence prove that:

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

9 Show that the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be represented in the form $x = a\cosh u$, $y = b\sinh u$.

10 Solve for real values of x:

$$3\cosh 2x = 3 + \sinh 2x$$

11 Prove that: $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$.

12 If $t = \tanh\frac{x}{2}$, prove that $\sinh x = \frac{2t}{1-t^2}$ and $\cosh x = \frac{1+t^2}{1-t^2}$.

Hence solve the equation:

$$7\sinh x + 20\cosh x = 24$$

13 If $x = \ln\tan\left\{\frac{\pi}{4} + \frac{\theta}{2}\right\}$, find e^x and e^{-x}, and hence show that $\sinh x = \tan\theta$.

14 Given that $\sinh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}$, determine $\sinh^{-1}(2+j)$ in the form $a+jb$.

15 If $\tan\frac{x}{2} = \tan A \tanh B$, prove that: $\tan x = \frac{\sin 2A \sinh 2B}{1 + \cos 2A \cosh 2B}$

16 Prove that $\sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$.

17 If $\lambda = \frac{at}{2}\left\{\frac{\sinh at + \sin at}{\cosh at - \cos at}\right\}$, calculate λ when $a = 0{\cdot}215$ and $t = 5$.

18 Prove that $\tanh^{-1}\left\{\frac{x^2-a^2}{x^2+a^2}\right\} = \ln\frac{x}{a}$.

Programme 4

Determinants

Frames 1 to 69

Learning outcomes

When you have completed this Programme you will be able to:

- Expand a 2×2 determinant
- Solve pairs of simultaneous linear equations in two variables using 2×2 determinants
- Expand a 3×3 determinant
- Solve three simultaneous linear equations in three variables using 3×3 determinants
- Determine the consistency of sets of simultaneous linear equations
- Use the properties of determinants to solve equations written in determinant form

Determinants

1

You are quite familiar with the method of solving a pair of simultaneous equations by elimination.

e.g. To solve $2x + 3y + 2 = 0$ (a)
$3x + 4y + 6 = 0$ (b)

we could first find the value of x by eliminating y. To do this, of course, we should multiply (a) by 4 and (b) by 3 to make the coefficient of y the same in each equation.

So $8x + 12y + 8 = 0$
$9x + 12y + 18 = 0$

Then by subtraction, we get $x + 10 = 0$, i.e. $x = -10$. By substituting back in either equation, we then obtain $y = 6$.

So, finally, $x = -10$, $y = 6$

That was trivial. You have done similar ones many times before. In just the same way, if:

$a_1x + b_1y + d_1 = 0$ (a)
$a_2x + b_2y + d_2 = 0$ (b)

then to eliminate y we make the coefficients of y in the two equations identical by multiplying (a) by .... and (b) by ....

2

(a) by b_2 and (b) by b_1

Correct, of course. So the equations:

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

become
$$a_1b_2x + b_1b_2y + b_2d_1 = 0$$
$$a_2b_1x + b_1b_2y + b_1d_2 = 0$$

Subtracting, we get:

$$(a_1b_2 - a_2b_1)x + b_2d_1 - b_1d_2 = 0$$

so that
$$(a_1b_2 - a_2b_1)x = b_1d_2 - b_2d_1$$

Then $x = \ldots\ldots\ldots\ldots$

3

$$\boxed{x = \frac{b_1 d_2 - b_2 d_1}{a_1 b_2 - a_2 b_1}}$$

In practice, this result can give a finite value for x only if the denominator is not zero. That is, the equations:

$$a_1 x + b_1 y + d_1 = 0$$
$$a_2 x + b_2 y + d_2 = 0$$

give a finite value for x provided that $(a_1 b_2 - a_2 b_1) \neq 0$.
Consider these equations:

$$3x + 2y - 5 = 0$$
$$4x + 3y - 7 = 0$$

In this case, $a_1 = 3$, $b_1 = 2$, $a_2 = 4$, $b_2 = 3$

$$\begin{aligned} a_1 b_2 - a_2 b_1 &= 3 \times 3 - 4 \times 2 \\ &= 9 - 8 = 1 \end{aligned}$$

This is not zero, so there $\left\{ \begin{array}{l} \text{will} \\ \text{will not} \end{array} \right\}$ be a finite value of x.

4

$\boxed{\text{will}}$

The expression $a_1 b_2 - a_2 b_1$ is therefore an important one in the solution of simultaneous equations. We have a shorthand notation for this:

$$a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

For $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ to represent $a_1 b_2 - a_2 b_1$ then we must multiply the terms diagonally to form the product terms in the expansion: we multiply $\begin{vmatrix} a_1 & \\ & b_2 \end{vmatrix}$ and then subtract the product $\begin{vmatrix} & b_1 \\ a_2 & \end{vmatrix}$ i.e. $+ \searrow$ and $- \nearrow$

e.g. $\begin{vmatrix} 3 & 7 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 3 & \\ & 2 \end{vmatrix} - \begin{vmatrix} & 7 \\ 5 & \end{vmatrix} = 3 \times 2 - 5 \times 7 = 6 - 35 = -29$

So $\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 6 & \\ & 2 \end{vmatrix} - \begin{vmatrix} & 5 \\ 1 & \end{vmatrix} = \ldots\ldots\ldots\ldots$

5

$$\boxed{\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix} = 12 - 5 = 7}$$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a *determinant* of the second order (since it has two rows and two columns) and represents $a_1b_2 - a_2b_1$. You can easily remember this as $+ \searrow - \nearrow$.

Just for practice, evaluate the following determinants:

$$\text{(a)} \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} 7 & 4 \\ 6 & 3 \end{vmatrix}, \quad \text{(c)} \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix}$$

Finish all three: then move on to Frame 6

6

(a) $\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 4 \times 3 - 5 \times 2 = 12 - 10 = \boxed{2}$

(b) $\begin{vmatrix} 7 & 4 \\ 6 & 3 \end{vmatrix} = 7 \times 3 - 6 \times 4 = 21 - 24 = \boxed{-3}$

(c) $\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = 2(-3) - 4 \times 1 = -6 - 4 = \boxed{-10}$

Now, in solving the equations $\begin{cases} a_1x + b_1y + d_1 = 0 \\ a_2x + b_2y + d_2 = 0 \end{cases}$

we found that $x = \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}$ and the numerator and the denominator can each be written as a determinant.

$b_1d_2 - b_2d_1 = \ldots\ldots\ldots\ldots;\quad a_1b_2 - a_2b_1 = \ldots\ldots\ldots\ldots$

7

$$\boxed{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}; \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

If we eliminate x from the original equations and find an expression for y, we obtain $y = -\left\{\dfrac{a_1d_2 - a_2d_1}{a_1b_2 - a_2b_1}\right\}$

So, for any pair of simultaneous equations:

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

we have $\quad x = \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1d_2 - a_2d_1}{a_1b_2 - a_2b_1}$

Each of these numerators and denominators can be expressed as a determinant.

So, $\quad x = \ldots\ldots\ldots\ldots$ and $y = \ldots\ldots\ldots\ldots$

8

$$x = \frac{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = -\frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\therefore \quad \frac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } \frac{y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

We can combine these results, thus:

$$\frac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Make a note of these results and then move on to the next frame

9

So if $\begin{cases} a_1x + b_1y + d_1 = 0 \\ a_2x + b_2y + d_2 = 0 \end{cases}$

Then $\dfrac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

Each variable is divided by a determinant. Let us see how we can get them from the original equations.

(a) Consider $\dfrac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}}$. Let us denote the determinant in the denominator by $\triangle_1$, i.e. $\triangle_1 = \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}$

To form $\triangle_1$ from the given equations, omit the x-terms and write down the coefficients and constant terms in the order in which they stand:

$$\begin{cases} a_1x + b_1y + d_1 = 0 \\ a_2x + b_2y + d_2 = 0 \end{cases} \text{gives } \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}$$

(b) Similarly for $\dfrac{-y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}$, let $\triangle_2 = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

To form $\triangle_2$ from the given equations, omit the y-terms and write down the coefficients and constant terms in the order in which they stand:

$$\begin{cases} a_1x + b_1y + d_1 = 0 \\ a_2x + b_2y + d_2 = 0 \end{cases} \text{gives } \triangle_2 = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

▶

(c) For the expression $\dfrac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$, denote the determinant by $\triangle_0$.

To form $\triangle_0$ from the given equations, omit the constant terms and write down the coefficients in the order in which they stand:

$$\begin{cases} a_1x + b_1y + d_1 = 0 \\ a_2x + b_2y + d_2 = 0 \end{cases} \text{gives} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Note finally that $\dfrac{x}{\triangle_1} = -\dfrac{y}{\triangle_2} = \dfrac{1}{\triangle_0}$

Now let us do some examples, so on to Frame 10

10

Example 1

To solve the equations $\begin{cases} 5x + 2y + 19 = 0 \\ 3x + 4y + 17 = 0 \end{cases}$

The key to the method is:

$$\frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{1}{\triangle_0}$$

To find $\triangle_0$, omit the constant terms:

$$\therefore\ \triangle_0 = \begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = 5 \times 4 - 3 \times 2 = 20 - 6 = 14$$

$$\therefore\ \triangle_0 = 14 \quad \text{(a)}$$

Now, to find $\triangle_1$, omit the x-terms:

$\therefore\ \triangle_1 = \dots\dots\dots\dots$

11

$$\boxed{\triangle_1 = -42}$$

Because $\triangle_1 = \begin{vmatrix} 2 & 19 \\ 4 & 17 \end{vmatrix} = 34 - 76 = -42 \quad \text{(b)}$

Similarly, to find $\triangle_2$, omit the y-terms:

$$\triangle_2 = \begin{vmatrix} 5 & 19 \\ 3 & 17 \end{vmatrix} = 85 - 57 = 28 \quad \text{(c)}$$

Substituting the values of $\triangle_1$, $\triangle_2$, $\triangle_0$ in the key, we get

$$\frac{x}{-42} = \frac{-y}{28} = \frac{1}{14}$$

from which $x = \dots\dots\dots\dots$ and $y = \dots\dots\dots\dots$

12

$$\boxed{x = \frac{-42}{14} = -3; \quad -y = \frac{28}{14}, \quad y = -2}$$

Now for another example.

Example 2

Solve by determinants $\begin{cases} 2x + 3y - 14 = 0 \\ 3x - 2y + 5 = 0 \end{cases}$

First of all, write down the key:

$$\frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{1}{\triangle_0}$$

(Note that the terms are alternately positive and negative.)

Then $\quad \triangle_0 = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13 \quad \text{(a)}$

Now you find $\triangle_1$ and $\triangle_2$ in the same way.

13

$$\boxed{\triangle_1 = -13; \quad \triangle_2 = 52}$$

Because we have $\begin{cases} 2x + 3y - 14 = 0 \\ 3x - 2y + 5 = 0 \end{cases}$

$$\therefore\ \triangle_1 = \begin{vmatrix} 3 & -14 \\ -2 & 5 \end{vmatrix} = \begin{vmatrix} 3 & \\ & 5 \end{vmatrix} - \begin{vmatrix} & -14 \\ -2 & \end{vmatrix}$$

$$= 15 - 28 = -13. \quad \therefore\ \triangle_1 = -13$$

$$\triangle_2 = \begin{vmatrix} 2 & -14 \\ 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & \\ & 5 \end{vmatrix} - \begin{vmatrix} & -14 \\ 3 & \end{vmatrix}$$

$$= 10 - (-42) = 52. \quad \therefore\ \triangle_2 = 52$$

So that $\quad \dfrac{x}{\triangle_1} = \dfrac{-y}{\triangle_2} = \dfrac{1}{\triangle_0}$

and $\quad \triangle_1 = -13; \quad \triangle_2 = 52; \quad \triangle_0 = -13$

$$\therefore\ x = \frac{\triangle_1}{\triangle_0} = \frac{-13}{-13} = 1 \qquad \therefore\ x = 1$$

$$-y = \frac{\triangle_2}{\triangle_0} = \frac{52}{-13} = -4 \qquad \therefore\ y = 4$$

Do not forget the key:

$$\frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{1}{\triangle_0}$$

with alternate plus and minus signs.

Make a note of this in your record book

14

Here is another one: do it on your own.

Example 3

Solve by determinants:

$$\begin{cases} 4x - 3y + 20 = 0 \\ 3x + 2y - 2 = 0 \end{cases}$$

First of all, write down the key.

Then off you go: find $\triangle_0$, $\triangle_1$ and $\triangle_2$ and hence determine the values of x and y.

When you have finished, move on to Frame 15

15

$$\boxed{x = -2; \quad y = 4}$$

Here is the working in detail:

$$\begin{cases} 4x - 3y + 20 = 0 \\ 3x + 2y - 2 = 0 \end{cases} \qquad \frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{1}{\triangle_0}$$

$$\triangle_0 = \begin{vmatrix} 4 & -3 \\ 3 & 2 \end{vmatrix} = 8 - (-9) = 8 + 9 = 17$$

$$\triangle_1 = \begin{vmatrix} -3 & 20 \\ 2 & -2 \end{vmatrix} = 6 - 40 = -34$$

$$\triangle_2 = \begin{vmatrix} 4 & 20 \\ 3 & -2 \end{vmatrix} = -8 - 60 = -68$$

$$x = \frac{\triangle_1}{\triangle_0} = \frac{-34}{17} = -2 \qquad \therefore\ x = -2$$

$$-y = \frac{\triangle_2}{\triangle_0} = \frac{-68}{17} = -4 \qquad \therefore\ y = 4$$

Now, by way of revision, complete the following:

(a) $\begin{vmatrix} 5 & 6 \\ 7 & 4 \end{vmatrix} = \ldots\ldots\ldots\ldots$

(b) $\begin{vmatrix} 5 & -2 \\ -3 & -4 \end{vmatrix} = \ldots\ldots\ldots\ldots$

(c) $\begin{vmatrix} a & d \\ b & c \end{vmatrix} = \ldots\ldots\ldots\ldots$

(d) $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \ldots\ldots\ldots\ldots$

Here are the results. You must have got them correct. **16**

(a) $20 - 42 = -22$
(b) $-20 - 6 = -26$
(c) $ac - bd$
(d) $ps - rq$

For the next section of the work, move on to Frame 17

Determinants of the third order

A determinant of the third order will contain 3 rows and 3 columns, thus: **17**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Each element in the determinant is associated with its *minor*, which is found by omitting the row and column containing the element concerned.

e.g. the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ obtained $\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$

the minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ obtained $\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$

the minor of c_1 is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ obtained $\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$

So, in the same way, the minor of a_2 is ....

18

Minor of a_2 is $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$

Because, to find the minor of a_2, we simply ignore the row and column containing a_2, i.e.

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{matrix}$$

Similarly, the minor of b_3 is ....

19

$$\boxed{\text{Minor of } b_3 \text{ is } \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}$$

i.e. omit the row and column containing b_3.

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}$$

Now on to Frame 20

20

Evaluation of a third-order determinant

To expand a determinant of the third order, we can write down each element along the top row, multiply it by its minor and give the terms a plus or minus sign alternately:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Then, of course, we already know how to expand a determinant of the second order by multiplying diagonally, $+ \searrow - \nearrow$

Example 1

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 5 & 7 \\ 2 & 4 & 8 \end{vmatrix} = 1\begin{vmatrix} 5 & 7 \\ 4 & 8 \end{vmatrix} - 3\begin{vmatrix} 4 & 7 \\ 2 & 8 \end{vmatrix} + 2\begin{vmatrix} 4 & 5 \\ 2 & 4 \end{vmatrix}$$

$$\begin{aligned} &= 1(5 \times 8 - 4 \times 7) - 3(4 \times 8 - 2 \times 7) + 2(4 \times 4 - 2 \times 5) \\ &= 1(40 - 28) - 3(32 - 14) + 2(16 - 10) \\ &= 1(12) - 3(18) + 2(6) \\ &= 12 - 54 + 12 = -30 \end{aligned}$$

21

Here is another.

Example 2

$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 6 & 7 \\ 2 & 9 & 2 \end{vmatrix} = 3\begin{vmatrix} 6 & 7 \\ 9 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 7 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 4 & 6 \\ 2 & 9 \end{vmatrix}$$

$$\begin{aligned} &= 3(12 - 63) - 2(8 - 14) + 5(36 - 12) \\ &= 3(-51) - 2(-6) + 5(24) \\ &= -153 + 12 + 120 = -21 \end{aligned}$$

Now here is one for you to do.

Example 3

Evaluate $\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix}$

Expand along the top row, multiply each element by its minor, and assign alternate + and − signs to the products.

When you are ready, move on to Frame 22 for the result

22

$\boxed{38}$

Because $\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix} = 2\begin{vmatrix} 6 & 3 \\ 9 & 1 \end{vmatrix} - 7\begin{vmatrix} 4 & 3 \\ 8 & 1 \end{vmatrix} + 5\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix}$

$$= 2(6 - 27) - 7(4 - 24) + 5(36 - 48)$$
$$= 2(-21) - 7(-20) + 5(-12)$$
$$= -42 + 140 - 60 = 38$$

We obtained the result above by expanding along the top row of the given determinant. If we expand down the first column in the same way, still assigning alternate + and − signs to the products, we get:

$$\begin{vmatrix} 2 & 7 & 5 \\ 4 & 6 & 3 \\ 8 & 9 & 1 \end{vmatrix} = 2\begin{vmatrix} 6 & 3 \\ 9 & 1 \end{vmatrix} - 4\begin{vmatrix} 7 & 5 \\ 9 & 1 \end{vmatrix} + 8\begin{vmatrix} 7 & 5 \\ 6 & 3 \end{vmatrix}$$
$$= 2(6 - 27) - 4(7 - 45) + 8(21 - 30)$$
$$= 2(-21) - 4(-38) + 8(-9)$$
$$= -42 + 152 - 72 = 38$$

which is the same result as that which we obtained before.

23

We can, if we wish, expand along any row or column in the same way, multiplying each element by its minor, so long as we assign to each product the appropriate + or − sign. The appropriate 'place signs' are given by

$$\begin{matrix} + & - & + & - & + & \dots & \dots \\ - & + & - & + & - & \dots & \dots \\ + & - & + & - & + & \dots & \dots \\ - & + & - & + & - & \dots & \dots \end{matrix}$$

etc., etc.

The key element (in the top left-hand corner) is always +. The others are then alternately + or −, as you proceed along any row or down any column.
So in the determinant:

$$\begin{vmatrix} 1 & 3 & 7 \\ 5 & 6 & 9 \\ 4 & 2 & 8 \end{vmatrix}$$

the 'place sign' of the element 9 is ....

24

$$\boxed{-}$$

Because in a third-order determinant, the 'place signs' are:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Remember that the top left-hand element always has a + place sign. The others follow from it.

Now consider this one: $\begin{vmatrix} 3 & 7 & 2 \\ 6 & 8 & 4 \\ 1 & 9 & 5 \end{vmatrix}$

If we expand down the middle column, we get

$$\begin{vmatrix} 3 & 7 & 2 \\ 6 & 8 & 4 \\ 1 & 9 & 5 \end{vmatrix} = -7\begin{vmatrix} 6 & 4 \\ 1 & 5 \end{vmatrix} + 8\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = \ldots\ldots\ldots\ldots$$

Finish it off. Then move on for the result

25

$$\boxed{-78}$$

Because $-7\begin{vmatrix} 6 & 4 \\ 1 & 5 \end{vmatrix} + 8\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}$

$= -7(30 - 4) + 8(15 - 2) - 9(12 - 12)$

$= -7(26) + 8(13) - 9(0) = -182 + 104 = -78$

So now you do this one:

Evaluate $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 1 & 3 \\ 5 & 7 & 2 \end{vmatrix}$ by expanding along the *bottom* row.

When you have done it, move to Frame 26 for the answer

26

$$\boxed{119}$$

We have $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 1 & 3 \\ 5 & 7 & 2 \end{vmatrix}$ and remember $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$= 5\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} - 7\begin{vmatrix} 2 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix} = 5(9 - 4) - 7(6 - 24) + 2(2 - 18)$

$= 5(5) - 7(-18) + 2(-16) = 25 + 126 - 32 = 119$

One more: Evaluate $\begin{vmatrix} 1 & 2 & 8 \\ 7 & 3 & 1 \\ 4 & 6 & 9 \end{vmatrix}$ by expanding along the *middle* row.

Result in Frame 27

27

$\boxed{143}$

Because $\begin{vmatrix} 1 & 2 & 8 \\ 7 & 3 & 1 \\ 4 & 6 & 9 \end{vmatrix} = -7\begin{vmatrix} 2 & 8 \\ 6 & 9 \end{vmatrix} + 3\begin{vmatrix} 1 & 8 \\ 4 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$

$$= -7(18 - 48) + 3(9 - 32) - 1(6 - 8)$$
$$= -7(-30) + 3(-23) - 1(-2)$$
$$= 210 - 69 + 2 = 143$$

We have seen how we can use second-order determinants to solve simultaneous equations in 2 unknowns.

We can now extend the method to solve simultaneous equations in 3 unknowns.

So move on to Frame 28

Simultaneous equations in three unknowns

28

Consider the equations:

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \\ a_3x + b_3y + c_3z + d_3 = 0 \end{cases}$$

If we find x, y and z by the elimination method, we obtain results that can be expressed in determinant form thus:

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

We can remember this more easily in this form:

$$\frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{z}{\triangle_3} = \frac{-1}{\triangle_0}$$

where $\triangle_1 =$ the det of the coefficients omitting the x-terms

$\triangle_2 =$ the det of the coefficients omitting the y-terms

$\triangle_3 =$ the det of the coefficients omitting the z-terms

$\triangle_0 =$ the det of the coefficients omitting the constant terms.

Notice that the signs are alternately plus and minus.

Let us work through a numerical example.

▶

Example 1

Find the value of x from the equations:

$$\begin{cases} 2x+3y-z-4=0 \\ 3x+y+2z-13=0 \\ x+2y-5z+11=0 \end{cases}$$

First the key: $\dfrac{x}{\triangle_1}=\dfrac{-y}{\triangle_2}=\dfrac{z}{\triangle_3}=\dfrac{-1}{\triangle_0}$

To find the value of x, we use $\dfrac{x}{\triangle_1}=\dfrac{-1}{\triangle_0}$, i.e. we must find $\triangle_1$ and $\triangle_0$.

(a) To find $\triangle_0$, omit the constant terms.

$$\therefore\ \triangle_0=\begin{vmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 2 & -5 \end{vmatrix}=2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}-3\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix}-1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$=-18+51-5=28$$

(b) Now you find $\triangle_1$, in the same way.

29

$$\boxed{\triangle_1=-56}$$

Because $$\triangle_1=\begin{vmatrix} 3 & -1 & -4 \\ 1 & 2 & -13 \\ 2 & -5 & 11 \end{vmatrix}=3(22-65)+1(11+26)-4(-5-4)$$

$$=3(-43)+1(37)-4(-9)$$

$$=-129+37+36$$

$$=-129+73=-56$$

But $\dfrac{x}{\triangle_1}=\dfrac{-1}{\triangle_0}\quad \therefore\ \dfrac{x}{-56}=\dfrac{-1}{28}$

$\therefore\ x=\dfrac{56}{28}=2\quad \therefore\ x=2$

Note that by this method we can evaluate any one of the variables, without necessarily finding the others. Let us do another example.

Example 2

Find y, given that:

$$\begin{cases} 2x+y-5z+11=0 \\ x-y+z-6=0 \\ 4x+2y-3z+8=0 \end{cases}$$

First, the key, which is ....

30

$$\boxed{\frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{z}{\triangle_3} = \frac{-1}{\triangle_0}}$$

To find y, we use $\dfrac{-y}{\triangle_2} = \dfrac{-1}{\triangle_0}$

Therefore, we must find $\triangle_2$ and $\triangle_0$.

The equations are $\begin{cases} 2x + y - 5z + 11 = 0 \\ x - y + z - 6 = 0 \\ 4x + 2y - 3z + 8 = 0 \end{cases}$

To find $\triangle_2$, omit the y-terms.

$$\therefore\ \triangle_2 = \begin{vmatrix} 2 & -5 & 11 \\ 1 & 1 & -6 \\ 4 & -3 & 8 \end{vmatrix} = 2\begin{vmatrix} 1 & -6 \\ -3 & 8 \end{vmatrix} + 5\begin{vmatrix} 1 & -6 \\ 4 & 8 \end{vmatrix} + 11\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix}$$

$$= 2(8 - 18) + 5(8 + 24) + 11(-3 - 4)$$

$$= -20 + 160 - 77 = 63$$

To find $\triangle_0$, omit the constant terms

$\therefore\ \triangle_0 = \ldots\ldots\ldots\ldots$

31

$$\boxed{\triangle_0 = -21}$$

Because

$$\triangle_0 = \begin{vmatrix} 2 & 1 & -5 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 5\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= 2(3 - 2) - 1(-3 - 4) - 5(2 + 4)$$

$$= 2 + 7 - 30 = -21$$

So we have $\dfrac{-y}{\triangle_2} = \dfrac{-1}{\triangle_0}$ $\quad\therefore\ y = \dfrac{\triangle_2}{\triangle_0} = \dfrac{63}{-21}$

$\therefore\ y = -3$

The important things to remember are:

(a) The key: $\dfrac{x}{\triangle_1} = \dfrac{-y}{\triangle_2} = \dfrac{z}{\triangle_3} = \dfrac{-1}{\triangle_0}$

with alternate + and − signs.

(b) To find $\triangle_1$, which is associated with x in this case, omit the x-terms and form a determinant with the remaining coefficients and constant terms. Similarly for $\triangle_2$, $\triangle_3$, $\triangle_0$.

Next frame

32

Revision exercise

Here is a short revision exercise on the work so far.

Find the following by the use of determinants:

1 $\left\{\begin{array}{l} x+2y-3z-3=0 \\ 2x-y-z-11=0 \\ 3x+2y+z+5=0 \end{array}\right\}$ Find y.

2 $\left\{\begin{array}{l} 3x-4y+2z+8=0 \\ x+5y-3z+2=0 \\ 5x+3y-z+6=0 \end{array}\right\}$ Find x and z.

3 $\left\{\begin{array}{l} 2x-2y-z-3=0 \\ 4x+5y-2z+3=0 \\ 3x+4y-3z+7=0 \end{array}\right\}$ Find x, y and z.

When you have finished them all, check your answers with those given in the next frame

33

Here are the answers:

1 $y=-4$

2 $x=-2;\quad z=5$

3 $x=2;\quad y=-1;\quad z=3$

If you have them *all* correct, turn straight on to *Frame 52*.

If you have not got them all correct, it is well worth spending a few minutes seeing where you may have gone astray, for one of the main applications of determinants is in the solution of simultaneous equations.

If you made any slips, move to Frame 34

34

The answer to question **1** in the revision test was $\boxed{y=-4}$

Did you get that one right? If so, move on straight away to Frame 41. If you did not manage to get it right, let us work through it in detail.

The equations were $\left\{\begin{array}{l} x+2y-3z-3=0 \\ 2x-y-z-11=0 \\ 3x+2y+z+5=0 \end{array}\right.$

Copy them down on your paper so that we can refer to them as we go along.

The first thing, always, is to write down the key to the solutions. In this case:

$$\frac{x}{\Delta_1}=\ldots=\ldots=\ldots$$

To fill in the missing terms, take each variable in turn, divide it by the associated determinant, and include the appropriate sign.

So what do we get?

On to Frame 35

35

$$\boxed{\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0}}$$

The signs go alternately + and −.

In this question, we have to find y, so we use the second and last terms in the key.

i.e. $\dfrac{-y}{\Delta_2} = \dfrac{-1}{\Delta_0} \quad \therefore \; y = \dfrac{\Delta_2}{\Delta_0}$.

So we have to find Δ_2 and Δ_0.

To find Δ_2, we ....

36

form a determinant of the coefficients omitting those of the y-terms

So $$\Delta_2 = \begin{vmatrix} 1 & -3 & -3 \\ 2 & -1 & -11 \\ 3 & 1 & 5 \end{vmatrix}$$

Expanding along the top row, this gives:

$$\Delta_2 = 1\begin{vmatrix} -1 & -11 \\ 1 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & -11 \\ 3 & 5 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$

We now evaluate each of these second-order determinants by the usual process of multiplying diagonally, remembering the sign convention that $+ \searrow$ and $- \nearrow$

So we get $\Delta_2 = $ ....

37

$$\boxed{\Delta_2 = 120}$$

Because

$$\begin{aligned} \Delta_2 &= 1(-5+11) + 3(10+33) - 3(2+3) \\ &= 6 + 3(43) - 3(5) \\ &= 6 + 129 - 15 = 135 - 15 = 120 \end{aligned}$$

$\therefore \; \Delta_2 = 120$

We also have to find Δ_0, i.e. the determinant of the coefficients omitting the constant terms.

So $$\Delta_0 = \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

38

$$\boxed{\triangle_0 = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{vmatrix}}$$

If we expand this along the top row, we get $\triangle_0 = \dots\dots\dots$

39

$$\boxed{\triangle_0 = 1\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}}$$

Now, evaluating the second-order determinants in the usual way gives $\triangle_0 = \dots\dots\dots$

40

$$\boxed{\triangle_0 = -30}$$

Because $\quad \triangle_0 = 1(-1+2) - 2(2+3) - 3(4+3)$
$\qquad\qquad = 1(1) - 2(5) - 3(7) = 1 - 10 - 21 = -30$

So $\quad \triangle_0 = -30$

So we have $\quad y = \dfrac{\triangle_2}{\triangle_0} = \dfrac{120}{-30} = -4 \qquad \therefore\ y = -4$

Every one is done in the same way.

Did you get revision exercise **2** correct?

If so, turn straight on to *Frame 50*.

If not, have another go at it, now that we have worked through **1** in detail.

When you have finished, move on to Frame 41

41

The answers to revision exercise **2** were $\boxed{\begin{array}{l} x = -2 \\ z = 5 \end{array}}$

Did you get those correct? If so, turn on right away to *Frame 50*. If not, follow through the working. Here it is:

The equations were:

$$\begin{cases} 3x - 4y + 2z + 8 = 0 \\ x + 5y - 3z + 2 = 0 \\ 5x + 3y - z + 6 = 0 \end{cases}$$

Copy them down on to your paper.

The key to the solutions is: $\dfrac{x}{\triangle_1} = \dots = \dots = \dots$

Fill in the missing terms and then move on to Frame 42

42

$$\boxed{\frac{x}{\triangle_1} = \frac{-y}{\triangle_2} = \frac{z}{\triangle_3} = \frac{-1}{\triangle_0}}$$

We have to find x and z. $\therefore$ We shall use:

$$\frac{x}{\triangle_1} = -\frac{1}{\triangle_0} \quad \text{i.e. } x = -\frac{\triangle_1}{\triangle_0}$$

and $$\frac{z}{\triangle_3} = \frac{-1}{\triangle_0} \quad \text{i.e. } z = -\frac{\triangle_3}{\triangle_0}$$

So we must find $\triangle_1$, $\triangle_3$ and $\triangle_0$.

(a) To find $\triangle_1$, form the determinant of coefficients omitting those of the x-terms.

$$\therefore \triangle_1 = \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

43

$$\boxed{\triangle_1 = \begin{vmatrix} -4 & 2 & 8 \\ 5 & -3 & 2 \\ 3 & -1 & 6 \end{vmatrix}}$$

Now expand along the top row.

$$\triangle_1 = -4\begin{vmatrix} -3 & 2 \\ -1 & 6 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} + 8\begin{vmatrix} 5 & -3 \\ 3 & -1 \end{vmatrix}$$
$$= \dots\dots\dots$$

Finish it off: then on to Frame 44

44

$$\boxed{\triangle_1 = 48}$$

Because
$$\begin{aligned} \triangle_1 &= -4(-18+2) - 2(30-6) + 8(-5+9) \\ &= -4(-16) - 2(24) + 8(4) \\ &= 64 - 48 + 32 = 96 - 48 = 48 \end{aligned}$$
$$\therefore \triangle_1 = 48$$

(b) To find $\triangle_3$, form the determinant of coefficients omitting the z-terms.

$$\therefore \triangle_3 = \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

45

$$\Delta_3 = \begin{vmatrix} 3 & -4 & 8 \\ 1 & 5 & 2 \\ 5 & 3 & 6 \end{vmatrix}$$

Expanding this along the top row gives

$\Delta_3 = \ldots\ldots\ldots\ldots$

46

$$\Delta_3 = 3\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + 8\begin{vmatrix} 1 & 5 \\ 5 & 3 \end{vmatrix}$$

Now evaluate the second-order determinants and finish it off. So that

$\Delta_3 = \ldots\ldots\ldots$

On to Frame 47

47

$$\Delta_3 = -120$$

Because
$$\begin{aligned} \Delta_3 &= 3(30-6) + 4(6-10) + 8(3-25) \\ &= 3(24) + 4(-4) + 8(-22) \\ &= 72 - 16 - 176 \\ &= 72 - 192 = -120 \end{aligned}$$
$$\therefore\ \Delta_3 = -120$$

(c) Now we want to find Δ_0.

$$\Delta_0 = \begin{vmatrix} \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

48

$$\Delta_0 = \begin{vmatrix} 3 & -4 & 2 \\ 1 & 5 & -3 \\ 5 & 3 & -1 \end{vmatrix}$$

Now expand this along the top row as we have done before. Then evaluate the second-order determinants which will appear and so find the value of Δ_0. Work it right through: so that

$\Delta_0 = \ldots\ldots\ldots$

49

$$\boxed{\triangle_0 = 24}$$

Because $\triangle_0 = 3\begin{vmatrix} 5 & -3 \\ 3 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & -3 \\ 5 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 5 & 3 \end{vmatrix}$

$$= 3(-5+9) + 4(-1+15) + 2(3-25)$$
$$= 3(4) + 4(14) + 2(-22)$$
$$= 12 + 56 - 44$$
$$= 68 - 44 = 24$$
$$\therefore \triangle_0 = 24$$

So we have: $\triangle_1 = 48, \quad \triangle_3 = -120, \quad \triangle_0 = 24$

Also we know that:

$$x = -\frac{\triangle_1}{\triangle_0} \text{ and } z = -\frac{\triangle_3}{\triangle_0}$$

So that $x = \ldots\ldots\ldots$ and $z = \ldots\ldots\ldots$

50

$$x = -\frac{48}{24} = -2 \qquad \boxed{x = -2}$$
$$z = -\frac{(-120)}{24} = 5 \qquad \boxed{z = 5}$$

Well, there you are. The method is the same every time – but take care not to make a slip with the signs.

Now what about revision exercise **3**. Did you get that right? If so, move on straight away to *Frame 52*.

If not, have another go at it. Here are the equations again: copy them down and then find x, y and z.

$$2x - 2y - z - 3 = 0$$
$$4x + 5y - 2z + 3 = 0$$
$$3x + 4y - 3z + 7 = 0$$

When you have finished this one, move on to the next frame and check your results

51

$$\boxed{x = 2, \quad y = -1, \quad z = 3}$$

Here are the main steps, so that you can check your own working:

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0}$$

$$\Delta_1 = \begin{vmatrix} -2 & -1 & -3 \\ 5 & -2 & 3 \\ 4 & -3 & 7 \end{vmatrix} = 54 \qquad \Delta_2 = \begin{vmatrix} 2 & -1 & -3 \\ 4 & -2 & 3 \\ 3 & -3 & 7 \end{vmatrix} = 27$$

$$\Delta_3 = \begin{vmatrix} 2 & -2 & -3 \\ 4 & 5 & 3 \\ 3 & 4 & 7 \end{vmatrix} = 81 \qquad \Delta_0 = \begin{vmatrix} 2 & -2 & -1 \\ 4 & 5 & -2 \\ 3 & 4 & -3 \end{vmatrix} = -27$$

$$\frac{x}{\Delta_1} = -\frac{1}{\Delta_0} \qquad \therefore\ x = -\frac{\Delta_1}{\Delta_0} = -\frac{54}{-27} = 2 \qquad \therefore\ x = 2$$

$$\frac{-y}{\Delta_2} = -\frac{1}{\Delta_0} \qquad \therefore\ y = \frac{\Delta_2}{\Delta_0} = \frac{27}{-27} = -1 \qquad \therefore\ y = -1$$

$$\frac{z}{\Delta_3} = -\frac{1}{\Delta_0} \qquad \therefore\ z = -\frac{\Delta_3}{\Delta_0} = -\frac{81}{-27} = +3 \qquad \therefore\ z = +3$$

All correct now?

On to Frame 52 then for the next section of the work

Consistency of a set of equations

52

Let us consider the following three equations in two unknowns.

$$3x - y - 4 = 0 \qquad \text{(a)}$$
$$2x + 3y - 8 = 0 \qquad \text{(b)}$$
$$x - 2y + 3 = 0 \qquad \text{(c)}$$

If we solve equations (b) and (c) in the usual way, we find that $x = 1$ and $y = 2$.

If we now substitute these values in the left-hand side of (a), we obtain $3x - y - 4 = 3 - 2 - 4 = -3$ (and not 0 as the equation states).

The solutions of (b) and (c) do not satisfy (a) and the three given equations do not have a common solution. They are thus not *consistent*. There are no values of x and y which satisfy all three equations.

If equations are consistent, they have a ....

53

common solution

Let us now consider the three equations:

$$\begin{aligned} 3x + y - 5 &= 0 \qquad &\text{(a)}\\ 2x + 3y - 8 &= 0 \qquad &\text{(b)}\\ x - 2y + 3 &= 0 \qquad &\text{(c)} \end{aligned}$$

The solutions of (b) and (c) are, as before, $x = 1$ and $y = 2$. Substituting these in (a) gives:

$$3x + y - 5 = 3 + 2 - 5 = 0$$

i.e. all three equations have the common solution $x = 1$, $y = 2$ and the equations are said to be ....

54

consistent

Now we will take the general case:

$$\begin{aligned} a_1x + b_1y + d_1 &= 0 \qquad &\text{(a)}\\ a_2x + b_2y + d_2 &= 0 \qquad &\text{(b)}\\ a_3x + b_3y + d_3 &= 0 \qquad &\text{(c)} \end{aligned}$$

If we solve equations (b) and (c):

i.e. $\begin{cases} a_2x + b_2y + d_2 = 0 \\ a_3x + b_3y + d_3 = 0 \end{cases}$ we get $\dfrac{x}{\triangle_1} = \dfrac{-y}{\triangle_2} = \dfrac{1}{\triangle_0}$

where $\triangle_1 = \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix}$, $\triangle_2 = \begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix}$, $\triangle_0 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

so that $x = \dfrac{\triangle_1}{\triangle_0}$ and $y = -\dfrac{\triangle_2}{\triangle_0}$

If these results also satisfy equation (a), then $a_1 \cdot \dfrac{\triangle_1}{\triangle_0} + b_1 \cdot \dfrac{-\triangle_2}{\triangle_0} + d_1 = 0$

i.e. $a_1.\triangle_1 - b_1.\triangle_2 + d_1.\triangle_0 = 0$

i.e. $a_1\begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix} + d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$

which is therefore the condition that the three given equations are *consistent*.

So three simultaneous equations in two unknowns are consistent if the determinant of coefficients is ....

55

$\boxed{\text{zero}}$

Example 1

Test for consistency:

$$\begin{cases} 2x + y - 5 &= 0 \\ x + 4y + 1 &= 0 \\ 3x - y - 10 &= 0 \end{cases}$$

For the equations to be consistent $\begin{vmatrix} 2 & 1 & -5 \\ 1 & 4 & 1 \\ 3 & -1 & -10 \end{vmatrix}$ must be zero

$$\begin{vmatrix} 2 & 1 & -5 \\ 1 & 4 & 1 \\ 3 & -1 & -10 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ -1 & -10 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -10 \end{vmatrix} - 5\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix}$$
$$= 2(-40+1) - 1(-10-3) - 5(-1-12)$$
$$= 2(-39) - (-13) - 5(-13)$$
$$= -78 + 13 + 65 = -78 + 78 = 0$$

The given equations therefore .... consistent.
(are/are not)

56

$\boxed{\text{are}}$

Example 2

Find the value of k for which the following equations are consistent:

$$\begin{cases} 3x + y + 2 = 0 \\ 4x + 2y - k = 0 \\ 2x - y + 3k = 0 \end{cases} \quad \text{For consistency: } \begin{vmatrix} 3 & 1 & 2 \\ 4 & 2 & -k \\ 2 & -1 & 3k \end{vmatrix} = 0$$

$$\therefore\ 3\begin{vmatrix} 2 & -k \\ -1 & 3k \end{vmatrix} - 1\begin{vmatrix} 4 & -k \\ 2 & 3k \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = 0$$

$$3(6k - k) - 1(12k + 2k) + 2(-4 - 4) = 0$$

$$\therefore\ 15k - 14k - 16 = 0 \quad \therefore\ k - 16 = 0 \quad \therefore\ k = 16$$

Now one for you, done in just the same way.

Example 3

$$\text{Given } \begin{cases} x + (k+1)y + 1 = 0 \\ 2kx + 5y - 3 = 0 \\ 3x + 7y + 1 = 0 \end{cases}$$

find the values of k for which the equations are consistent.

57

$$\boxed{k = 2 \text{ or } -\frac{1}{2}}$$

The condition for consistency is that:

$$\begin{vmatrix} 1 & k+1 & 1 \\ 2k & 5 & -3 \\ 3 & 7 & 1 \end{vmatrix} = 0$$

$$\therefore\ 1\begin{vmatrix} 5 & -3 \\ 7 & 1 \end{vmatrix} - (k+1)\begin{vmatrix} 2k & -3 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2k & 5 \\ 3 & 7 \end{vmatrix} = 0$$

$$(5+21) - (k+1)(2k+9) + (14k-15) = 0$$

$$26 - 2k^2 - 11k - 9 + 14k - 15 = 0$$

$$-2k^2 + 3k + 2 = 0$$

$$\therefore\ 2k^2 - 3k - 2 = 0 \quad \therefore\ (2k+1)(k-2) = 0$$

$$\therefore\ k = 2 \text{ or } k = -\frac{1}{2}$$

Finally, one more for you to do.

Example 4

Find the values of k for consistency when $\begin{cases} x + y - k = 0 \\ kx - 3y + 11 = 0 \\ 2x + 4y - 8 = 0 \end{cases}$

58

$$\boxed{k = 1 \text{ or } -\frac{1}{2}}$$

Because

$$\begin{vmatrix} 1 & 1 & -k \\ k & -3 & 11 \\ 2 & 4 & -8 \end{vmatrix} = 0$$

$$1\begin{vmatrix} -3 & 11 \\ 4 & -8 \end{vmatrix} - 1\begin{vmatrix} k & 11 \\ 2 & -8 \end{vmatrix} - k\begin{vmatrix} k & -3 \\ 2 & 4 \end{vmatrix} = 0$$

$$\therefore\ (24-44) - (-8k-22) - k(4k+6) = 0$$

$$\therefore\ -20 + 8k + 22 - 4k^2 - 6k = 0$$

$$-4k^2 + 2k + 2 = 0$$

$$\therefore\ 2k^2 - k - 1 = 0 \quad \therefore\ (2k+1)(k-1) = 0$$

$$\therefore\ k = 1 \text{ or } k = -\frac{1}{2}$$

Properties of determinants

59

Expanding a determinant in which the elements are large numbers can be a very tedious affair. It is possible, however, by knowing something of the properties of determinants, to simplify the working. So here are some of the main properties. Make a note of them in your record book for future reference.

1 *The value of a determinant remains unchanged if rows are changed to columns and columns to rows.*

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2 *If two rows (or two columns) are interchanged, the sign of the determinant is changed.*

$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

3 *If two rows (or two columns) are identical, the value of the determinant is zero.*

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$$

4 *If the elements of any one row (or column) are all multiplied by a common factor, the determinant is multiplied by that factor.*

$$\begin{vmatrix} ka_1 & kb_2 \\ a_2 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

5 *If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.*

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Note: The properties stated above are general and apply not only to second-order determinants but to determinants of any order.

Move on for some examples

60

Example 1

Evaluate $\begin{vmatrix} 427 & 429 \\ 369 & 371 \end{vmatrix}$

Of course we could evaluate this by the usual method

$(427)(371) - (369)(429)$

which is rather deadly! On the other hand, we could apply our knowledge of the properties of determinants, thus:

$$\begin{vmatrix} 427 & 429 \\ 369 & 371 \end{vmatrix} = \begin{vmatrix} 427 & 429-427 \\ 369 & 371-369 \end{vmatrix} \quad \text{(Rule 5) Subtract column 1 from column 2}$$

$$= \begin{vmatrix} 427 & 2 \\ 369 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 58 & 0 \\ 369 & 2 \end{vmatrix} \quad \text{(Rule 5) Subtract row 2 from row 1}$$

$$= (58)(2) - (0) = 116$$

Naturally, the more zero elements we can arrange, the better.

For another example, move to Frame 61

61

Example 2

Evaluate

$$\begin{vmatrix} 1 & 2 & 2 \\ 4 & 3 & 5 \\ 4 & 2 & 7 \end{vmatrix} \quad \text{Column 2 minus column 3 will give us one zero}$$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 4 & -2 & 5 \\ 4 & -5 & 7 \end{vmatrix} \quad \text{Column 3 minus twice(column 1) will give another zero}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 4 & -2 & -3 \\ 4 & -5 & -1 \end{vmatrix} \quad \text{Now expand along the top row}$$

$$= \begin{vmatrix} -2 & -3 \\ -5 & -1 \end{vmatrix}$$

We could take a factor (-1) from the top row and another factor (-1) from the bottom row (rule 4)

$$= (-1)(-1)\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$$

$$= 1(2 - 15) = -13$$

Next frame

62

Example 3

Evaluate $\begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}$

You do that one, but by way of practice, apply as many of the listed properties as possible. It is quite fun.

When you have finished it, move on to Frame 63

63

The answer is $\boxed{32}$, but what we are more interested in is the method of applying the properties, so follow it through. This is one way of doing it; not the only way by any means.

$$\begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}$$

We can take out a factor 2 from each row, giving a factor 2^3, i.e. 8 outside the determinant

$$= 8\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Column 2 minus column 3 will give one zero in the top row

$$= 8\begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

Column 1 minus twice(column 3) will give another zero in the same row

$$= 8\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -3 & -1 & 2 \end{vmatrix}$$

Expanding along the top row will now reduce this to a second-order determinant

$$= 8\begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix}$$

Now row 2 + row 1

$$= 8\begin{vmatrix} -1 & 1 \\ -4 & 0 \end{vmatrix}$$

$$= -8\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -8(-4) = 32$$

64

Here is another type of problem.

Example 4

Solve the equation $\begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0$

In this type of question, we try to establish common factors wherever possible. ▶

For example, if we add row 2 and row 3 to row 1, we get:

$$\begin{vmatrix} (x+2) & (x+2) & (x+2) \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0$$

Taking out the common factor $(x+2)$ gives:

$$(x+2)\begin{vmatrix} 1 & 1 & 1 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0$$

Now if we take column 1 from column 2 and also from column 3, what do we get?

When you have done it, move on to the next frame

65

We now have $(x+2)\begin{vmatrix} 1 & 0 & 0 \\ 5 & x-4 & -4 \\ -3 & -1 & x+1 \end{vmatrix} = 0$

Expanding along the top row reduces this to a second-order determinant:

$$(x+2)\begin{vmatrix} x-4 & -4 \\ -1 & x+1 \end{vmatrix} = 0$$

If we now multiply out the determinant, we get

$$(x+2)[(x-4)(x+1)-4] = 0$$

$$\therefore\ (x+2)(x^2-3x-8) = 0$$

$$\therefore\ x+2 = 0 \text{ or } x^2-3x-8 = 0$$

which finally gives $x = -2$ or $x = \dfrac{3 \pm \sqrt{41}}{2}$

Finally, here is one for you to do on your own.

Example 5

Solve the equation:

$$\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$

Check your working with that given in the next frame

66

$$\boxed{x = -4 \text{ or } 1 \pm \sqrt{6}}$$

Here is one way of doing the problem:

$$\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$ Adding row 2 and row 3 to row 1, gives

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$ Taking out the common factor $(x+4)$

$$(x+4)\begin{vmatrix} 1 & 1 & 1 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$$ Take column 3 from column 1 and from column 2

$$(x+4)\begin{vmatrix} 0 & 0 & 1 \\ x+1 & 1 & 1 \\ -x-3 & 2-x & x \end{vmatrix} = 0$$ This now reduces to second order

$$(x+4)\begin{vmatrix} x+1 & 1 \\ -x-3 & 2-x \end{vmatrix} = 0$$ Subtract column 2 from column 1

$$(x+4)\begin{vmatrix} x & 1 \\ -5 & 2-x \end{vmatrix} = 0$$ We now finish it off

$$\therefore \ (x+4)(2x - x^2 + 5) = 0$$

$$\therefore \ x+4 = 0 \text{ or } x^2 - 2x - 5 = 0$$

which gives $x = -4$ or $x = 1 \pm \sqrt{6}$

You have now reached the end of this Programme on determinants except for the **Can You?** checklist and **Test exercise** which follow. Before you work through them, brush up on any parts of the work about which you are at all uncertain.

Can You?

Checklist 4

67

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Expand a 2×2 determinant? 1 to 6

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve pairs of simultaneous linear equations in two variables using 2×2 determinants? 7 to 16

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Expand a 3×3 determinant? 17 to 27

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve three simultaneous linear equations in three variables using 3×3 determinants? 28 to 51

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the consistency of sets of simultaneous linear equations? 52 to 58

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the properties of determinants to solve equations written in determinant form? 59 to 66

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 4

68

If you have worked steadily through the Programme, you should have no difficulty with this exercise. Take your time and work carefully. There is no extra credit for speed. Off you go then. They are all quite straightforward.

1 Evaluate

(a) $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

2 By determinants, find the value of x, given:

$$\begin{cases} 2x + 3y - z + 3 = 0 \\ x - 4y + 2z - 14 = 0 \\ 4x + 2y - 3z + 6 = 0 \end{cases}$$

▶

3 Use determinants to solve completely:

$$\begin{cases} x - 3y + 4z - 5 = 0 \\ 2x + y + z - 3 = 0 \\ 4x + 3y + 5z - 1 = 0 \end{cases}$$

4 Find the values of k for which the following equations are consistent:

$$\begin{cases} 3x + 5y + k = 0 \\ 2x + y - 5 = 0 \\ (k+1)x + 2y - 10 = 0 \end{cases}$$

5 Solve the equation:

$$\begin{vmatrix} x+1 & -5 & -6 \\ -1 & x & 2 \\ -3 & 2 & x+1 \end{vmatrix} = 0$$

Now you can continue with the next Programme

Further problems 4

69

1 Evaluate: (a) $\begin{vmatrix} 3 & 5 & 7 \\ 11 & 9 & 13 \\ 15 & 17 & 19 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 428 & 861 \\ 2 & 535 & 984 \\ 3 & 642 & 1107 \end{vmatrix}$

2 Evaluate: (a) $\begin{vmatrix} 25 & 3 & 35 \\ 16 & 10 & -18 \\ 34 & 6 & 38 \end{vmatrix}$ (b) $\begin{vmatrix} 155 & 226 & 81 \\ 77 & 112 & 39 \\ 74 & 111 & 37 \end{vmatrix}$

3 Solve by determinants:

$$4x - 5y + 7z = -14$$
$$9x + 2y + 3z = 47$$
$$x - y - 5z = 11$$

4 Use determinants to solve the equations:

$$4x - 3y + 2z = -7$$
$$6x + 2y - 3z = 33$$
$$2x - 4y - z = -3$$

5 Solve by determinants:

$$3x + 2y - 2z = 16$$
$$4x + 3y + 3z = 2$$
$$2x - y + z = 0$$

6 Find the values of λ for which the following equations are consistent:

$$5x + (\lambda + 1)y - 5 = 0$$
$$(\lambda - 1)x + 7y + 5 = 0$$
$$3x + 5y + 1 = 0$$

7 Determine the values of k for which the following equations have solutions:

$$\begin{aligned} 4x - (k-2)y - 5 &= 0 \\ 2x + y - 10 &= 0 \\ (k+1)x - 4y - 9 &= 0 \end{aligned}$$

8 (a) Find the values of k which satisfy the equation:

$$\begin{vmatrix} k & 1 & 0 \\ 1 & k & 1 \\ 0 & 1 & k \end{vmatrix} = 0$$

(b) Factorize

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

9 Solve the equation:

$$\begin{vmatrix} x & 2 & 3 \\ 2 & x+3 & 6 \\ 3 & 4 & x+6 \end{vmatrix} = 0$$

10 Find the values of x that satisfy the equation:

$$\begin{vmatrix} x & 3+x & 2+x \\ 3 & -3 & -1 \\ 2 & -2 & -2 \end{vmatrix} = 0$$

11 Express

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ (b+c)^2 & (c+a)^2 & (a+b)^2 \end{vmatrix}$$

as a product of linear factors.

12 A resistive network gives the following equations:

$$\begin{aligned} 2(i_3 - i_2) + 5(i_3 - i_1) &= 24 \\ (i_2 - i_3) + 2i_2 + (i_2 - i_1) &= 0 \\ 5(i_1 - i_3) + 2(i_1 - i_2) + i_1 &= 6 \end{aligned}$$

Simplify the equations and use determinants to find the value of i_2 correct to two significant figures.

13 Show that $(a+b+c)$ is a factor of the determinant

$$\begin{vmatrix} b+c & a & a^3 \\ c+a & b & b^3 \\ a+b & c & c^3 \end{vmatrix}$$

and express the determinant as a product of five factors.

14 Find values of k for which the following equations are consistent:

$$\begin{aligned} x + (1+k)y + 1 &= 0 \\ (2+k)x + 5y - 10 &= 0 \\ x + 7y + 9 &= 0 \end{aligned}$$

15 Express $\begin{vmatrix} 1+x^2 & yz & 1 \\ 1+y^2 & zx & 1 \\ 1+z^2 & xy & 1 \end{vmatrix}$ as a product of four linear factors.

16 Solve the equation $\begin{vmatrix} x+1 & x+2 & 3 \\ 2 & x+3 & x+1 \\ x+3 & 1 & x+2 \end{vmatrix} = 0$

17 If x, y, z satisfy the equations:

$$\begin{aligned} \left(\frac{1}{2}M_1 + M_2\right)x - M_2y &= W \\ -M_2x + 2M_2y + (M_1 - M_2)z &= 0 \\ -M_2y + \left(\frac{1}{2}M_1 + M_2\right)z &= 0 \end{aligned}$$

evaluate x in terms of W, M_1 and M_2.

18 Three currents, i_1, i_2, i_3, in a network are related by the following equations:

$$\begin{aligned} 2i_1 + 3i_2 + 8i_3 &= 30 \\ 6i_1 - i_2 + 2i_3 &= 4 \\ 3i_1 - 12i_2 + 8i_3 &= 0 \end{aligned}$$

By the use of determinants, find the value of i_1 and hence solve completely the three equations.

19 If
$$\begin{aligned} k(x-a) + 2x - z &= 0 \\ k(y-a) + 2y - z &= 0 \\ k(z-a) - x - y + 2z &= 0 \end{aligned}$$

show that $x = \dfrac{ak(k+3)}{k^2+4k+2}$.

20 Find the angles between $\theta = 0$ and $\theta = \pi$ that satisfy the equation:

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 2\theta \end{vmatrix} = 0$$

Matrices

Frames 1 to 58

Learning outcomes

When you have completed this Programme you will be able to:

- Define a matrix
- Understand what is meant by the equality of two matrices
- Add and subtract two matrices
- Multiply a matrix by a scalar and multiply two matrices together
- Obtain the transpose of a matrix
- Recognize special types of matrix
- Obtain the determinant, cofactors and adjoint of a square matrix
- Obtain the inverse of a non-singular matrix
- Use matrices to solve a set of linear equations using inverse matrices
- Use the Gaussian elimination method to solve a set of linear equations
- Evaluate eigenvalues and eigenvectors

Matrices – definitions

1

A *matrix* is a set of real or complex numbers (or *elements*) arranged in rows and columns to form a rectangular array.

A matrix having m rows and n columns is called an $m \times n$ (i.e. 'm by n') matrix and is referred to as having *order* $m \times n$.

A matrix is indicated by writing the array within brackets

e.g. $\begin{pmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{pmatrix}$ is a 2×3 matrix, i.e. a '2 by 3' matrix, where

5, 7, 2, 6, 3, 8 are the elements of the matrix.

Note that, in describing the matrix, the number of rows is stated first and the number of columns second.

$\begin{pmatrix} 5 & 6 & 4 \\ 2 & -3 & 2 \\ 7 & 8 & 7 \\ 6 & 7 & 5 \end{pmatrix}$ is a matrix of order 4×3, i.e. 4 rows and 3 columns.

So the matrix $\begin{pmatrix} 6 & 4 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$ is of order ....

and the matrix $\begin{pmatrix} 2 & 5 & 3 & 4 \\ 6 & 7 & 4 & 9 \end{pmatrix}$ is of order ....

2

$\boxed{3 \times 2: \quad 2 \times 4}$

A matrix is simply an array of numbers: there is no arithmetical connection between the elements and it therefore differs from a determinant in that the elements cannot be multiplied together in any way to find a numerical value of the matrix. A matrix has no numerical value. Also, in general, rows and columns cannot be interchanged as was the case with determinants.

Row matrix: A row matrix consists of 1 row only.

e.g. $(4 \quad 3 \quad 7 \quad 2)$ is a row matrix of order 1×4.

Column matrix: A column matrix consists of 1 column only.

e.g. $\begin{pmatrix} 6 \\ 3 \\ 8 \end{pmatrix}$ is a column matrix of order 3×1.

To conserve space in printing, a column matrix is sometimes written on one line but with 'curly' brackets, e.g. $\{6 \; 3 \; 8\}$ is the same column matrix of order 3×1.

Move on to the next frame

So, from what we have already said: **3**

(a) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is a .... matrix of order ....

(b) $(4 \quad 0 \quad 7 \quad 3)$ is a .... matrix of order ....

(c) $\{2 \quad 6 \quad 9\}$ is a .... matrix of order ....

(a) column, 2×1	(b) row, 1×4	(c) column, 3×1

4

We use a simple row matrix in stating the x- and y-coordinates of a point relative to the x- and y-axes. For example, if P is the point (3, 5) then the 3 is the x-coordinate and the 5 the y-coordinate. In matrices generally, however, no commas are used to separate the elements.

Single element matrix: A single number may be regarded as a 1×1 matrix, i.e. having 1 row and 1 column.

Double suffix notation: Each element in a matrix has its own particular 'address' or location which can be defined by a system of double suffixes, the first indicating the row, the second the column, thus:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$\therefore$ a_{23} indicates the element in the second row and third column.

Therefore, in the matrix

$$\begin{pmatrix} 6 & -5 & 1 & -3 \\ 2 & -4 & 8 & 3 \\ 4 & -7 & -6 & 5 \\ -2 & 9 & 7 & -1 \end{pmatrix}$$

the location of (a) the element 3 can be stated as ....
(b) the element -1 can be stated as ....
(c) the element 9 can be stated as ....

(a) a_{24}	(b) a_{44}	(c) a_{42}

5

Move on

Matrix notation

6

Where there is no ambiguity, a whole matrix can be denoted by a single general element enclosed in brackets, or by a single letter printed in bold type. This is a very neat shorthand and saves much space and writing. For example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \text{ can be denoted by } (a_{ij}) \text{ or } (a) \text{ or by } \mathbf{A}.$$

$$\text{Similarly } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ can be denoted by } (x_i) \text{ or } (x) \text{ or simply by } \mathbf{x}.$$

For an $(m \times n)$ matrix, we use a bold capital letter, e.g. **A**. For a row or column matrix, we use a lower-case bold letter, e.g. **x**. (In handwritten work, we can indicate bold-face type by a wavy line placed under the letter, e.g. $\underset{\sim}{A}$ or $\underset{\sim}{x}$.)

So, if **B** represents a 2×3 matrix, write out the elements b_{ij} in the matrix, using the double suffix notation. This gives ....

7

$$\boxed{\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}}$$

Next frame

Equal matrices

8

By definition, two matrices are said to be equal if corresponding elements throughout are equal. Thus, the two matrices must also be of the same order.

$$\text{So, if } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 5 \\ 2 & 3 & 7 \end{pmatrix}$$

then $a_{11} = 4$; $a_{12} = 6$; $a_{13} = 5$; $a_{21} = 2$; etc.

Therefore, if $(a_{ij}) = (x_{ij})$ then $a_{ij} = x_{ij}$ for all values of i and j.

$$\text{So, if } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} 5 & -7 & 3 \\ 1 & 2 & 6 \\ 0 & 4 & 8 \end{pmatrix}$$

then $d =$ ....; $b =$ ....; $a - k =$ ....

9

$$\boxed{d = 1; \quad b = -7; \quad a - k = -3}$$

Addition and subtraction of matrices

10

To be added or subtracted, two matrices must be of the *same order*. The sum or difference is then determined by adding or subtracting corresponding elements.

$$\text{e.g.}\quad \begin{pmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{pmatrix}$$

$$\text{and}\quad \begin{pmatrix} 6 & 5 & 12 \\ 9 & 4 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 7 & 1 \\ 2 & 10 & -5 \end{pmatrix} = \begin{pmatrix} 6-3 & 5-7 & 12-1 \\ 9-2 & 4-10 & 8+5 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 11 \\ 7 & -6 & 13 \end{pmatrix}$$

So, (a) $\begin{pmatrix} 6 & 5 & 4 & 1 \\ 2 & 3 & -7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 2 & 3 \\ 6 & -1 & 0 & 5 \end{pmatrix} = \dots\dots\dots\dots$

(b) $\begin{pmatrix} 8 & 3 & 6 \\ 5 & 2 & 7 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \dots\dots\dots\dots$

11

(a) $\begin{pmatrix} 7 & 9 & 6 & 4 \\ 8 & 2 & -7 & 13 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 1 & 3 \\ 1 & -3 & 1 \\ -6 & -8 & -5 \end{pmatrix}$

Multiplication of matrices

12

1 Scalar multiplication

To multiply a matrix by a single number (i.e. a scalar), each individual element of the matrix is multiplied by that factor:

$$\text{e.g.}\ 4 \times \begin{pmatrix} 3 & 2 & 5 \\ 6 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 8 & 20 \\ 24 & 4 & 28 \end{pmatrix}$$

i.e. in general, $k(a_{ij}) = (ka_{ij})$.

It also means that, in reverse, we can take a common factor out of every element – not just one row or one column as in determinants.

Therefore, $\begin{pmatrix} 10 & 25 & 45 \\ 35 & 15 & 50 \end{pmatrix}$ can be written $\dots\dots\dots\dots$

13

$$\boxed{5 \times \begin{pmatrix} 2 & 5 & 9 \\ 7 & 3 & 10 \end{pmatrix}}$$

2 Multiplication of two matrices

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

$$\text{e.g. if } \mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \text{ and } \mathbf{b} = (b_i) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\text{then } \mathbf{A}.\mathbf{b} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{pmatrix}$$

i.e. each element in the top row of **A** is multiplied by the corresponding element in the first column of **b** and the products added. Similarly, the second row of the product is found by multiplying each element in the second row of **A** by the corresponding element in the first column of **b**.

Example 1

$$\begin{pmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{pmatrix} . \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \times 8 + 7 \times 5 + 6 \times 9 \\ 2 \times 8 + 3 \times 5 + 1 \times 9 \end{pmatrix} = \begin{pmatrix} 32 + 35 + 54 \\ 16 + 15 + 9 \end{pmatrix} = \begin{pmatrix} 121 \\ 40 \end{pmatrix}$$

$$\text{Similarly } \begin{pmatrix} 2 & 3 & 5 & 1 \\ 4 & 6 & 0 & 7 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ 2 \\ 9 \end{pmatrix} = \dots\dots\dots\dots$$

14

$$\boxed{\begin{pmatrix} 6 + 12 + 10 + 9 \\ 12 + 24 + 0 + 63 \end{pmatrix} = \begin{pmatrix} 37 \\ 99 \end{pmatrix}}$$

In just the same way, if $\mathbf{A} = \begin{pmatrix} 3 & 6 & 8 \\ 1 & 0 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$ then

$\mathbf{A}.\mathbf{b} = \dots\dots\dots\dots$

15

$$\boxed{\begin{pmatrix} 85 \\ 17 \end{pmatrix}}$$

The same process is carried out for each row and column. ▶

Example 2

$$\text{If } \mathbf{A} = (a_{ij}) = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix} \text{ and } \mathbf{B} = (b_{ij}) = \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$$

$$\text{then } \mathbf{A}.\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 8+5\times 2 & 1\times 4+5\times 5 & 1\times 3+5\times 8 & 1\times 1+5\times 6 \\ 2\times 8+7\times 2 & 2\times 4+7\times 5 & 2\times 3+7\times 8 & 2\times 1+7\times 6 \\ 3\times 8+4\times 2 & 3\times 4+4\times 5 & 3\times 3+4\times 8 & 3\times 1+4\times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8+10 & 4+25 & 3+40 & 1+30 \\ 16+14 & 8+35 & 6+56 & 2+42 \\ 24+8 & 12+20 & 9+32 & 3+24 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{pmatrix}$$

Note that multiplying a (3×2) matrix and a (2×4) matrix gives a product matrix of order (3×4)

i.e. order $(3\times 2) \times$ order $(2 \times 4) \rightarrow$ order (3×4).

(same)

In general then, the product of an $(l \times m)$ matrix and an $(m \times n)$ matrix has order $(l \times n)$.

$$\text{If } \mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}$$

then $\mathbf{A}.\mathbf{B} = \ldots\ldots\ldots\ldots$

16

$$\boxed{\begin{pmatrix} 30 & 56 \\ 23 & 99 \end{pmatrix}}$$

$$\text{since } \mathbf{A}.\mathbf{B} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix} \cdot \begin{pmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 14-8+24 & 2+36+18 \\ 21-18+20 & 3+81+15 \end{pmatrix} = \begin{pmatrix} 30 & 56 \\ 23 & 99 \end{pmatrix}$$

Example 3

It follows that a matrix can be squared only if it is itself a square matrix, i.e. the number of rows equals the number of columns.

$$\text{If } \mathbf{A} = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 16+35 & 28+14 \\ 20+10 & 35+4 \end{pmatrix} = \begin{pmatrix} 51 & 42 \\ 30 & 39 \end{pmatrix}$$

Remember that multiplication of matrices is defined only when ....

17

the number of columns in the first = the number of rows in the second

That is correct. $\begin{pmatrix} 1 & 5 & 6 \\ 4 & 9 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 5 \\ 8 & 7 & 1 \end{pmatrix}$ has no meaning.

$$\left.\begin{array}{r} \text{If } \mathbf{A} \text{ is an } (m \times n) \text{ matrix} \\ \text{and } \mathbf{B} \text{ is an } (n \times m) \text{ matrix} \end{array}\right\} \text{then products } \mathbf{A}.\mathbf{B} \text{ and } \mathbf{B}.\mathbf{A} \text{ are possible.}$$

Example

$$\text{If } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$$

$$\text{then } \mathbf{A}.\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 7+16+27 & 10+22+36 \\ 28+40+54 & 40+55+72 \end{pmatrix} = \begin{pmatrix} 50 & 68 \\ 122 & 167 \end{pmatrix}$$

$$\text{and } \mathbf{B}.\mathbf{A} = \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 7+40 & 14+50 & 21+60 \\ 8+44 & 16+55 & 24+66 \\ 9+48 & 18+60 & 27+72 \end{pmatrix} = \begin{pmatrix} 47 & 64 & 81 \\ 52 & 71 & 90 \\ 57 & 78 & 99 \end{pmatrix}$$

Note that, in matrix multiplication, $\mathbf{A}.\mathbf{B} \neq \mathbf{B}.\mathbf{A}$, i.e. multiplication is not commutative. The order of the factors is important!

In the product **A.B**, **B** is *pre-multiplied* by **A**
and **A** is *post-multiplied* by **B**.

▶

So, if $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \dots\dots\dots\dots$ and $\mathbf{B}.\mathbf{A} = \dots\dots\dots\dots$

18

$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix}; \quad \mathbf{B}.\mathbf{A} = \begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$$

Transpose of a matrix

19

If the rows and columns of a matrix are interchanged:

i.e. the first row becomes the first column,
the second row becomes the second column,
the third row becomes the third column, etc.,

then the new matrix so formed is called the *transpose* of the original matrix. If $\mathbf{A}$ is the original matrix, its transpose is denoted by $\tilde{\mathbf{A}}$ or $\mathbf{A}^{\mathrm{T}}$. We shall use the latter.

$\therefore$ If $\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{pmatrix}$, then $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{pmatrix}$

Therefore, given that

$\mathbf{A} = \begin{pmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 3 & 7 \\ 1 & 5 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \dots\dots\dots\dots$ and $(\mathbf{A}.\mathbf{B})^{\mathrm{T}} = \dots\dots\dots\dots$

20

$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} 35 & 79 \\ 20 & 32 \end{pmatrix}; \quad \mathbf{A}.\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 35 & 20 \\ 79 & 32 \end{pmatrix}$$

Special matrices

21

(a) *Square matrix* is a matrix of order $m \times m$.

e.g. $\begin{pmatrix} 1 & 2 & 5 \\ 6 & 8 & 9 \\ 1 & 7 & 4 \end{pmatrix}$ is a 3×3 matrix

A square matrix (a_{ij}) is *symmetric* if $a_{ij} = a_{ji}$, e.g. $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix}$

i.e. it is symmetrical about the leading diagonal.

Note that $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$.

A square matrix (a_{ij}) is skew-symmetric if $a_{ij} = -a_{ji}$ e.g. $\begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{pmatrix}$

In that case, $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$.

(b) *Diagonal matrix* is a square matrix with all elements zero except those on the leading diagonal, thus $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(c) *Unit matrix* is a diagonal matrix in which the elements on the leading diagonal are all unity, i.e. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The unit matrix is denoted by **I**.

If $\mathbf{A} = \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $\mathbf{A.I} =$ ....

22

$$\boxed{\begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix} \text{ i.e. } \mathbf{A.I} = \mathbf{A}}$$

▶

Similarly, if we form the product **I.A** we obtain:

$$\mathbf{I.A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5+0+0 & 2+0+0 & 4+0+0 \\ 0+1+0 & 0+3+0 & 0+8+0 \\ 0+0+7 & 0+0+9 & 0+0+6 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix} = \mathbf{A}$$

$$\mathbf{A.I} = \mathbf{I.A} = \mathbf{A}$$

Therefore, the unit matrix **I** behaves very much like the unit factor in ordinary algebra and arithmetic.

(d) *Null matrix*: A null matrix is one whose elements are all zero.

i.e. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and is denoted by **0**.

If $\mathbf{A.B} = \mathbf{0}$, we cannot say that therefore $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$

for if $\mathbf{A} = \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 9 \\ 4 & -6 \\ 2 & 4 \end{pmatrix}$

then $\mathbf{A.B} = \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 9 \\ 4 & -6 \\ 2 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 2+4-6 & 18-6-12 \\ 6+12-18 & 54-18-36 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

That is, $\mathbf{A.B} = \mathbf{0}$, but clearly $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$.

Now a short revision exercise. Do these without looking back.

1 If $\mathbf{A} = \begin{pmatrix} 4 & 6 & 5 & 7 \\ 3 & 1 & 9 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 8 & 3 & -1 \\ 5 & 2 & -4 & 6 \end{pmatrix}$

determine (a) $\mathbf{A} + \mathbf{B}$ and (b) $\mathbf{A} - \mathbf{B}$.

2 If $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 7 \\ 6 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 9 & 2 \\ 4 & 0 & 8 \end{pmatrix}$

determine (a) $5\mathbf{A}$; (b) $\mathbf{A.B}$; (c) $\mathbf{B.A}$.

3 If $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 5 & 7 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 0 & 7 \\ 2 & 3 \end{pmatrix}$ then $\mathbf{A.B} = \ldots\ldots\ldots\ldots$

4 Given that $\mathbf{A} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$ determine (a) $\mathbf{A}^{\mathrm{T}}$ and (b) $\mathbf{A.A}^{\mathrm{T}}$.

When you have completed them, check your results with the next frame

23

Here are the solutions. Check your results.

1 (a) $\mathbf{A}+\mathbf{B}=\begin{pmatrix}6 & 14 & 8 & 6\\ 8 & 3 & 5 & 10\end{pmatrix}$; (b) $\mathbf{A}-\mathbf{B}=\begin{pmatrix}2 & -2 & 2 & 8\\ -2 & -1 & 13 & -2\end{pmatrix}$

2 (a) $5\mathbf{A}=\begin{pmatrix}20 & 15\\ 10 & 35\\ 30 & 5\end{pmatrix}$ (b) $\mathbf{A}.\mathbf{B}=\begin{pmatrix}32 & 36 & 32\\ 38 & 18 & 60\\ 34 & 54 & 20\end{pmatrix}$ (c) $\mathbf{B}.\mathbf{A}=\begin{pmatrix}50 & 80\\ 64 & 20\end{pmatrix}$

3 $\mathbf{A}.\mathbf{B}=\begin{pmatrix}2 & 6\\ 5 & 7\\ 4 & 1\end{pmatrix}\cdot\begin{pmatrix}3 & 2\\ 0 & 7\\ 2 & 3\end{pmatrix}$ is not possible since the number of columns in the first must be equal to the number of rows in the second.

4 $\mathbf{A}=\begin{pmatrix}4 & 2 & 6\\ 1 & 8 & 7\end{pmatrix}\quad\therefore\ \mathbf{A}^{\mathrm{T}}=\begin{pmatrix}4 & 1\\ 2 & 8\\ 6 & 7\end{pmatrix}$

$$\mathbf{A}.\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}4 & 2 & 6\\ 1 & 8 & 7\end{pmatrix}\cdot\begin{pmatrix}4 & 1\\ 2 & 8\\ 6 & 7\end{pmatrix}=\begin{pmatrix}16+4+36 & 4+16+42\\ 4+16+42 & 1+64+49\end{pmatrix}$$
$$=\begin{pmatrix}56 & 62\\ 62 & 114\end{pmatrix}$$

Now move on to the next frame

Determinant of a square matrix

24

The determinant of a square matrix is the determinant having the same elements as those of the matrix. For example:

the determinant of $\begin{pmatrix}5 & 2 & 1\\ 0 & 6 & 3\\ 8 & 4 & 7\end{pmatrix}$ is $\begin{vmatrix}5 & 2 & 1\\ 0 & 6 & 3\\ 8 & 4 & 7\end{vmatrix}$ and the value of this

determinant is $5(42-12)-2(0-24)+1(0-48)$
$=5(30)-2(-24)+1(-48)=150+48-48=150$

Note that the transpose of the matrix is $\begin{pmatrix}5 & 0 & 8\\ 2 & 6 & 4\\ 1 & 3 & 7\end{pmatrix}$ and the

determinant of the transpose is $\begin{vmatrix}5 & 0 & 8\\ 2 & 6 & 4\\ 1 & 3 & 7\end{vmatrix}$ the value of which is

$5(42-12)-0(14-4)+8(6-6)=5(30)=150.$

That is, the determinant of a square matrix has the same value as that of the determinant of the transposed matrix.

A matrix whose determinant is zero is called a *singular* matrix. ▶

The determinant of the matrix $\begin{pmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 8 & 6 \end{pmatrix}$ has the value ....

and the determinant value of the diagonal matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ has the value ....

25

$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 8 & 6 \end{vmatrix} = 3(-30) - 2(15) + 5(25) = 5$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2(20) + 0 + 0 = 40$$

Cofactors

If $\mathbf{A} = (a_{ij})$ is a square matrix, we can form a determinant of its elements:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

Each element gives rise to a *cofactor*, which is simply the minor of the element in the determinant together with its 'place sign', which was described in detail in the previous programme.

For example, the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ is

$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$ which has a value of 45.

The minor of the element 2 is $\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = 0 - 24 = -24.$

The place sign is $+$. Therefore the cofactor of the element 2 is $+(-24)$ i.e. -24.

Similarly, the minor of the element 3 is $\begin{pmatrix} 4 & 6 \\ 1 & 0 \end{pmatrix} = 0 - 6 = -6.$

The place sign is $-$. Therefore the cofactor of the element 3 is $-(-6) = 6$.

In each case the minor is found by striking out the line and column containing the element in question and forming a determinant of the remaining elements. The appropriate place signs are given by:

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \\ + & - & + & & \\ \vdots & & & & \end{pmatrix}$$

alternate plus and minus from the top left-hand corner which carries a $+$

▶

Therefore, in the example above, the minor of the element 6 is $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ i.e. $8 - 3 = 5$. The place sign is $-$. Therefore the cofactor of the element 6 is -5.

So, for the matrix $\begin{pmatrix} 7 & 1 & -2 \\ 6 & 5 & 4 \\ 3 & 8 & 9 \end{pmatrix}$, the cofactor of the element 3 is ....

and that of the element 4 is ....

26

Cofactor of 3 is $4 - (-10) = 14$
Cofactor of 4 is $-(56 - 3) = -53$

Adjoint of a square matrix

If we start afresh with $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$, its determinant

$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$ from which we can form a new matrix $\mathbf{C}$ of the cofactors.

$\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ where A_{11} is the cofactor of a_{11}
A_{ij} is the cofactor of a_{ij} etc.

$$A_{11} = +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = +(0 - 24) = -24 \qquad A_{12} = -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = -(0 - 6) = 6$$

$$A_{13} = +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = +(16 - 1) = 15$$

$$A_{21} = -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} = -(0 - 20) = 20 \qquad A_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = +(0 - 5) = -5$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -(8 - 3) = -5$$

$$A_{31} = +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = +(18 - 5) = 13 \qquad A_{32} = -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} = -(12 - 20) = 8$$

$$A_{33} = +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = +(2 - 12) = -10$$

$\therefore$ The matrix of cofactors is $\mathbf{C} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$

and the transpose of $\mathbf{C}$, i.e. $\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$

This is called the *adjoint* of the original matrix $\mathbf{A}$ and is written adj $\mathbf{A}$. ▶

Therefore, to find the adjoint of a square matrix **A**:

(a) we form the matrix **C** of cofactors,

(b) we write the transpose of **C**, i.e. $\mathbf{C}^T$.

Hence the adjoint of $\begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{pmatrix}$ is …………

27

$$\boxed{\operatorname{adj} \mathbf{A} = \mathbf{C}^T = \begin{pmatrix} -21 & 0 & 7 \\ 7 & 11 & -17 \\ 14 & -22 & -1 \end{pmatrix}}$$

Inverse of a square matrix

28

The adjoint of a square matrix is important, since it enables us to form the inverse of the matrix. If each element of the adjoint of **A** is divided by the value of the determinant of **A**, i.e. $|\mathbf{A}|$, (provided $|\mathbf{A}| \neq 0$), the resulting matrix is called the *inverse* of **A** and is denoted by $\mathbf{A}^{-1}$.

For the matrix which we used in the last frame, $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix} = 2(0-24) - 3(0-6) + 5(16-1) = 45,$$

$$\text{the matrix of cofactors } \mathbf{C} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

$$\text{and the adjoint of } \mathbf{A}\text{, i.e. } \mathbf{C}^T = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Then the inverse of **A** is given by

$$\mathbf{A}^{-1} = \begin{pmatrix} -\dfrac{24}{45} & \dfrac{20}{45} & \dfrac{13}{45} \\ \dfrac{6}{45} & -\dfrac{5}{45} & \dfrac{8}{45} \\ \dfrac{15}{45} & -\dfrac{5}{45} & -\dfrac{10}{45} \end{pmatrix} = \frac{1}{45}\begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

▶

Therefore, to form the inverse of a square matrix **A**:

(a) Evaluate the determinant of **A**, i.e. $|\mathbf{A}|$

(b) Form a matrix **C** of the cofactors of the elements of $|\mathbf{A}|$

(c) Write the transpose of **C**, i.e. $\mathbf{C}^T$, to obtain the adjoint of **A**

(d) Divide each element of $\mathbf{C}^T$ by $|\mathbf{A}|$

(e) The resulting matrix is the inverse $\mathbf{A}^{-1}$ of the original matrix **A**.

Let us work through an example in detail:

To find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$

(a) Evaluate the determinant of **A**, i.e. $|\mathbf{A}|$. $|\mathbf{A}| = \ldots\ldots\ldots\ldots$

29

$$\boxed{|\mathbf{A}| = 28}$$

Because

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{vmatrix} = 1(2-0) - 2(8-30) + 3(0-6) = 28$$

(b) Now form the matrix of the cofactors. $\mathbf{C} = \ldots\ldots\ldots\ldots$

30

$$\boxed{\mathbf{C} = \begin{pmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{pmatrix}}$$

Because

$A_{11} = +(2-0) = 2$; $A_{12} = -(8-30) = 22$; $A_{13} = +(0-6) = -6$

$A_{21} = -(4-0) = -4$; $A_{22} = +(2-18) = -16$; $A_{23} = -(0-12) = 12$

$A_{31} = +(10-3) = 7$; $A_{32} = -(5-12) = 7$; $A_{33} = +(1-8) = -7$

(c) Next we have to write down the transpose of **C** to obtain the adjoint of **A**.

adj $\mathbf{A} = \mathbf{C}^T = \ldots\ldots\ldots\ldots$

31

$$\boxed{\text{adj } \mathbf{A} = \mathbf{C}^T = \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}}$$

(d) Finally, we divide the elements of adj **A** by the value of $|\mathbf{A}|$, i.e. 28, to arrive at $\mathbf{A}^{-1}$, the inverse of **A**.

$\therefore \mathbf{A}^{-1} = \ldots\ldots\ldots\ldots$

32

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{28} & -\frac{4}{28} & \frac{7}{28} \\ \frac{22}{28} & -\frac{16}{28} & \frac{7}{28} \\ -\frac{6}{28} & \frac{12}{28} & -\frac{7}{28} \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$$

Every one is done in the same way. Work the next one right through on your own.

Determine the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{pmatrix}$

$\mathbf{A}^{-1} = \ldots\ldots\ldots\ldots$

33

$$\mathbf{A}^{-1} = \frac{1}{38}\begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$$

Here are the details:

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{vmatrix} = 2(8) - 7(-6) + 4(-5) = 38$$

Cofactors:

$A_{11} = +(8 - 0) = 8;$ $\quad A_{12} = -(24 - 30) = 6;$ $\quad A_{13} = +(0 - 5) = -5$

$A_{21} = -(56 - 0) = -56;$ $\quad A_{22} = +(16 - 20) = -4;$ $\quad A_{23} = -(0 - 35) = 35$

$A_{31} = +(42 - 4) = 38;$ $\quad A_{32} = -(12 - 12) = 0;$ $\quad A_{33} = +(2 - 21) = -19$

$$\therefore\ \mathbf{C} = \begin{pmatrix} 8 & 6 & -5 \\ -56 & -4 & 35 \\ 38 & 0 & -19 \end{pmatrix} \quad \therefore\ \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$$

$$\text{then } \mathbf{A}^{-1} = \frac{1}{38}\begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$$

Now let us find some uses for the inverse.

▶

Product of a square matrix and its inverse

From a previous example, we have seen that when $\mathbf{A}=\begin{pmatrix}1&2&3\\4&1&5\\6&0&2\end{pmatrix}$

$$\mathbf{A}^{-1}=\frac{1}{28}\begin{pmatrix}2&-4&7\\22&-16&7\\-6&12&-7\end{pmatrix}$$

$$\text{Then } \mathbf{A}^{-1}.\mathbf{A}=\frac{1}{28}\begin{pmatrix}2&-4&7\\22&-16&7\\-6&12&-7\end{pmatrix}\cdot\begin{pmatrix}1&2&3\\4&1&5\\6&0&2\end{pmatrix}$$

$$=\frac{1}{28}\begin{pmatrix}2-16+42&4-4+0&6-20+14\\22-64+42&44-16+0&66-80+14\\-6+48-42&-12+12+0&-18+60-14\end{pmatrix}$$

$$=\frac{1}{28}\begin{pmatrix}28&0&0\\0&28&0\\0&0&28\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}\qquad\therefore\ \mathbf{A}^{-1}.\mathbf{A}=\mathbf{I}$$

$$\text{Also } \mathbf{A}.\mathbf{A}^{-1}=\begin{pmatrix}1&2&3\\4&1&5\\6&0&2\end{pmatrix}\times\frac{1}{28}\begin{pmatrix}2&-4&7\\22&-16&7\\-6&12&-7\end{pmatrix}$$

$$=\frac{1}{28}\begin{pmatrix}1&2&3\\4&1&5\\6&0&2\end{pmatrix}\cdot\begin{pmatrix}2&-4&7\\22&-16&7\\-6&12&-7\end{pmatrix}=\ldots\ldots\ldots\ldots$$

Finish it off

34

$$\boxed{\mathbf{A}.\mathbf{A}^{-1}=\frac{1}{28}\begin{pmatrix}28&0&0\\0&28&0\\0&0&28\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}}$$

$\therefore\ \mathbf{A}.\mathbf{A}^{-1}=\mathbf{A}^{-1}.\mathbf{A}=\mathbf{I}$

That is, the product of a square matrix and its inverse, in whatever order the factors are written, is the unit matrix of the same matrix order.

Solution of a set of linear equations

35

Consider the set of linear equations:

$$\begin{aligned}
&a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots\ldots+a_{1n}x_n=b_1\\
&a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots\ldots+a_{2n}x_n=b_2\\
&\ \vdots\qquad\ \vdots\qquad\qquad\qquad\qquad\qquad\quad\ \vdots\qquad\vdots\\
&a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots\ldots\ldots+a_{nn}x_n=b_n
\end{aligned}$$

From our knowledge of matrix multiplication, this can be written in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{i.e. } \mathbf{A.x} = \mathbf{b}$$

$$\text{where } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad \text{and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

If we multiply both sides of the matrix equation by the inverse of $\mathbf{A}$, we have:

$$\mathbf{A}^{-1}.\mathbf{A}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

But $\mathbf{A}^{-1}.\mathbf{A} = \mathbf{I}$ $\quad\therefore\ \mathbf{I}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$ $\quad$ i.e. $\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$

Therefore, if we form the inverse of the matrix of coefficients and pre-multiply matrix $\mathbf{b}$ by it, we shall determine the matrix of the solutions of $\mathbf{x}$.

Example

To solve the set of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 - 4x_2 - 2x_3 &= 2 \\ 5x_1 + 3x_2 + 5x_3 &= -1 \end{aligned}$$

First write the set of equations in matrix form, which gives ....

36

$$\boxed{\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}}$$

i.e. $\mathbf{A.x} = \mathbf{b}$ $\qquad\therefore\ \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$

So the next step is to find the inverse of $\mathbf{A}$ where $\mathbf{A}$ is the matrix of the coefficients of $\mathbf{x}$. We have already seen how to determine the inverse of a matrix, so in this case $\mathbf{A}^{-1} =$ ....

37

$$\boxed{\mathbf{A}^{-1} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}}$$

Because

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{vmatrix} = -14 - 50 + 29 = 29 - 64 \quad \therefore \ |\mathbf{A}| = -35$$

Cofactors:

$$\begin{aligned} &A_{11} = +(-20+6) = -14; \quad A_{12} = -(15+10) = -25; \quad A_{13} = +(9+20) = 29 \\ &A_{21} = -(10-3) = -7; \qquad A_{22} = +(5-5) = 0; \qquad A_{23} = -(3-10) = 7 \\ &A_{31} = +(-4+4) = 0; \qquad A_{32} = -(-2-3) = 5; \qquad A_{33} = +(-4-6) = -10 \end{aligned}$$

$$\therefore \ \mathbf{C} = \begin{pmatrix} -14 & -25 & 29 \\ -7 & 0 & 7 \\ 0 & 5 & -10 \end{pmatrix} \quad \therefore \ \text{adj } \mathbf{A} = \mathbf{C}^T = \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

$$\text{Now } |\mathbf{A}| = -35 \quad \therefore \ \mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

$$\therefore \ \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \ldots\ldots\ldots\ldots$$

Multiply it out

38

$$\boxed{\mathbf{x} = -\frac{1}{35}\begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}}$$

$$\text{So finally } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \therefore \ x_1 = 2; \ x_2 = 3; \ x_3 = -4$$

Once you have found the inverse, the rest is simply $\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$. Here is another example to solve in the same way:

$$\begin{aligned} \text{If} \quad 2x_1 \ -x_2 \ +3x_3 &= 2 \\ x_1 \ +3x_2 \ -x_3 &= 11 \\ 2x_1 \ -2x_2 \ +5x_3 &= 3 \end{aligned}$$

then $\quad x_1 = \ldots\ldots\ldots\ldots; \ x_2 = \ldots\ldots\ldots\ldots; \ x_3 = \ldots\ldots\ldots\ldots$

$$\boxed{x_1 = -1;\ x_2 = 5;\ x_3 = 3}$$

The essential intermediate results are as follows:

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix} \text{ i.e. } \mathbf{A}.\mathbf{x} = \mathbf{b} \quad \therefore\ \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

$$\det \mathbf{A} = |\mathbf{A}| = 9$$

$$\mathbf{C} = \begin{pmatrix} 13 & -7 & -8 \\ -1 & 4 & 2 \\ -8 & 5 & 7 \end{pmatrix} \quad \therefore\ \text{adj } \mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^{\mathrm{T}}}{|\mathbf{A}|} = \frac{1}{9}\begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} = \frac{1}{9}\begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -9 \\ 45 \\ 27 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} \qquad \therefore\ x_1 = -1;\ x_2 = 5;\ x_3 = 3$$

Gaussian elimination method for solving a set of linear equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ i.e. } \mathbf{A}.\mathbf{x} = \mathbf{b}$$

All the information for solving the set of equations is provided by the matrix of coefficients **A** and the column matrix **b**. If we write the elements of **b** within the matrix **A**, we obtain the *augmented matrix* **B** of the given set of equations.

$$\text{i.e. } \mathbf{B} = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{array}\right)$$

(a) We then eliminate the elements other than a_{11} from the first column by subtracting a_{21}/a_{11} times the first row from the second row and a_{31}/a_{11} times the first row from the third row, etc.

▶

(b) This gives a new matrix of the form:

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & c_{22} & c_{23} & \dots & c_{2n} & d_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c_{n2} & c_{n3} & \dots & c_{nn} & d_n \end{array}\right)$$

The process is then repeated to eliminate c_{i2} from the third and subsequent rows.

A specific example will explain the method, so move on to the next frame

40

To solve
$$x_1 + 2x_2 - 3x_3 = 3$$
$$2x_1 - x_2 - x_3 = 11$$
$$3x_1 + 2x_2 + x_3 = -5$$

This can be written $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$

The augmented matrix becomes $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$

Now subtract $\dfrac{2}{1}$ times the first row from the second row

and $\dfrac{3}{1}$ times the first row from the third row.

This gives $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{array}\right)$

Now subtract $\dfrac{-4}{-5}$, i.e. $\dfrac{4}{5}$, times the second row from the third row.

The matrix becomes $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & 0 & 6 & -18 \end{array}\right)$

Note that as a result of these steps, the matrix of coefficients of x has been reduced to a triangular matrix.

Finally, we detach the right-hand column back to its original position:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -18 \end{pmatrix}$$

Then, by 'back-substitution', starting from the bottom row we get:

$$6x_3 = -18 \quad \therefore \quad x_3 = -3$$
$$-5x_2 + 5x_3 = 5 \quad \therefore \quad -5x_2 = 5 + 15 = 20 \quad \therefore \quad x_2 = -4$$
$$x_1 + 2x_2 - 3x_3 = 3 \quad \therefore \quad x_1 - 8 + 9 = 3 \quad \therefore \quad x_1 = 2$$
$$\therefore \quad x_1 = 2;\ x_2 = -4;\ x_3 = -3$$

▶

Note that when dealing with the augmented matrix, we may, if we wish:

(a) interchange two rows

(b) multiply any row by a non-zero factor

(c) add (or subtract) a constant multiple of any one row to (or from) another.

These operations are permissible since we are really dealing with the coefficients of both sides of the equations.

Now for another example: move on to the next frame

41

Solve the following set of equations:

$$x_1 - 4x_2 - 2x_3 = 21$$
$$2x_1 + x_2 + 2x_3 = 3$$
$$3x_1 + 2x_2 - x_3 = -2$$

First write the equations in matrix form, which is ....

42

$$\boxed{\begin{pmatrix} 1 & -4 & -2 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ -2 \end{pmatrix}}$$

The augmented matrix is then ....

43

$$\boxed{\left(\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right)}$$

We can now eliminate the x_1 coefficients from the second and third rows by .... and ....

44

subtracting 2 times the first row from the second row and 3 times the first row from the third row.

So the matrix now becomes $\left(\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{array}\right)$

and the next stage is to subtract from the third row .... times the second row.

45

$$\boxed{\frac{14}{9}}$$

If we do this, the matrix becomes $\begin{pmatrix} 1 & -4 & -2 & | & 21 \\ 0 & 9 & 6 & | & -39 \\ 0 & 0 & -4{\cdot}33 & | & -4{\cdot}33 \end{pmatrix}$

Re-forming the matrix equation $\begin{pmatrix} 1 & -4 & -2 \\ 0 & 9 & 6 \\ 0 & 0 & -4{\cdot}33 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 21 \\ -39 \\ -4{\cdot}33 \end{pmatrix}$

Now, starting from the bottom row, you can finish it off.

$x_1 = \dots\dots\dots\dots$; $x_2 = \dots\dots\dots\dots$; $x_3 = \dots\dots\dots\dots$

46

$$\boxed{x_1 = 3;\ x_2 = -5;\ x_3 = 1}$$

Now for something rather different.

Eigenvalues and eigenvectors

47

In many applications of matrices to technological problems involving coupled oscillations and vibrations, equations of the form

$$\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$$

occur, where $\mathbf{A} = [a_{ij}]$ is a square matrix and λ is a number (scalar). Clearly, $\mathbf{x} = 0$ is a solution for any value of λ and is not normally useful. For non-trivial solutions, i.e. $\mathbf{x} \neq \mathbf{0}$, the values of λ are called the *eigenvalues*, *characteristic values* or *latent roots* of the matrix $\mathbf{A}$ and the corresponding solutions of the given equations $\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$ are called the *eigenvectors* or characteristic vectors of $\mathbf{A}$.

Expressed as a set of separate equations, we have:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

i.e.
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= \lambda x_2 \\ \vdots \quad\quad \vdots \quad\quad\quad & \ \ \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

▶

Bringing the right-hand-side terms to the left-hand side, this simplifies to:

$$(a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots + a_{2n}x_n = 0$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\lambda)x_n = 0$$

i.e. $$\begin{pmatrix} (a_{11}-\lambda) & a_{12} & \ldots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & (a_{nn}-\lambda) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$ becomes $\mathbf{A}.\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$

and then $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

Note that the unit matrix is introduced since we can subtract only a matrix from another matrix.

For this set of homogeneous linear equations (i.e. right-hand constants all zero) to have a non-trivial solution, $|\mathbf{A} - \lambda\mathbf{I}|$ must be zero (see Programme 4, Frame 54).

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} (a_{11}-\lambda) & a_{12} & \ldots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & (a_{nn}-\lambda) \end{vmatrix} = 0$$

$|\mathbf{A} - \lambda\mathbf{I}|$ is called the *characteristic determinant* of $\mathbf{A}$ and $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the *characteristic equation*. On expanding the determinant, this gives a polynomial of degree n and solution of the characteristic equation gives the values of λ, i.e. the eigenvalues of $\mathbf{A}$.

Example 1

To find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$.

$\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$ i.e. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

Characteristic determinant: $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} (4-\lambda) & -1 \\ 2 & (1-\lambda) \end{vmatrix}$

Characteristic equation: $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$\therefore\ (4-\lambda)(1-\lambda) + 2 = 0 \quad \therefore\ 4 - 5\lambda + \lambda^2 + 2 = 0$

$\therefore\ \lambda^2 - 5\lambda + 6 = 0\ \therefore\ (\lambda - 2)(\lambda - 3) = 0$

$\therefore\ \lambda = 2$ or $3\ \therefore\ \lambda_1 = 2;\ \lambda_2 = 3.$

Example 2

To find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & -2 \\ 1 & 4 & -2 \\ 2 & 10 & 5 \end{pmatrix}$

The characteristic determinant is ....

48

$$\begin{vmatrix} (2-\lambda) & 3 & -2 \\ 1 & (4-\lambda) & -2 \\ 2 & 10 & (-5-\lambda) \end{vmatrix}$$

Expanding this, we get:

$$(2-\lambda)\{(4-\lambda)(-5-\lambda)+20\} - 3\{(-5-\lambda)+4\} - 2\{10-2(4-\lambda)\}$$
$$=(2-\lambda)\{-20+\lambda+\lambda^2+20\} + 3(1+\lambda) - 2(2+2\lambda)$$
$$=(2-\lambda)\{\lambda^2+\lambda\} + 3(1+\lambda) - 4(1+\lambda)$$
$$=(2-\lambda)\lambda(\lambda+1) - (1+\lambda) = (1+\lambda)(2\lambda-\lambda^2-1) = -(1+\lambda)(1-\lambda)^2$$

$\therefore$ Characteristic equation: $(1+\lambda)(1-\lambda)^2 = 0 \;\; \therefore \;\; \lambda = -1, 1, 1$

$\therefore \;\; \lambda_1 = -1; \;\; \lambda_2 = 1; \;\; \lambda_3 = 1$

Now one for you to do. Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

Work through the steps in the same manner.

$\lambda = \ldots\ldots\ldots\ldots$

49

$$\boxed{\lambda_1 = -1; \quad \lambda_2 = 1; \quad \lambda_3 = 2}$$

Here is the working:

Characteristic equation: $\begin{vmatrix} (1-\lambda) & -1 & 0 \\ 1 & (2-\lambda) & 1 \\ -2 & 1 & (-1-\lambda) \end{vmatrix} = 0$

$\therefore \;\; (1-\lambda)\{(2-\lambda)(-1-\lambda) - 1\} + 1(-1-\lambda+2) + 0 = 0$

$(1-\lambda)\{\lambda^2-\lambda-3\} + 1 - \lambda = 0$

$\therefore \;\; 1-\lambda = 0$ or $\lambda^2 - \lambda - 2 = 0$

$\therefore \;\; \lambda = 1$ or $(\lambda+1)(\lambda-2) = 0$ i.e. $\lambda = -1$ or 2

$\therefore \;\; \lambda_1 = -1; \;\; \lambda_2 = 1; \;\; \lambda_3 = 2$

▶

Eigenvectors

Each eigenvalue obtained has corresponding to it a solution of $\mathbf{x}$ called an *eigenvector*. In matrices, the term 'vector' indicates a row matrix or column matrix.

Example 1

Consider the equation $\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

The characteristic equation is $\begin{vmatrix} (4-\lambda) & 1 \\ 3 & (2-\lambda) \end{vmatrix} = 0$

$$\therefore\ (4-\lambda)(2-\lambda) - 3 = 0 \ \therefore\ \lambda^2 - 6\lambda + 5 = 0$$
$$\therefore\ (\lambda - 1)(\lambda - 5) = 0 \ \therefore\ \lambda = 1 \text{ or } 5$$
$$\lambda_1 = 1;\ \lambda_2 = 5$$

For $\lambda_1 = 1$, the equation $\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$ becomes:

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left.\begin{aligned} 4x_1 + x_2 &= x_1 \\ 3x_1 + 2x_2 &= x_2 \end{aligned}\right\} \text{ Either of these gives } x_2 = -3x_1$$

This result merely tells us that whatever value x_1 has, the value of x_2 is -3 times it. Therefore, the eigenvector $x_1 = \begin{pmatrix} k \\ -3k \end{pmatrix}$ is the general form of an infinite number of such eigenvectors. The simplest eigenvector is therefore $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

For $\lambda_2 = 5$, a similar result can be obtained. Determine the eigenvector in the same way.

$\mathbf{x}_2 = \ldots\ldots\ldots\ldots$

50

$\mathbf{x}_2 = \begin{pmatrix} k \\ k \end{pmatrix}$ is the general solution; $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution

Because, when $\lambda_2 = 5$, $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}$

$$\therefore\ 4x_1 + x_2 = 5x_1 \ \therefore\ x_1 = x_2 \ \therefore\ x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is a solution}$$

Therefore, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 5$.

▶

Example 2

Determine the eigenvalues and eigenvectors for the equation

$$\mathbf{A}.\mathbf{x} = \lambda\mathbf{x} \text{ where } \mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$$

The characteristic equation is $\begin{vmatrix} (2-\lambda) & 0 & 1 \\ -1 & (4-\lambda) & -1 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$$\therefore\ (2-\lambda)\{-\lambda(4-\lambda)+2\} + 1\{-2+(4-\lambda)\} = 0$$

$$\therefore\ (2-\lambda)\{\lambda^2 - 4\lambda + 2\} + (2-\lambda) = 0$$

$$\therefore\ (2-\lambda)\{\lambda^2 - 4\lambda + 3\} = 0 \qquad \therefore\ \lambda = \ldots\ldots\ldots\ldots$$

51

$$\boxed{\lambda = 1, 2, 3}$$

For $\lambda_1 = 1$: $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ using $(\mathbf{A} - \lambda\mathbf{I}).\mathbf{x} = \mathbf{0}$

$x_1 + x_3 = 0\ \therefore\ x_3 = -x_1$

$-x_1 + 2x_2 - x_3 = 0\ \therefore\ -x_1 + 2x_2 + x_1 = 0\ \therefore\ x_2 = 0$

$\therefore\ \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

For $\lambda_2 = 2$: $\begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Therefore, an eigenvector corresponding to $\lambda_2 = 2$ is

$\mathbf{x}_2 = \ldots\ldots\ldots\ldots$

52

$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}$$

Because $x_3 = 0$ and $-x_1 + 2x_2 - x_3 = 0\ \therefore\ x_1 = 2x_2$.

For $\lambda_3 = 3$, we can find an eigenvector in the same way. This gives

$\mathbf{x}_3 = \ldots\ldots\ldots\ldots$

53

$$\boxed{\mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

Because with $\lambda_3 = 3$: $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & -3 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore \ -x_1 + x_3 = 0 \ \therefore \ x_3 = x_1$

$-x_1 + x_2 - x_3 = 0 \ \therefore \ -2x_1 + x_2 = 0 \ \therefore \ x_2 = 2x_1.$

So, collecting our results together, we have:

$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

$\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 2$

$\mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_3 = 3$

Here is one for you to do on your own. The method is the same as before.

If $\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ and the eigenvalues are known to be $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$, determine corresponding eigenvectors.

$\mathbf{x}_1 = \ldots\ldots\ldots\ldots \quad \mathbf{x}_2 = \ldots\ldots\ldots\ldots \quad \mathbf{x}_3 = \ldots\ldots\ldots\ldots$

54

$$\boxed{\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}; \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$$

Using $\begin{pmatrix} (1-\lambda) & -1 & 0 \\ 1 & (2-\lambda) & 1 \\ -2 & 1 & (-1-\lambda) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

simple substitution of the values of λ in turn and the knowledge of how to multiply matrices together give the results indicated.

As we have seen, a basic knowledge of matrices provides a neat and concise way of dealing with sets of linear equations. In practice, the numerical coefficients are not always simple numbers, neither is the number of equations in the set limited to three. In more extensive problems, recourse to computing facilities is a great help, but the underlying methods are still the same.

All that now remains is to check down the **Revision summary** and the **Can You?** checklist. Then you can work through the **Test exercise**.

55

Revision summary

1 *Matrix* – a rectangular array of numbers (elements).

2 *Order* – a matrix order of $(m \times n)$ denotes m rows and n columns.

3 *Row matrix* – one row only.

4 *Column matrix* – one column only.

5 *Double suffix notation* – a_{34} denotes element in 3rd row and 4th column.

6 *Equal matrices* – corresponding elements equal.

7 *Addition and subtraction of matrices* – add or subtract corresponding elements. Therefore, for addition or subtraction, matrices must be of the same order.

8 *Multiplication of matrices*

(a) *Scalar multiplier* – every element multiplied by the same constant, i.e. $k[a_{ij}] = [ka_{ij}]$.

(b) *Matrix multiplier* – product **A**.**B** possible only if the number of columns in **A** equals the number of rows in **B**.

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \cdot \begin{pmatrix} g & j \\ h & k \\ i & l \end{pmatrix} = \begin{pmatrix} ag+bh+ci & aj+bk+cl \\ dg+eh+fi & dj+ek+fl \end{pmatrix}$$

9 *Square matrix* – of order $(m \times m)$

(a) *Symmetric* if $a_{ij} = a_{ji}$, e.g. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 6 & 1 \\ 4 & 1 & 5 \end{pmatrix}$

(b) *Skew symmetric* if $a_{ij} = -a_{ji}$, e.g. $\begin{pmatrix} 0 & 2 & 4 \\ -2 & 0 & 1 \\ -4 & -1 & 0 \end{pmatrix}$

10 *Diagonal matrix* – all elements zero except those on the leading diagonal.

11 *Unit matrix* – a diagonal matrix with elements on the leading diagonal all unity, i.e. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ denoted by **I**.

12 *Null matrix* – all elements zero.

13 *Transpose of a matrix* – rows and columns interchanged. Transpose of **A** is $\mathbf{A}^T$.

14 *Cofactors* – minors of the elements of $|\mathbf{A}|$ together with the respective 'place signs' of the elements.

15 *Adjoint of a square matrix* **A** – form matrix **C** of the cofactors of the elements of $|\mathbf{A}|$, then the adjoint of **A** is $\mathbf{C}^T$, i.e. the transpose of **C**. $\therefore$ adj $\mathbf{A} = \mathbf{C}^T$.

▶

16 *Inverse of a square matrix* **A**

$$\mathbf{A}^{-1}=\frac{\text{adj }\mathbf{A}}{|\mathbf{A}|}=\frac{\mathbf{C}^{\mathrm{T}}}{|\mathbf{A}|}$$

17 Product of a square matrix and its inverse

$$\mathbf{A}.\mathbf{A}^{-1}=\mathbf{A}^{-1}.\mathbf{A}=\mathbf{I}$$

18 *Solution of a set of linear equations*

$$\mathbf{A}.\mathbf{x}=\mathbf{b}\quad \mathbf{x}=\mathbf{A}^{-1}.\mathbf{b}$$

19 *Gaussian elimination method* – reduce augmented matrix to triangular form, then use 'back substitution'.

20 *Eigenvalues* – values of λ for which $\mathbf{A}.\mathbf{x}=\lambda\mathbf{x}$.

21 *Eigenvectors* – solutions of $\mathbf{x}$ corresponding to particular values of λ.

Now for the **Can You?** *checklist*

☑ Can You?

Checklist 5

56

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Define a matrix? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 7
• Understand what is meant by the equality of two matrices? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	8 to 9
• Add and subtract two matrices? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	10 to 11
• Multiply a matrix by a scalar and multiply two matrices together? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	12 to 18
• Obtain the transpose of a matrix? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	19 to 20
• Recognize special types of matrix? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	21 to 23
• Obtain the determinant, cofactors and adjoint of a square matrix? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	24 to 27

▶

	Frames
• Obtain the inverse of a non-singular matrix? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	28 to 34
• Use matrices to solve a set of linear equations using inverse matrices? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	35 to 39
• Use the Gaussian elimination method to solve a set of linear equations? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	39 to 46
• Evaluate eigenvalues and eigenvectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	47 to 54

Test exercise 5

57

The questions are all straightforward and based on the work covered. You will have no trouble.

1 If $\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 & 3 \\ 1 & 7 & 0 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 & 2 & 7 \\ 9 & 1 & 6 & 3 \end{pmatrix}$ determine (a) $\mathbf{A}+\mathbf{B}$ and (b) $\mathbf{A}-\mathbf{B}$.

2 Given that $\mathbf{A} = \begin{pmatrix} 6 & 0 & 4 \\ 1 & 5 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 9 \\ 8 & 0 \\ -4 & 7 \end{pmatrix}$ determine (a) $3\mathbf{A}$, (b) $\mathbf{A}.\mathbf{B}$, (c) $\mathbf{B}.\mathbf{A}$.

3 If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 0 & 6 \end{pmatrix}$, form the transpose $\mathbf{A}^{\mathrm{T}}$ and determine the matrix product $\mathbf{A}^{\mathrm{T}}.\mathbf{I}$.

4 Show that the square matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{pmatrix}$ is a singular matrix.

5 If $\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 5 \\ 1 & 7 & 0 \end{pmatrix}$, determine (a) $|\mathbf{A}|$ and (b) adj $\mathbf{A}$.

6 Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$

▶

7 Express the following set of linear equations in matrix form:

$$2x_1 + 4x_2 - 5x_3 = -7$$
$$x_1 - 3x_2 + x_3 = 10$$
$$3x_1 + 5x_2 + 3x_3 = 2$$

8 Solve the following set of linear equations by the matrix method:

$$x_1 + 3x_2 + 2x_3 = 3$$
$$2x_1 - x_2 - 3x_3 = -8$$
$$5x_1 + 2x_2 + x_3 = 9$$

9 For the following set of simultaneous equations:
(a) form the augmented coefficient matrix
(b) solve the equations by Gaussian elimination.

$$x_1 + 2x_2 + 3x_3 = 5$$
$$3x_1 - x_2 + 2x_3 = 8$$
$$4x_1 - 6x_2 - 4x_3 = -2$$

10 If $\mathbf{A}.\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ determine the eigenvalues of the matrix $\mathbf{A}$ and an eigenvector corresponding to each eigenvalue.

Further problems 5

58

1 If $\mathbf{A} = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 5 & 8 \end{pmatrix}$, determine:

(a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$, (c) $\mathbf{A}.\mathbf{B}$, (d) $\mathbf{B}.\mathbf{A}$

2 If $\mathbf{A} = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$ $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $j = \sqrt{-1}$, express (a) $\mathbf{A}.\mathbf{B}$, (b) $\mathbf{B}.\mathbf{C}$, (c) $\mathbf{C}.\mathbf{A}$ and (d) $\mathbf{A}^2$ in terms of other matrices.

3 If $\mathbf{A} = \begin{pmatrix} 1 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, determine:

(a) $\mathbf{B}^{-1}$, (b) $\mathbf{A}.\mathbf{B}$, (c) $\mathbf{B}^{-1}.\mathbf{A}$

4 Determine the value of k for which the following set of homogeneous equations has non-trivial solutions:

$$4x_1 + 3x_2 - x_3 = 0$$
$$7x_1 - x_2 - 3x_3 = 0$$
$$3x_1 - 4x_2 + kx_3 = 0$$

▶

5 Express the following sets of simultaneous equations in matrix form:

(a) $\begin{aligned} 2x_1 - 3x_2 - x_3 &= 2 \\ x_1 + 4x_2 + 2x_3 &= 3 \\ x_1 - x_2 + x_3 &= 5 \end{aligned}$

(b) $\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 10 \\ 2x_1 + 3x_2 + x_4 &= 8 \\ x_1 - 4x_3 - 2x_4 &= 3 \\ -x_2 + 3x_3 + x_4 &= -7 \end{aligned}$

In **6** to **10** solve, where possible, the sets of equations by a matrix method:

6 $\begin{aligned} 2i_1 + i_2 + i_3 &= 8 \\ 5i_1 - 3i_2 + 2i_3 &= 3 \\ 7i_1 + i_2 + 3i_3 &= 20 \end{aligned}$

7 $\begin{aligned} 3x + 2y + 4z &= 3 \\ x + y + z &= 2 \\ 2x - y + 3z &= -3 \end{aligned}$

8 $\begin{aligned} 4i_1 - 5i_2 + 6i_3 &= 3 \\ 8i_1 - 7i_2 - 3i_3 &= 9 \\ 7i_1 - 8i_2 + 9i_3 &= 6 \end{aligned}$

9 $\begin{aligned} 3x + 2y + 5z &= 1 \\ x - y + z &= 4 \\ 6x + 4y + 10z &= 7 \end{aligned}$

10 $\begin{aligned} 3x_1 + 2x_2 - 2x_3 &= 16 \\ 4x_1 + 3x_2 + 3x_3 &= 2 \\ -2x_1 + x_2 - x_3 &= 1 \end{aligned}$

In **11** to **13** form the augmented matrix and solve the sets of equations by Gaussian elimination:

11 $\begin{aligned} 5i_1 - i_2 + 2i_3 &= 3 \\ 2i_1 + 4i_2 + i_3 &= 8 \\ i_1 + 3i_2 - 3i_3 &= 2 \end{aligned}$

12 $\begin{aligned} i_1 + 2i_2 + 3i_3 &= -4 \\ 2i_1 + 6i_2 - 3i_3 &= 33 \\ 4i_1 - 2i_2 + i_3 &= 3 \end{aligned}$

13 $\begin{aligned} 7i_1 - 4i_2 &= 12 \\ -4i_1 + 12i_2 - 6i_3 &= 0 \\ -6i_2 + 14i_3 &= 0 \end{aligned}$

14 In a star-connected circuit, currents i_1, i_2, i_3 flowing through impedances Z_1, Z_2, Z_3, are given by:

$$\begin{aligned} i_1 + i_2 + i_3 &= 0 \\ Z_1 i_1 - Z_2 i_2 &= e_1 - e_2 \\ Z_2 i_1 - Z_3 i_3 &= e_2 - e_3 \end{aligned}$$

If $Z_1 = 10$; $Z_2 = 8$; $Z_3 = 3$; $e_1 - e_2 = 65$; $e_2 - e_3 = 160$; apply matrix methods to determine the values of i_1, i_2, i_3.

15 Currents of i_1, i_2, i_3 in a network are related by the following equations:

$$\begin{aligned} Z_1 i_1 + Z_3 i_3 &= V \\ Z_2 i_2 - Z_3 i_3 &= 0 \\ i_1 - i_2 - i_3 &= 0 \end{aligned}$$

Determine expressions for i_1, i_2, i_3, in terms of Z_1, Z_2, Z_3 and V.

16 to **20** refer to the vector equation $\mathbf{A.x} = \lambda\mathbf{x}$. For the coefficient matrix $\mathbf{A}$ given in each case, determine the eigenvalues and an eigenvector corresponding to each eigenvalue:

16 $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{pmatrix}$

17 $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}$

18 $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$

19 $\mathbf{A} = \begin{pmatrix} 1 & -4 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

20 $\mathbf{A} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Programme 6

Vectors

Frames 1 to 63

Learning outcomes

When you have completed this Programme you will be able to:

- Define a vector
- Represent a vector by a directed straight line
- Add vectors
- Write a vector in terms of component vectors
- Write a vector in terms of component unit vectors
- Set up a coordinate system for representing vectors
- Obtain the direction cosines of a vector
- Calculate the scalar product of two vectors
- Calculate the vector product of two vectors
- Determine the angle between two vectors
- Evaluate the direction ratios of a vector

Introduction: scalar and vector quantities

1

Physical quantities can be divided into two main groups, scalar quantities and vector quantities.

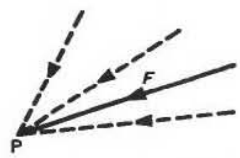
(a) A *scalar quantity* is one that is defined completely by a single number with appropriate units, e.g. length, area, volume, mass, time, etc. Once the units are stated, the quantity is denoted entirely by its size or *magnitude*.

(b) A *vector quantity* is defined completely when we know not only its magnitude (with units) but also the direction in which it operates, e.g. force, velocity, acceleration. A vector quantity necessarily involves *direction* as well as magnitude.

So (a) a speed of 10 km/h is a scalar quantity, but
(b) a velocity of '10 km/h due north' is a .... quantity.

2

vector



A force F acting at a point P is a vector quantity, since to define it completely we must give:

(a) its magnitude, and also
(b) its ....

3

direction

So that:

(a) A temperature of 100°C is a .... quantity.
(b) An acceleration of 9·8 m/s^2 vertically downwards is a .... quantity.
(c) The weight of a 7 kg mass is a .... quantity.
(d) The sum of £500 is a .... quantity.
(e) A north-easterly wind of 20 knots is a .... quantity.

4

(a) scalar (b) vector (c) vector (d) scalar (e) vector

Since, in (b), (c) and (e) the complete description of the quantity includes not only its magnitude, but also its ....

5

direction

Move on to Frame 6

Vector representation

6

A vector quantity can be represented graphically by a line, drawn so that:

(a) the *length* of the line denotes the magnitude of the quantity, according to some stated vector scale

(b) the *direction* of the line denotes the direction in which the vector quantity acts. The sense of the direction is indicated by an arrowhead.

e.g. A horizontal force of 35 N acting to the right, would be indicated by a line $\longrightarrow$ and if the chosen vector scale were 1 cm $\equiv$ 10 N, the line would be .... cm long.

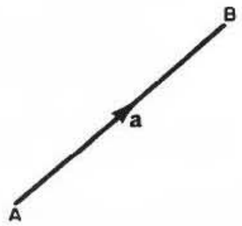
7

3·5

The vector quantity AB is referred to as

$\overline{AB}$ or $\mathbf{a}$

The magnitude of the vector quantity is written $|\overline{AB}|$, or $|\mathbf{a}|$, or simply AB or a.



Note that $\overline{BA}$ would represent a vector quantity of the same magnitude but with opposite sense.

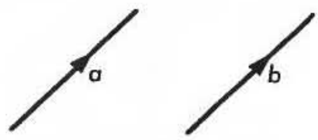
On to Frame 8

Two equal vectors

8

If two vectors, $\mathbf{a}$ and $\mathbf{b}$, are said to be equal, they have the same magnitude and the same direction.

If $\mathbf{a} = \mathbf{b}$, then



(a) $a = b$ (magnitudes equal)

(b) the direction of $\mathbf{a}$ = direction of $\mathbf{b}$, i.e. the two vectors are parallel and in the same sense.

Similarly, if two vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{b} = -\mathbf{a}$, what can we say about:

(a) their magnitudes,

(b) their directions?

9

$$\boxed{\begin{array}{l}\text{(a) Magnitudes are equal} \\ \text{(b) The vectors are parallel but opposite in sense}\end{array}}$$

i.e. if $\mathbf{b} = -\mathbf{a}$, then

10

Types of vectors

(a) A *position vector* $\overline{AB}$ occurs when the point A is fixed.

(b) A *line vector* is such that it can slide along its line of action, e.g. a mechanical force acting on a body.

(c) A *free vector* is not restricted in any way. It is completely defined by its magnitude and direction and can be drawn as any one of a set of equal-length parallel lines.

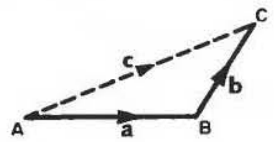
Most of the vectors we shall consider will be free vectors

So on now to Frame 11

11

Addition of vectors

The sum of two vectors, $\overline{AB}$ and $\overline{BC}$, is defined as the single or equivalent or resultant vector $\overline{AC}$



i.e. $\overline{AB} + \overline{BC} = \overline{AC}$

or $\mathbf{a} + \mathbf{b} = \mathbf{c}$

To find the sum of two vectors **a** and **b** then, we draw them as a chain, starting the second where the first ends: the sum **c** is given by the single vector joining the start of the first to the end of the second.

e.g. if $\mathbf{p} \equiv$ a force of 40 N, acting in the direction due east

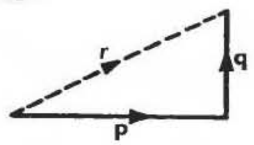
$\mathbf{q} \equiv$ a force of 30 N, acting in the direction due north

then the magnitude of the vector sum r of these will forces will be ....

12

$$\boxed{r = 50 \text{ N}}$$

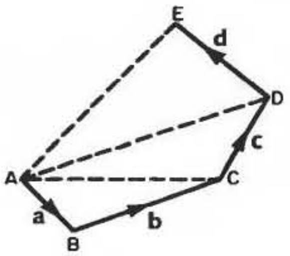
Because



$$\begin{aligned} r^2 &= p^2 + q^2 \\ &= 1600 + 900 = 2500 \\ r &= \sqrt{2500} = 50 \text{ N} \end{aligned}$$

▶

The sum of a number of vectors $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\ldots$

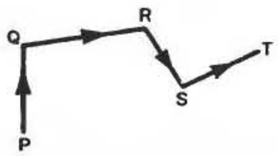


(a) Draw the vectors as a chain.

(b) Then:

$$\mathbf{a}+\mathbf{b}=\overline{\mathrm{AC}}$$
$$\overline{\mathrm{AC}}+\mathbf{c}=\overline{\mathrm{AD}}$$
$$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c}=\overline{\mathrm{AD}}$$
$$\overline{\mathrm{AD}}+\mathbf{d}=\overline{\mathrm{AE}}$$
$$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=\overline{\mathrm{AE}}$$

i.e. the sum of all vectors, **a**, **b**, **c**, **d**, is given by the single vector joining the start of the first to the end of the last – in this case, $\overline{\mathrm{AE}}$. This follows directly from our previous definition of the sum of two vectors.



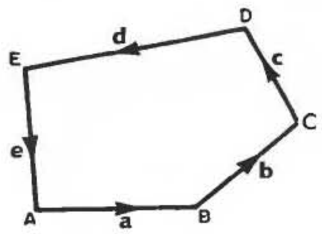
Similarly:

$\overline{\mathrm{PQ}}+\overline{\mathrm{QR}}+\overline{\mathrm{RS}}+\overline{\mathrm{ST}}=\ldots\ldots\ldots\ldots$

13

$\boxed{\overline{\mathrm{PT}}}$

Now suppose that in another case, we draw the vector diagram to find the sum of **a**, **b**, **c**, **d**, **e**, and discover that the resulting diagram is, in fact, a closed figure.



What is the sum of the vectors **a**, **b**, **c**, **d**, **e** in this case?

Think carefully and when you have decided, move on to Frame 14

14

$\boxed{\text{Sum of the vectors} = \mathbf{0}}$

Because we said in the previous case, that the vector sum was given by the single equivalent vector joining the beginning of the first vector to the end of the last.

But, if the vector diagram is a closed figure, the end of the last vector coincides with the beginning of the first, so that the resultant sum is a vector with no magnitude.

▶

Now for some examples:

Find the vector sum $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF}$.

Without drawing a diagram, we can see that the vectors are arranged in a chain, each beginning where the previous one left off. The sum is therefore given by the vector joining the beginning of the first vector to the end of the last.

$$\therefore \text{ Sum} = \overline{AF}$$

In the same way:

$$\overline{AK} + \overline{KL} + \overline{LP} + \overline{PQ} = \ldots\ldots\ldots\ldots$$

15

$$\boxed{\overline{AQ}}$$

Right. Now what about this one?

Find the sum of $\overline{AB} - \overline{CB} + \overline{CD} - \overline{ED}$

We must beware of the negative vectors. Remember that $-\overline{CB} = \overline{BC}$, i.e. the same magnitude and direction but in the opposite sense.
Also $-\overline{ED} = \overline{DE}$

$$\therefore \overline{AB} - \overline{CB} + \overline{CD} - \overline{ED} = \overline{AB} + \overline{BC} + \overline{CD} + \overline{DE}$$
$$= \overline{AE}$$

Now you do this one:

Find the vector sum $\overline{AB} + \overline{BC} - \overline{DC} - \overline{AD}$

When you have the result, move on to Frame 16

16

$$\boxed{\mathbf{0}}$$

Because

$$\overline{AB} + \overline{BC} - \overline{DC} - \overline{AD} = \overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}$$

and the lettering indicates that the end of the last vector coincides with the beginning of the first. The vector diagram is thus a closed figure and therefore the sum of the vectors is **0**.
Now here are some for you to do:

(a) $\overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST} = \ldots\ldots\ldots\ldots$

(b) $\overline{AC} + \overline{CL} - \overline{ML} = \ldots\ldots\ldots\ldots$

(c) $\overline{GH} + \overline{HJ} + \overline{JK} + \overline{KL} + \overline{LG} = \ldots\ldots\ldots\ldots$

(d) $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DB} = \ldots\ldots\ldots\ldots$

When you have finished all four, check with the results in the next frame

17

Here are the results:

(a) $\overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST} = \overline{PT}$

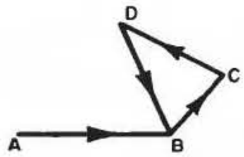
(b) $\overline{AC} + \overline{CL} - \overline{ML} = \overline{AC} + \overline{CL} + \overline{LM} = \overline{AM}$

(c) $\overline{GH} + \overline{HJ} + \overline{JK} + \overline{KL} + \overline{LG} = \mathbf{0}$

[Since the end of the last vector coincides with the beginning of the first.]

(d) $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DB} = \overline{AB}$

The last three vectors form a closed figure and therefore the sum of these three vectors is zero, leaving only $\overline{AB}$ to be considered.



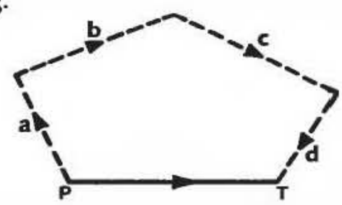
Now on to Frame 18

Components of a given vector

18

Just as $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE}$ can be replaced by $\overline{AE}$, so any single vector $\overline{PT}$ can be replaced by any number of component vectors so long as they form a chain in the vector diagram, beginning at P and ending at T.

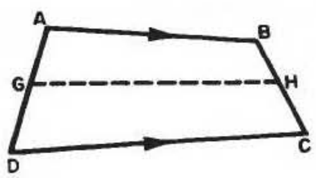
e.g.



$\overline{PT} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$

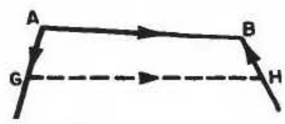
Example 1

ABCD is a quadrilateral, with G and H the mid-points of DA and BC respectively. Show that $\overline{AB} + \overline{DC} = 2\overline{GH}$.



We can replace vector $\overline{AB}$ by any chain of vectors so long as they start at A and end at B e.g. we could say

$\overline{AB} = \overline{AG} + \overline{GH} + \overline{HB}$



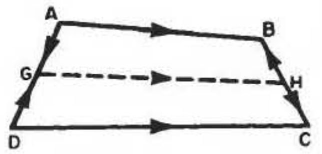
Similarly, we could say

$\overline{DC} = \ldots\ldots\ldots\ldots$

19

$$\boxed{\overline{DC} = \overline{DG} + \overline{GH} + \overline{HC}}$$

So we have:



$$\overline{AB} = \overline{AG} + \overline{GH} + \overline{HB}$$
$$\overline{DC} = \overline{DG} + \overline{GH} + \overline{HC}$$

$$\begin{aligned}\therefore\ \overline{AB} + \overline{DC} &= \overline{AG} + \overline{GH} + \overline{HB} + \overline{DG} + \overline{GH} + \overline{HC}\\ &= 2\overline{GH} + (\overline{AG} + \overline{DG}) + (\overline{HB} + \overline{HC})\end{aligned}$$

Now, G is the mid-point of AD. Therefore, vectors $\overline{AG}$ and $\overline{DG}$ are equal in length but opposite in sense.

$$\therefore\ \overline{DG} = -\overline{AG}$$
$$\text{Similarly } \overline{HC} = -\overline{HB}$$
$$\begin{aligned}\therefore\ \overline{AB} + \overline{DC} &= 2\overline{GH} + (\overline{AG} - \overline{AG}) + (\overline{HB} - \overline{HB})\\ &= 2\overline{GH}\end{aligned}$$

Next frame

20

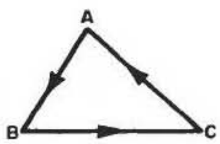
Example 2

Points L, M, N are mid-points of the sides AB, BC, CA of the triangle ABC. Show that:

(a) $\overline{AB} + \overline{BC} + \overline{CA} = \mathbf{0}$

(b) $2\overline{AB} + 3\overline{BC} + \overline{CA} = 2\overline{LC}$

(c) $\overline{AM} + \overline{BN} + \overline{CL} = \mathbf{0}$



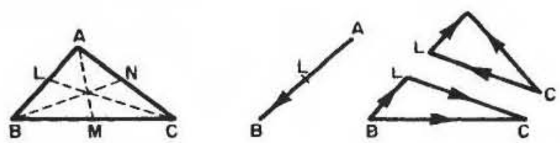
(a) We can dispose of the first part straight away without any trouble. We can see from the vector diagram that $\overline{AB} + \overline{BC} + \overline{CA} = \mathbf{0}$ since these three vectors form a ....

21

$$\boxed{\text{closed figure}}$$

Now for part (b):

To show that $2\overline{AB} + 3\overline{BC} + \overline{CA} = 2\overline{LC}$



▶

From the figure:

$$\overline{AB} = 2\overline{AL}; \quad \overline{BC} = \overline{BL} + \overline{LC}; \quad \overline{CA} = \overline{CL} + \overline{LA}$$

$$\therefore \; 2\overline{AB} + 3\overline{BC} + \overline{CA} = 4\overline{AL} + 3\overline{BL} + 3\overline{LC} + \overline{CL} + \overline{LA}$$

$$\text{Now } \overline{BL} = -\overline{AL}; \quad \overline{CL} = -\overline{LC}; \quad \overline{LA} = -\overline{AL}$$

Substituting these in the previous line, gives

$2\overline{AB} + 3\overline{BC} + \overline{CA} = \dots\dots\dots\dots$

22

$$\boxed{2\overline{LC}}$$

Because

$$\begin{aligned} 2\overline{AB} + 3\overline{BC} + \overline{CA} &= 4\overline{AL} + 3\overline{BL} + 3\overline{LC} + \overline{CL} + \overline{LA} \\ &= 4\overline{AL} - 3\overline{AL} + 3\overline{LC} - \overline{LC} - \overline{AL} \\ &= 4\overline{AL} - 4\overline{AL} + 3\overline{LC} - \overline{LC} \\ &= 2\overline{LC} \end{aligned}$$

Now part (c):

To prove that $\overline{AM} + \overline{BN} + \overline{CL} = \mathbf{0}$

From the figure in Frame 21, we can say:

$$\overline{AM} = \overline{AB} + \overline{BM}$$
$$\overline{BN} = \overline{BC} + \overline{CN}$$

Similarly $\overline{CL} = \dots\dots\dots\dots$

23

$$\boxed{\overline{CL} = \overline{CA} + \overline{AL}}$$

$$\begin{aligned} \text{So } \overline{AM} + \overline{BN} + \overline{CL} &= \overline{AB} + \overline{BM} + \overline{BC} + \overline{CN} + \overline{CA} + \overline{AL} \\ &= (\overline{AB} + \overline{BC} + \overline{CA}) + (\overline{BM} + \overline{CN} + \overline{AL}) \\ &= (\overline{AB} + \overline{BC} + \overline{CA}) + \frac{1}{2}(\overline{BC} + \overline{CA} + \overline{AB}) \\ &= \dots\dots\dots\dots \end{aligned}$$

Finish it off

24

$$\boxed{\overline{AM} + \overline{BN} + \overline{CL} = \mathbf{0}}$$

Because $\overline{AM} + \overline{BN} + \overline{CL} = (\overline{AB} + \overline{BC} + \overline{CA}) + \frac{1}{2}(\overline{BC} + \overline{CA} + \overline{AB})$

Now $\overline{AB} + \overline{BC} + \overline{CA}$ is a closed figure $\;\therefore$ Vector sum $= \mathbf{0}$

and $\overline{BC} + \overline{CA} + \overline{AB}$ is a closed figure $\;\therefore$ Vector sum $= \mathbf{0}$

$\therefore \; \overline{AM} + \overline{BN} + \overline{CL} = \mathbf{0}$

Here is another. ▶

Example 3

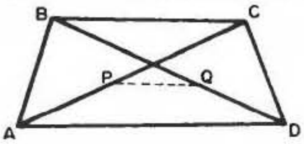
ABCD is a quadrilateral in which P and Q are the mid-points of the diagonals AC and BD respectively.

Show that $\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 4\overline{PQ}$

First, just draw the figure.

Then move on to Frame 25

25



To prove that $\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 4\overline{PQ}$

Taking the vectors on the left-hand side, one at a time, we can write:

$\overline{AB} = \overline{AP} + \overline{PQ} + \overline{QB}$

$\overline{AD} = \overline{AP} + \overline{PQ} + \overline{QD}$

$\overline{CB} = \dots\dots\dots\dots$

$\overline{CD} = \dots\dots\dots\dots$

26

$$\boxed{\overline{CB} = \overline{CP} + \overline{PQ} + \overline{QB};\quad \overline{CD} = \overline{CP} + \overline{PQ} + \overline{QD}}$$

Adding all four lines together, we have:

$$\begin{aligned}\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} &= 4\overline{PQ} + 2\overline{AP} + 2\overline{CP} + 2\overline{QB} + 2\overline{QD}\\ &= 4\overline{PQ} + 2(\overline{AP} + \overline{CP}) + 2(\overline{QB} + \overline{QD})\end{aligned}$$

Now what can we say about $(\overline{AP} + \overline{CP})$?

27

$$\boxed{\overline{AP} + \overline{CP} = \mathbf{0}}$$

Because P is the mid-point of AC $\therefore$ AP = PC

$\therefore\ \overline{CP} = -\overline{PC} = -\overline{AP}$

$\therefore\ \overline{AP} + \overline{CP} = \overline{AP} - \overline{AP} = \mathbf{0}.$

In the same way, $(\overline{QB} + \overline{QD}) = \dots\dots\dots\dots$

28

$$\boxed{\overline{QB} + \overline{QD} = \mathbf{0}}$$

Since Q is the mid-point of BD $\therefore$ $\overline{QD} = -\overline{QB}$

$\therefore\ \overline{QB} + \overline{QD} = \overline{QB} - \overline{QB} = \mathbf{0}$

$\therefore\ \overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 4\overline{PQ} + \mathbf{0} + \mathbf{0} = 4\overline{PQ}$

29

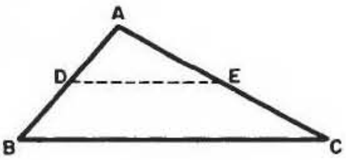
Here is one more.

Example 4

Prove by vectors that the line joining the mid-points of two sides of a triangle is parallel to the third side and half its length.

Let D and E by the mid-points of AB and AC respectively.

We have $\overline{DE} = \overline{DA} + \overline{AE}$



Now express $\overline{DA}$ and $\overline{AE}$ in terms of $\overline{BA}$ and $\overline{AC}$ respectively and see if you can get the required results.

Then on to Frame 30

30

Here is the working. Check through it.

$$\overline{DE} = \overline{DA} + \overline{AE}$$
$$= \frac{1}{2}\overline{BA} + \frac{1}{2}\overline{AC} = \frac{1}{2}(\overline{BA} + \overline{AC})$$
$$\overline{DE} = \frac{1}{2}\overline{BC}$$

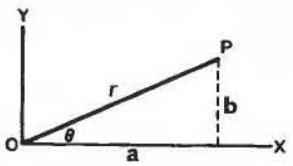
$\therefore$ $\overline{DE}$ is half the magnitude (length) of $\overline{BC}$ and acts in the same direction.

i.e. DE and BC are parallel.

Now for the next section of the work: move on to Frame 31

Components of a vector in terms of unit vectors

31



The vector $\overline{OP}$ is defined by its magnitude (r) and its direction (θ). It could also be defined by its two components in the OX and OY directions.

i.e. $\overline{OP}$ is equivalent to a vector $\mathbf{a}$ in the OX direction + a vector $\mathbf{b}$ in the OY direction.

i.e. $\overline{OP} = \mathbf{a}$ (along OX) + $\mathbf{b}$ (along OY)

If we now define $\mathbf{i}$ to be a *unit vector* in the OX direction,

then $\mathbf{a} = a\mathbf{i}$

Similarly, if we define $\mathbf{j}$ to be a *unit vector* in the OY direction,

then $\mathbf{b} = b\mathbf{j}$

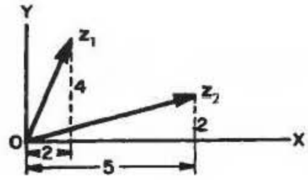
So that the vector OP can be written as:

$\mathbf{r} = a\mathbf{i} + b\mathbf{j}$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the OX and OY directions.

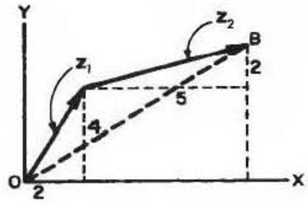
32

Let $\mathbf{z}_1 = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{z}_2 = 5\mathbf{i} + 2\mathbf{j}$



To find $\mathbf{z}_1 + \mathbf{z}_2$, draw the two vectors in a chain.

$$\mathbf{z}_1 + \mathbf{z}_2 = \overline{OB} = (2+5)\mathbf{i} + (4+2)\mathbf{j} = 7\mathbf{i} + 6\mathbf{j}$$



i.e. total up the vector components along OX, and total up the vector components along OY

Of course, we can do this without a diagram:

If $\mathbf{z}_1 = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{z}_2 = 4\mathbf{i} + 3\mathbf{j}$

$$\begin{aligned}\mathbf{z}_1 + \mathbf{z}_2 &= 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{i} + 3\mathbf{j}\\ &= 7\mathbf{i} + 5\mathbf{j}\end{aligned}$$

And in much the same way, $\mathbf{z}_2 - \mathbf{z}_1 =$ ....

33

$$\boxed{\mathbf{z}_2 - \mathbf{z}_1 = \mathbf{i} + \mathbf{j}}$$

Because
$$\begin{aligned}\mathbf{z}_2 - \mathbf{z}_1 &= (4\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j})\\ &= 4\mathbf{i} + 3\mathbf{j} - 3\mathbf{i} - 2\mathbf{j}\\ &= 1\mathbf{i} + 1\mathbf{j}\\ &= \mathbf{i} + \mathbf{j}\end{aligned}$$

Similarly, if $\mathbf{z}_1 = 5\mathbf{i} - 2\mathbf{j}$; $\mathbf{z}_2 = 3\mathbf{i} + 3\mathbf{j}$; $\mathbf{z}_3 = 4\mathbf{i} - 1\mathbf{j}$

then (a) $\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 =$ ....

and (b) $\mathbf{z}_1 - \mathbf{z}_2 - \mathbf{z}_3 =$ ....

When you have the results, move on to Frame 34

34

$$\boxed{\text{(a) } 12\mathbf{i} \qquad \text{(b) } -2\mathbf{i}-4\mathbf{j}}$$

Here is the working:

(a) $\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3 = 5\mathbf{i}-2\mathbf{j}+3\mathbf{i}+3\mathbf{j}+4\mathbf{i}-1\mathbf{j}$

$= (5+3+4)\mathbf{i}+(3-2-1)\mathbf{j} = 12\mathbf{i}$

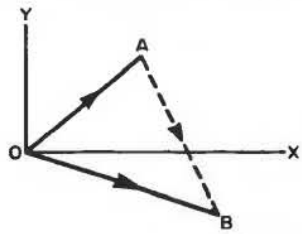
(b) $\mathbf{z}_1-\mathbf{z}_2-\mathbf{z}_3 = (5\mathbf{i}-2\mathbf{j})-(3\mathbf{i}+3\mathbf{j})-(4\mathbf{i}-1\mathbf{j})$

$= (5-3-4)\mathbf{i}+(-2-3+1)\mathbf{j} = -2\mathbf{i}-4\mathbf{j}$

Now this one.

If $\overline{\text{OA}} = 3\mathbf{i}+5\mathbf{j}$ and $\overline{\text{OB}} = 5\mathbf{i}-2\mathbf{j}$, find $\overline{\text{AB}}$.

As usual, a diagram will help. Here it is:



First of all, from the diagram, write down a relationship between the vectors. Then express them in terms of the unit vectors.

$\overline{\text{AB}} = \ldots\ldots\ldots\ldots$

35

$$\boxed{\overline{\text{AB}} = 2\mathbf{i}-7\mathbf{j}}$$

Because we have

$\overline{\text{OA}}+\overline{\text{AB}} = \overline{\text{OB}}$ (from diagram)

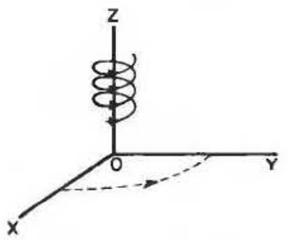
$\therefore\ \overline{\text{AB}} = \overline{\text{OB}}-\overline{\text{OA}}$

$= (5\mathbf{i}-2\mathbf{j})-(3\mathbf{i}+5\mathbf{j}) = 2\mathbf{i}-7\mathbf{j}$

On to Frame 36

Vectors in space

36



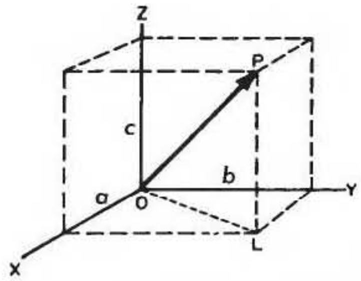
The axes of reference are defined by the 'right-hand' rule.

OX, OY, OZ form a right-handed set if rotation from OX to OY takes a right-handed corkscrew action along the positive direction of OZ.

Similarly, rotation from OY to OZ gives right-hand corkscrew action along the positive direction of ....

37

$\boxed{\text{OX}}$



Vector $\overline{\text{OP}}$ is defined by its components

a along OX
b along OY
c along OZ

Let $\mathbf{i}$ = unit vector in OX direction
$\mathbf{j}$ = unit vector in OY direction
$\mathbf{k}$ = unit vector in OZ direction

Then $\quad \overline{\text{OP}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Also $\quad \text{OL}^2 = a^2 + b^2$ and $\text{OP}^2 = \text{OL}^2 + c^2$

$$\text{OP}^2 = a^2 + b^2 + c^2$$

So, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $r = \sqrt{a^2 + b^2 + c^2}$

This gives us an easy way of finding the magnitude of a vector expressed in terms of the unit vectors.

Now you can do this one:

If $\overline{\text{PQ}} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, then $|\overline{\text{PQ}}| = \ldots\ldots\ldots\ldots$

38

$$\boxed{|\overline{\text{PQ}}| = \sqrt{29} = 5{\cdot}385}$$

Because

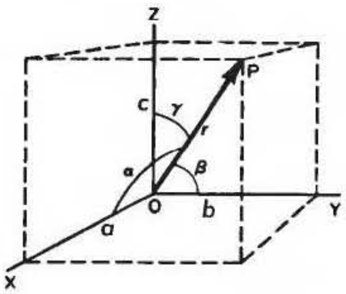
$$\begin{aligned} \overline{\text{PQ}} &= 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \\ |\overline{\text{PQ}}| &= \sqrt{4^2 + 3^2 + 2^2} \\ &= \sqrt{16 + 9 + 4} \\ &= \sqrt{29} \\ &= 5{\cdot}385 \end{aligned}$$

Now move on to Frame 39

Direction cosines

39

The direction of a vector in three dimensions is determined by the angles which the vector makes with the three axes of reference.



Let $\overline{OP} - \mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Then

$$\frac{a}{r} = \cos\alpha \qquad \therefore \; a = r\cos\alpha$$
$$\frac{b}{r} = \cos\beta \qquad b = r\cos\beta$$
$$\frac{c}{r} = \cos\gamma \qquad c = r\cos\gamma$$

Also $a^2 + b^2 + c^2 = r^2$

$\therefore \; r^2\cos^2\alpha + r^2\cos^2\beta + r^2\cos^2\gamma = r^2$

$\therefore \; \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

If $l = \cos\alpha$

$m = \cos\beta$

$n = \cos\gamma$ then $l^2 + m^2 + n^2 = 1$

Note: $[l, m, n]$ written in square brackets are called the *direction cosines* of the vector $\overline{OP}$ and are the values of the cosines of the angles which the vector makes with the three axes of reference.

So for the vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$l = \frac{a}{r}; \; m = \frac{b}{r}; \; n = \frac{c}{r}; \text{ and, of course } r = \sqrt{a^2 + b^2 + c^2}$$

So, with that in mind, find the direction cosines $[l, m, n]$ of the vector

$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$

Then to Frame 40

40

$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$

$\therefore \; a = 3, \; b = -2, \; c = 6, \; r = \sqrt{9 + 4 + 36}$

$\therefore \; r = \sqrt{49} = 7$

$\therefore \; l = \dfrac{3}{7}; \; m = -\dfrac{2}{7}; \; n = \dfrac{6}{7}$

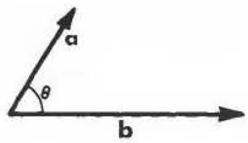
Just as easy as that!

On to the next frame

Scalar product of two vectors

41

If **a** and **b** are two vectors, the *scalar product* of **a** and **b** is defined as the *scalar* (number) $ab\cos\theta$ where a and b are the magnitudes of the vectors **a** and **b** and θ is the angle between them.

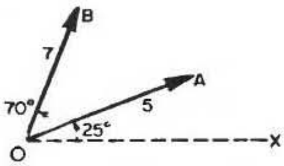


The scalar product is denoted by **a.b** (often called the 'dot product' for obvious reasons).

$$\therefore\ \mathbf{a}.\mathbf{b} = ab\cos\theta$$
$$= a \times \text{projection of } \mathbf{b} \text{ on } \mathbf{a}$$
$$= b \times \text{projection of } \mathbf{a} \text{ on } \mathbf{b}$$

In both cases the result is a *scalar* quantity.

For example:

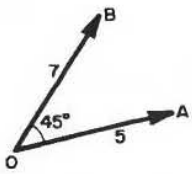


$\overline{OA}.\overline{OB} = \ldots\ldots\ldots\ldots$

42

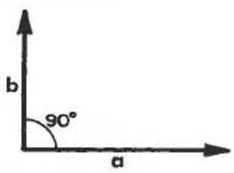
$$\boxed{\overline{OA}.\overline{OB} = \frac{35\sqrt{2}}{2}}$$

Because we have:



$$\overline{OA}.\overline{OB} = OA.OB.\cos\theta$$
$$= 5.7.\cos 45°$$
$$= 35.\frac{1}{\sqrt{2}} = \frac{35\sqrt{2}}{2}$$

Now what about this case:



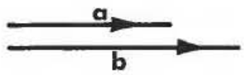
The scalar product of **a** and **b** = **a.b** = ...

43

$\boxed{0}$

Because in this case $\mathbf{a}.\mathbf{b} = ab\cos 90^\circ = ab0 = 0$. So the scalar product of any two vectors at right-angles to each other is always zero.

And in this case now, with two vectors in the same direction, $\theta = 0^\circ$



so $\mathbf{a}.\mathbf{b} = \dots\dots\dots\dots$

44

$\boxed{ab}$

Because $\mathbf{a}.\mathbf{b} = ab\cos 0^\circ = ab.1 = ab$

Now suppose our two vectors are expressed in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Let $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

and $\quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\begin{aligned}\text{Then } \mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j} + a_2b_3\mathbf{j}.\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}\end{aligned}$$

This can now be simplified.

Because $\mathbf{i}.\mathbf{i} = (1)(1)(\cos 0^\circ) = 1$

$\therefore \ \mathbf{i}.\mathbf{i} = 1; \ \mathbf{j}.\mathbf{j} = 1; \ \mathbf{k}.\mathbf{k} = 1 \qquad$ (a)

Also $\mathbf{i}.\mathbf{j} = (1)(1)(\cos 90^\circ) = 0$

$\therefore \ \mathbf{i}.\mathbf{j} = 0; \ \mathbf{j}.\mathbf{k} = 0; \ \mathbf{k}.\mathbf{i} = 0 \qquad$ (b)

So, using the results (a) and (b), we get:

$$\mathbf{a}.\mathbf{b} = \dots\dots\dots\dots$$

45

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Because

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= a_1b_1.1 + a_1b_2.0 + a_1b_3.0 + a_2b_1.0 + a_2b_2.1 + a_2b_3.0 \\ &\quad + a_3b_1.0 + a_3b_2.0 + a_3b_3.1\end{aligned}$$

$$\therefore \ \mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

i.e. we just sum the products of the coefficients of the unit vectors along the corresponding axes.

▶

For example:

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 1\mathbf{j} + 6\mathbf{k}$

then $\mathbf{a}.\mathbf{b} = 2 \times 4 + 3 \times 1 + 5 \times 6$

$= 8 + 3 + 30$

$= 41 \qquad \therefore\ \mathbf{a}.\mathbf{b} = 41$

One for you: If $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}$; $\mathbf{q} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

then $\mathbf{p}.\mathbf{q} = \dots\dots\dots\dots$

46

$\boxed{-4}$

Because

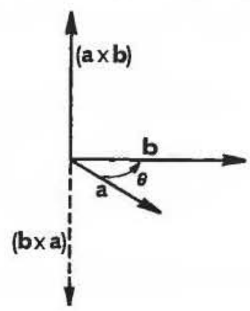
$$\mathbf{p}.\mathbf{q} = 3 \times 2 + (-2) \times 3 + 1 \times (-4)$$
$$= 6 - 6 - 4$$
$$= -4 \qquad \therefore\ \mathbf{p}.\mathbf{q} = -4$$

Now on to Frame 47

Vector product of two vectors

47

The vector product of **a** and **b** is written $\mathbf{a} \times \mathbf{b}$ (often called the 'cross product') and is defined as a *vector* having magnitude $ab \sin\theta$ where θ is the angle between the two given vectors. The product vector acts in a direction perpendicular to both **a** and **b** in such a sense that **a, b** and $\mathbf{a} \times \mathbf{b}$ form a right-handed set – in that order.



$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta$$

Note that $\mathbf{b} \times \mathbf{a}$ reverses the direction of rotation and the product vector would now act downwards, i.e.

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$$

If $\theta = 0^\circ$, then $|\mathbf{a} \times \mathbf{b}| = \dots\dots\dots\dots$

and if $\theta = 90^\circ$, then $|\mathbf{a} \times \mathbf{b}| = \dots\dots\dots\dots$

48

$$\boxed{\begin{array}{l}\theta = 0^\circ, \quad |\mathbf{a} \times \mathbf{b}| = 0 \\ \theta = 90^\circ, |\mathbf{a} \times \mathbf{b}| = ab\end{array}}$$

If **a** and **b** are given in terms of the unit vectors **i**, **j** and **k**:

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} = {} & a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} \\ & + a_2b_3\mathbf{j} \times \mathbf{k} + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}\end{aligned}$$

But $|\mathbf{i} \times \mathbf{i}| = (1)(1)(\sin 0^\circ) = 0 \quad \therefore \ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ (a)

Also $|\mathbf{i} \times \mathbf{j}| = (1)(1)(\sin 90^\circ) = 1$ and $\mathbf{i} \times \mathbf{j}$ is in the direction of **k**, i.e. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (same magnitude and same direction). Therefore:

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \qquad \text{(b)}\end{aligned}$$

And remember too that therefore:

$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i})$

$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j})$

$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k})$ since the sense of rotation is reversed

Now with the results of (a) and (b), and this last reminder, you can simplify the expression for $\mathbf{a} \times \mathbf{b}$.

Remove the zero terms and tidy up what is left.

....

Then on to Frame 49

49

$$\boxed{\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}}$$

Because

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= a_1b_1\mathbf{0} + a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_2\mathbf{0} + a_2b_3\mathbf{i} \\ &\quad + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i}) + a_3b_3\mathbf{0} \\ \mathbf{a} \times \mathbf{b} &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

and you may recognize this as the pattern of a determinant where the first row is made up of the vectors **i**, **j** and **k**.

▶

So now we have that:

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

and that is the easiest way to write out the vector product of two vectors.

Notes: (a) The top row consists of the unit vectors in order **i**, **j**, **k**.
(b) The second row consists of the coefficients of **a**.
(c) The third row consists of the coefficients of **b**.

For example, if $\mathbf{p} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, first write down the determinant that represents the vector product $\mathbf{p} \times \mathbf{q}$.

50

$$\boxed{\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ 1 & 5 & -2 \end{vmatrix} \begin{array}{l} \text{Unit vectors} \\ \text{Coefficients of } \mathbf{p} \\ \text{Coefficients of } \mathbf{q} \end{array}}$$

And now, expanding the determinant, we get:

$\mathbf{p} \times \mathbf{q} = \dots\dots\dots\dots$

51

$$\boxed{\mathbf{p} \times \mathbf{q} = -23\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}}$$

Because

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ 1 & 5 & -2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}$$
$$= \mathbf{i}(-8 - 15) - \mathbf{j}(-4 - 3) + \mathbf{k}(10 - 4)$$
$$= -23\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

So, by way of revision:

(a) *Scalar product* ('dot product')

$\mathbf{a}.\mathbf{b} = ab\cos\theta$ a scalar quantity

(b) *Vector product* ('cross product')

$\mathbf{a} \times \mathbf{b}$ = vector of magnitude $ab\sin\theta$, acting in a direction to make **a**, **b** and $\mathbf{a} \times \mathbf{b}$ a right-handed set. Also:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

And here is one final example on this point.

Find the vector product of **p** and **q** where:

$\mathbf{p} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

52

$$\boxed{\mathbf{p} \times \mathbf{q} = -6\mathbf{i} + 7\mathbf{j} + 23\mathbf{k}}$$

Because

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 2 & 5 & -1 \end{vmatrix}$$
$$= \mathbf{i}\begin{vmatrix} -4 & 2 \\ 5 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix}$$
$$= \mathbf{i}(4 - 10) - \mathbf{j}(-3 - 4) + \mathbf{k}(15 + 8)$$
$$= -6\mathbf{i} + 7\mathbf{j} + 23\mathbf{k}$$

Remember that the order in which the vectors appear in the vector product is important. It is a simple matter to verify that:

$$\mathbf{q} \times \mathbf{p} = 6\mathbf{i} - 7\mathbf{j} - 23\mathbf{k} = -(\mathbf{p} \times \mathbf{q})$$

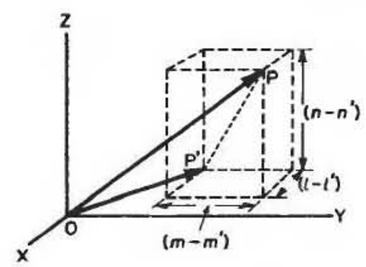
On to Frame 53

Angle between two vectors

53

Let $\mathbf{a}$ be one vector with direction cosines $[l, m, n]$
Let $\mathbf{b}$ be the other vector with direction cosines $[l', m', n']$

We have to find the angle between these two vectors.



Let $\overline{\text{OP}}$ and $\overline{\text{OP}}'$ be *unit* vectors parallel to $\mathbf{a}$ and $\mathbf{b}$ respectively. Then P has coordinates (l, m, n) and P' has coordinates (l', m', n').

Then

$$(\text{PP}')^2 = (l - l')^2 + (m - m')^2 + (n - n')^2$$
$$= l^2 - 2.l.l' + l'^2 + m^2 - 2.m.m' + m'^2 + n^2 - 2n.n' + n'^2$$
$$= (l^2 + m^2 + n^2) + (l'^2 + m'^2 + n'^2) - 2(ll' + mm' + nn')$$

But $(l^2 + m^2 + n^2) = 1$ and $(l'^2 + m'^2 + n'^2) = 1$ as was proved earlier.

$$\therefore\ (\text{PP}')^2 = 2 - 2(ll' + mm' + nn') \qquad \text{(a)}$$

▶

Also, by the cosine rule:

$$\begin{aligned}(\text{PP}')^2 &= \text{OP}^2 + \text{OP}'^2 - 2.\text{OP}.\text{OP}'.\cos\theta \\ &= 1 + 1 - 2.1.1.\cos\theta \\ &= 2 - 2\cos\theta \qquad \text{(b)}\end{aligned} \qquad \left\{\begin{array}{l}\overline{\text{OP}} \text{ and } \overline{\text{OP}'} \text{ are} \\ \text{unit vectors}\end{array}\right\}$$

So from (a) and (b), we have:

$$(\text{PP}')^2 = 2 - 2(ll' + mm' + nn')$$

and $(\text{PP}')^2 = 2 - 2\cos\theta$

$\therefore \quad \cos\theta = \dots\dots\dots\dots$

54

$$\boxed{\cos\theta = ll' + mm' + nn'}$$

That is, just sum the products of the corresponding direction cosines of the two given vectors.

So, if $[l, m, n] = [0{\cdot}54, 0{\cdot}83, -0{\cdot}14]$
and $[l', m', n'] = [0{\cdot}25, 0{\cdot}60, 0{\cdot}76]$

the angle between the vectors is $\theta = \dots\dots\dots\dots$

55

$$\boxed{\theta = 58^\circ 13'}$$

Because, we have:

$$\begin{aligned}\cos\theta &= ll' &&+ mm' &&+ nn' \\ &= (0{\cdot}54)(0{\cdot}25) &&+ (0{\cdot}83)(0{\cdot}60) &&+ (-0{\cdot}14)(0{\cdot}76) \\ &= 0{\cdot}1350 &&+ 0{\cdot}4980 &&- 0{\cdot}1064 \\ &= 0{\cdot}6330 &&- 0{\cdot}1064 && &&= 0{\cdot}5266\end{aligned}$$

$\theta = 58^\circ 13'$

Note: For *parallel vectors*, $\theta = 0^\circ \quad \therefore \quad ll' + mm' + nn' = 1$

For *perpendicular vectors*, $\theta = 90^\circ, \quad \therefore \quad ll' + mm' + nn' = 0$

Now an example for you to work:

Find the angle between the vectors

$$\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \text{ and } \mathbf{q} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

First of all, find the direction cosines of $\mathbf{p}$. So do that.

56

$$\boxed{l = \frac{2}{\sqrt{29}} \qquad m = \frac{3}{\sqrt{29}} \qquad n = \frac{4}{\sqrt{29}}}$$

Because

$$p = |\mathbf{p}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$\therefore\; l = \frac{a}{p} = \frac{2}{\sqrt{29}}$$

$$m = \frac{b}{p} = \frac{3}{\sqrt{29}}$$

$$n = \frac{c}{p} = \frac{4}{\sqrt{29}}$$

$$\therefore\; [l, m, n] = \left[\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right]$$

Now find the direction cosines $[l', m', n']$ of $\overline{\mathrm{Q}}$ in just the same way.

When you have done that move on to the next frame

57

$$\boxed{l' = \frac{4}{\sqrt{29}} \qquad m' = \frac{-3}{\sqrt{29}} \qquad n' = \frac{2}{\sqrt{29}}}$$

Because

$$q = |\mathbf{q}| = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{16 + 9 + 4} = \sqrt{29}$$

$$\therefore\; [l', m', n'] = \left[\frac{4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right]$$

We already know that, for $\overline{\mathrm{P}}$:

$$[l, m, n] = \left[\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right]$$

So, using $\cos\theta = ll' + mm' + nn'$, you can finish it off and find the angle θ. Off you go.

58

$$\boxed{\theta = 76^\circ 2'}$$

Because

$$\cos\theta = \frac{2}{\sqrt{29}} \cdot \frac{4}{\sqrt{29}} + \frac{3}{\sqrt{29}} \cdot \frac{(-3)}{\sqrt{29}} + \frac{4}{\sqrt{29}} \cdot \frac{2}{\sqrt{29}}$$

$$= \frac{8}{29} - \frac{9}{29} + \frac{8}{29}$$

$$= \frac{7}{29} = 0{\cdot}2414 \;\therefore\; \theta = 76^\circ 2'$$

Now on to Frame 59

Direction ratios

59

If $\overline{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, we know that:

$$|\overline{OP}| = r = \sqrt{a^2 + b^2 + c^2}$$

and that the direction cosines of $\overline{OP}$ are given by:

$$l = \frac{a}{r}, \quad m = \frac{b}{r}, \quad n = \frac{c}{r}$$

We can see that the components, a, b, c, are proportional to the direction cosines, l, m, n, respectively and they are sometimes referred to as the *direction ratios* of the vector $\overline{OP}$.

Note: The direction ratios can be converted into the direction cosines by dividing each of them by r (the magnitude of the vector).

Now read through the summary of the work we have covered in this Programme.

So move on to Frame 60

60

Revision summary

1 A *scalar* quantity has magnitude only; a *vector* quantity has both magnitude and direction.

2 The axes of reference, OX, OY, OZ, are chosen so that they form a right-handed set. The symbols **i**, **j**, **k** denote *unit vectors* in the directions OX, OY, OZ, respectively.

If $\overline{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $|\overline{OP}| = r = \sqrt{a^2 + b^2 + c^2}$

3 The *direction cosines* $[l, m, n]$ are the cosines of the angles between the vector and the axes OX, OY, OZ respectively.

For any vector: $l = \frac{a}{r}$, $m = \frac{b}{r}$, $n = \frac{c}{r}$; and $l^2 + m^2 + n^2 = 1$

4 *Scalar product* ('dot product')

$\mathbf{a}.\mathbf{b} = ab\cos\theta$ where θ is the angle between **a** and **b**.
If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
then $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

5 *Vector product* ('cross product')
$\mathbf{a} \times \mathbf{b} = (ab\sin\theta)$ in direction perpendicular to **a** and **b**, so that **a**, **b** and $(\mathbf{a} \times \mathbf{b})$ form a right-handed set.

$$\text{Also } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

▶

6 *Angle between two vectors*

$\cos\theta = ll' + mm' + nn'$

For perpendicular vectors, $ll' + mm' + nn' = 0$

Now you are ready for the **Can You?** checklist and **Test exercise**.

So off you go

Can You?

Checklist 6

61

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Define a vector? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 5
• Represent a vector by a directed straight line? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	6 to 10
• Add vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	11 to 17
• Write a vector in terms of component vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	18 to 30
• Write a vector in terms of component unit vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	31 to 35
• Set up a coordinate system for representing vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	36 to 38
• Obtain the direction cosines of a vector? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	39 to 40
• Calculate the scalar product of two vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	41 to 46
• Calculate the vector product of two vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	47 to 52
• Determine the angle between two vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	53 to 58
• Evaluate the direction ratios of a vector? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	59

Test exercise 6

62 Take your time: the problems are all straightforward so avoid careless slips. Diagrams often help where appropriate.

1. If $\overline{OA} = 4\mathbf{i} + 3\mathbf{j}$, $\overline{OB} = 6\mathbf{i} - 2\mathbf{j}$, $\overline{OC} = 2\mathbf{i} - \mathbf{j}$, find $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, and deduce the lengths of the sides of the triangle ABC.
2. Find the direction cosines of the vector joining the two points (4, 2, 2) and (7, 6, 14).
3. If $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, find (a) $\mathbf{a}.\mathbf{b}$ and (b) $\mathbf{a} \times \mathbf{b}$.
4. If $\mathbf{a} = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors, determine:
   (a) the value of $\mathbf{a}.\mathbf{b}$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$
   (b) the magnitude and the direction cosines of the product vector $(\mathbf{a} \times \mathbf{b})$ and also the angle which this product vector makes with the vector $\mathbf{c}$.

Further problems 6

63

1. The centroid of the triangle OAB is denoted by G. If O is the origin and $\overline{OA} = 4\mathbf{i} + 3\mathbf{j}$, $\overline{OB} = 6\mathbf{i} - \mathbf{j}$, find $\overline{OG}$ in terms of the unit vectors, $\mathbf{i}$ and $\mathbf{j}$.
2. Find the direction cosines of the vectors whose direction ratios are (3, 4, 5) and (1, 2, −3). Hence find the angle between the two vectors.
3. Find the modulus and the direction cosines of each of the vectors $3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$, $\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$ and $6\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}$. Find also the modulus and the direction cosines of their sum.
4. If $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, determine the scalar and vector products, and the angle between the two given vectors.
5. If $\overline{OA} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\overline{OB} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, determine:
   (a) the value of $\overline{OA}.\overline{OB}$
   (b) the product $\overline{OA} \times \overline{OB}$ in terms of the unit vectors
   (c) the cosine of the angle between $\overline{OA}$ and $\overline{OB}$
6. Find the cosine of the angle between the vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$.
7. Find the scalar product $(\mathbf{a}.\mathbf{b})$ and the vector product $(\mathbf{a} \times \mathbf{b})$, when
   (a) $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
   (b) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

▶

8 Find the unit vector perpendicular to each of the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the mutually perpendicular unit vectors. Calculate the sine of the angle between the two vectors.

9 If A is the point $(1, -1, 2)$, B is the point $(-1, 2, 2)$ and C is the point $(4, 3, 0)$, find the direction cosines of $\overline{BA}$ and $\overline{BC}$, and hence show that the angle $ABC = 69°14'$.

10 If $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, determine the magnitude and direction cosines of the product vector $(\mathbf{a} \times \mathbf{b})$ and show that it is perpendicular to a vector $\mathbf{c} = 9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

11 $\mathbf{a}$ and $\mathbf{b}$ are vectors defined by $\mathbf{a} = 8\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are mutually perpendicular unit vectors.
(a) Calculate $\mathbf{a}.\mathbf{b}$ and show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.
(b) Find the magnitude and the direction cosines of the product vector $\mathbf{a} \times \mathbf{b}$.

12 If the position vectors of P and Q are $\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ respectively, find $\overline{PQ}$ and determine its direction cosines.

13 If position vectors, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, are defined by $\overline{OA} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\overline{OB} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, $\overline{OC} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, determine:
(a) the vector $\overline{AB}$
(b) the vector $\overline{BC}$
(c) the vector product $\overline{AB} \times \overline{BC}$
(d) the unit vector perpendicular to the plane ABC.

Differentiation

Frames 1 to 39

Learning outcomes

When you have completed this Programme you will be able to:

- Differentiate by using a list of standard derivatives
- Apply the chain rule
- Apply the product and quotient rules
- Perform logarithmic differentiation
- Differentiate implicit functions
- Differentiate parametric equations

Standard derivatives

1

Here is a revision list of the standard derivatives which you have no doubt used many times before. Copy out the list into your record book and memorize those with which you are less familiar – possibly **4**, **6**, **10**, **11** and **12**. Here they are:

	$y = f(x)$	$\frac{dy}{dx}$
1	x^n	nx^{n-1}
2	e^x	e^x
3	e^{kx}	ke^{kx}
4	a^x	$a^x . \ln a$
5	$\ln x$	$\frac{1}{x}$
6	$\log_a x$	$\frac{1}{x . \ln a}$
7	$\sin x$	$\cos x$
8	$\cos x$	$-\sin x$
9	$\tan x$	$\sec^2 x$
10	$\cot x$	$-\text{cosec}^2 x$
11	$\sec x$	$\sec x . \tan x$
12	$\text{cosec}\, x$	$-\text{cosec}\, x . \cot x$
13	$\sinh x$	$\cosh x$
14	$\cosh x$	$\sinh x$

The last two are proved in Frame 2, so move on

2

The derivatives of $\sinh x$ and $\cosh x$ are easily obtained by remembering the exponential definitions, and also that:

$$\frac{d}{dx}\{e^x\} = e^x \text{ and } \frac{d}{dx}\{e^{-x}\} = -e^{-x}$$

(a) $y = \sinh x \qquad y = \dfrac{e^x - e^{-x}}{2}$

$$\therefore \frac{dy}{dx} = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\therefore \frac{d}{dx}(\sinh x) = \cosh x$$

(b) $y = \cosh x \qquad y = \dfrac{e^x + e^{-x}}{2}$

$$\therefore \frac{dy}{dx} = \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\therefore \frac{d}{dx}(\cosh x) = \sinh x$$

Note that there is no minus sign involved as there is when differentiating the trig function $\cos x$.

We will find the derivative of $\tanh x$ later on.

Move on to Frame 3

3

Let us see if you really do know those basic derivatives. First of all cover up the list you have copied and then write down the derivatives of the following. All very easy.

1	x^5	**11**	$\cos x$
2	$\sin x$	**12**	$\sinh x$
3	e^{3x}	**13**	$\text{cosec } x$
4	$\ln x$	**14**	a^3
5	$\tan x$	**15**	$\cot x$
6	2^x	**16**	a^x
7	$\sec x$	**17**	x^{-4}
8	$\cosh x$	**18**	$\log_a x$
9	$\log_{10} x$	**19**	$\sqrt{x}$
10	e^x	**20**	$e^{x/2}$

When you have finished them all, move on to the next frame to check your results

4

Here are the results. Check yours carefully and make a special note of any where you may have slipped up.

1	$5x^4$	**11**	$-\sin x$
2	$\cos x$	**12**	$\cosh x$
3	$3e^{3x}$	**13**	$-\operatorname{cosec} x.\cot x$
4	$1/x$	**14**	0
5	$\sec^2 x$	**15**	$-\operatorname{cosec}^2 x$
6	$2^x \ln 2$	**16**	$a^x \ln a$
7	$\sec x.\tan x$	**17**	$-4x^{-5}$
8	$\sinh x$	**18**	$1/(x \ln a)$
9	$1/(x \ln 10)$	**19**	$\frac{1}{2}x^{-\frac{1}{2}} = 1/(2\sqrt{x})$
10	e^x	**20**	$\frac{1}{2}e^{x/2}$

If by chance you have not got them all correct, it is well worth while returning to Frame 1, or to the list you copied, and brushing up where necessary. These are the tools for all that follows.

When you are sure you know the basic results, move on

Functions of a function

5

$\sin x$ is a function of x since the value of $\sin x$ depends on the value of the angle x. Similarly, $\sin(2x+5)$ is a function of the angle $(2x+5)$ since the value of the sine depends on the value of this angle.

i.e. $\sin(2x+5)$ is a function of $(2x+5)$

But $(2x+5)$ is itself a function of x, since its value depends on x.

i.e. $(2x+5)$ is a function of x

$\sin(2x+5)$ is therefore a function of a function of x and such expressions are referred to generally as *functions of a function*.

So $e^{\sin y}$ is a function of a function of ....

6

$\boxed{y}$

Because $e^{\sin y}$ depends on the value of the index $\sin y$ and $\sin y$ depends on y. Therefore $e^{\sin y}$ is a function of a function of y.

We very often need to find the derivatives of such functions of a function. We could do them from first principles.

As a first example, differentiate with respect to x, $y = \cos(5x - 4)$.

Let $u = (5x - 4)$ $\therefore$ $y = \cos u$ $\therefore$ $\dfrac{dy}{du} = -\sin u = -\sin(5x - 4)$.

But this gives us $\dfrac{dy}{du}$, not $\dfrac{dy}{dx}$. To convert our result into the required derivative, we use $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$, i.e. we multiply $\dfrac{dy}{du}$ (which we have) by $\dfrac{du}{dx}$ to obtain $\dfrac{dy}{dx}$ (which we want); $\dfrac{du}{dx}$ is found from the substitution $u = (5x - 4)$, i.e. $\dfrac{du}{dx} = 5$.

$$\therefore \quad \frac{d}{dx}\{\cos(5x - 4)\} = -\sin(5x - 4) \times 5 = -5\sin(5x - 4)$$

So now find from first principles the derivative of $y = e^{\sin x}$. (As before, put $u = \sin x$.)

7

$$\boxed{\frac{d}{dx}\{e^{\sin x}\} = \cos x . e^{\sin x}}$$

Because $y = e^{\sin x}$. Put $u = \sin x$ $\therefore$ $y = e^u$ $\therefore$ $\dfrac{dy}{du} = e^u$

But $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ and $\dfrac{du}{dx} = \cos x$

$$\therefore \quad \frac{d}{dx}\{e^{\sin x}\} = e^{\sin x} . \cos x$$

This is quite general.

If $y = f(u)$ and $u = F(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$, i.e. if $y = \ln F$, where F is a function of x, then:

$$\frac{dy}{dx} = \frac{dy}{dF} \cdot \frac{dF}{dx} = \frac{1}{F} \cdot \frac{dF}{dx}$$

So, if $y = \ln \sin x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\sin x} . \cos x = \cot x$

It is of utmost importance not to forget this factor $\dfrac{dF}{dx}$, so beware!

8

Just two more examples:

(a) $y = \tan(5x - 4)$ Basic standard form is $y = \tan x$, $\dfrac{dy}{dx} = \sec^2 x$

In this case $(5x - 4)$ replaces the single x

$$\therefore \quad \frac{dy}{dx} = \sec^2(5x - 4) \times \text{the derivative of the function } (5x - 4)$$

$$= \sec^2(5x - 4) \times 5 = 5\sec^2(5x - 4)$$

▶

(b) $y = (4x-3)^5$ Basic standard form is $y = x^5$, $\dfrac{dy}{dx} = 5x^4$

Here $(4x-3)$ replaces the single x

$$\therefore \frac{dy}{dx} = 5(4x-3)^4 \times \text{the derivative of the function } (4x-3)$$
$$= 5(4x-3)^4 \times 4 = 20(4x-3)^4$$

So what about this one?

If $y = \cos(7x+2)$, then $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

9

$$\boxed{\frac{dy}{dx} = -7\sin(7x+2)}$$

Right, now you differentiate these:

1 $y = (4x-5)^6$
2 $y = e^{3-x}$
3 $y = \sin 2x$
4 $y = \cos(x^2)$
5 $y = \ln(3-4\cos x)$

The results are in Frame 10. Check to see that yours are correct

10

1 $y = (4x-5)^6$ $\quad \dfrac{dy}{dx} = 6(4x-5)^5.4 = 24(4x-5)^5$

2 $y = e^{3-x}$ $\quad \dfrac{dy}{dx} = e^{3-x}(-1) = -e^{3-x}$

3 $y = \sin 2x$ $\quad \dfrac{dy}{dx} = \cos 2x.2 = 2\cos 2x$

4 $y = \cos(x^2)$ $\quad \dfrac{dy}{dx} = -\sin(x^2).2x = -2x\sin(x^2)$

5 $y = \ln(3-4\cos x)$ $\quad \dfrac{dy}{dx} = \dfrac{1}{3-4\cos x}.(4\sin x) = \dfrac{4\sin x}{3-4\cos x}$

Now do these

6 $y = e^{\sin 2x}$
7 $y = \sin^2 x$
8 $y = \ln\cos 3x$
9 $y = \cos^3(3x)$
10 $y = \log_{10}(2x-1)$

Take your time to do them.

When you are satisfied with your results, check them against the results in Frame 11

11

6 $y = e^{\sin 2x}$ $\qquad \frac{dy}{dx} = e^{\sin 2x}.2\cos 2x = 2\cos 2x.e^{\sin 2x}$

7 $y = \sin^2 x$ $\qquad \frac{dy}{dx} = 2\sin x \cos x = \sin 2x$

8 $y = \ln\cos 3x$ $\qquad \frac{dy}{dx} = \frac{1}{\cos 3x}(-3\sin 3x) = -3\tan 3x$

9 $y = \cos^3(3x)$ $\qquad \frac{dy}{dx} = 3\cos^2(3x).(-3\sin 3x) = -9\sin 3x\cos^2 3x$

10 $y = \log_{10}(2x-1)$ $\qquad \frac{dy}{dx} = \frac{1}{(2x-1)\ln 10}.2 = \frac{2}{(2x-1)\ln 10}$

All correct? Now on with the Programme. Next frame

12

Of course, we may need to differentiate functions which are products or quotients of two of the functions.

1 Products

If $y = uv$, where u and v are functions, of x, then you already know that:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

e.g. If $y = x^3.\sin 3x$

$$\text{then} \quad \frac{dy}{dx} = x^3.3\cos 3x + 3x^2\sin 3x$$
$$= 3x^2(x\cos 3x + \sin 3x)$$

Every one is done the same way. To differentiate a product:

(a) put down the first, differentiate the second; plus

(b) put down the second, differentiate the first.

So what is the derivative of $e^{2x}\ln 5x$?

13

$$\boxed{\frac{dy}{dx} = e^{2x}\left(\frac{1}{x} + 2\ln 5x\right)}$$

Because $y = e^{2x}\ln 5x$, i.e. $u = e^{2x}$, $v = \ln 5x$

$$\frac{dy}{dx} = e^{2x}\frac{1}{5x}.5 + 2e^{2x}\ln 5x$$
$$= e^{2x}\left(\frac{1}{x} + 2\ln 5x\right)$$

Now here is a short test for you to do. Find $\frac{dy}{dx}$ when:

1 $y = x^2\tan x$ $\qquad$ 2 $y = e^{5x}(3x+1)$ $\qquad$ 3 $y = x\cos 2x$

4 $y = x^3\sin 5x$ $\qquad$ 5 $y = x^2\ln\sinh x$

When you have completed all five, move on to Frame 14

14

1 $y = x^2 \tan x \qquad \therefore \dfrac{dy}{dx} = x^2 \sec^2 x + 2x \tan x$

$= x(x \sec^2 x + 2 \tan x)$

2 $y = e^{5x}(3x+1) \qquad \therefore \dfrac{dy}{dx} = e^{5x}.3 + 5e^{5x}(3x+1)$

$= e^{5x}(3 + 15x + 5) = e^{5x}(8 + 15x)$

3 $y = x \cos 2x \qquad \therefore \dfrac{dy}{dx} = x(-2\sin 2x) + 1.\cos 2x$

$= \cos 2x - 2x \sin 2x$

4 $y = x^3 \sin 5x \qquad \therefore \dfrac{dy}{dx} = x^3.5\cos 5x + 3x^2 \sin 5x$

$= x^2(5x \cos 5x + 3 \sin 5x)$

5 $y = x^2 \ln \sinh x \qquad \therefore \dfrac{dy}{dx} = x^2 \dfrac{1}{\sinh x} \cosh x + 2x \ln \sinh x$

$= x(x \coth x + 2 \ln \sinh x)$

So much for the product. What about the quotient?

Next frame

15

2 Quotients

In the case of the quotient, if u and v are functions of x, and $y = \dfrac{u}{v}$

then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Here are two examples:

If $y = \dfrac{\sin 3x}{x+1}$, $\dfrac{dy}{dx} = \dfrac{(x+1)3\cos 3x - \sin 3x.1}{(x+1)^2}$

If $y = \dfrac{\ln x}{e^{2x}}$, $\dfrac{dy}{dx} = \dfrac{e^{2x}\dfrac{1}{x} - \ln x.2e^{2x}}{e^{4x}}$

$= \dfrac{e^{2x}\left(\dfrac{1}{x} - 2\ln x\right)}{e^{4x}}$

$= \dfrac{\dfrac{1}{x} - 2\ln x}{e^{2x}}$

If you can differentiate the separate functions, the rest is easy.

You do this one. If $y = \dfrac{\cos 2x}{x^2}$, $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

16

$$\boxed{\frac{d}{dx}\left\{\frac{\cos 2x}{x^2}\right\} = \frac{-2(x\sin 2x + \cos 2x)}{x^3}}$$

Because
$$\frac{d}{dx}\left\{\frac{\cos 2x}{x^2}\right\} = \frac{x^2(-2\sin 2x) - \cos 2x.2x}{x^4}$$
$$= \frac{-2x(x\sin 2x + \cos 2x)}{x^4}$$
$$= \frac{-2(x\sin 2x + \cos 2x)}{x^3}$$

So: for $y = uv$,
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{(a)}$$

for $y = \dfrac{u}{v}$
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \text{(b)}$$

Be sure that you remember these.

You can prove the derivative of $\tan x$ by the quotient method, for if $y = \tan x$, $y = \dfrac{\sin x}{\cos x}$.

Then by the quotient rule, $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$ (work it through in detail)

17

$$\boxed{\frac{dy}{dx} = \sec^2 x}$$

Because $y = \dfrac{\sin x}{\cos x}$ $\quad\therefore\quad \dfrac{dy}{dx} = \dfrac{\cos x.\cos x + \sin x.\sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

In the same way we can obtain the derivative of $\tanh x$

$$y = \tanh x = \frac{\sinh x}{\cosh x} \quad\therefore\quad \frac{dy}{dx} = \frac{\cosh x.\cosh x - \sinh x.\sinh x}{\cosh^2 x}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x \quad\therefore\quad \frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

Add this last result to your list of derivatives in your record book.

So what is the derivative of $\tanh(5x + 2)$?

18

$$\boxed{\frac{d}{dx}\left\{\tanh(5x+2)\right\} = 5\text{sech}^2(5x+2)}$$

▶

Because we have: If $\dfrac{d}{dx}\left\{\tanh x\right\} = \operatorname{sech}^2 x$

then $\dfrac{d}{dx}\left\{\tanh(5x+2)\right\} = \operatorname{sech}^2(5x+2) \times \text{derivative of } (5x+2)$

$= \operatorname{sech}^2(5x+2) \times 5 = 5\operatorname{sech}^2(5x+2)$

Fine. Now move on to Frame 19 for the next part of the Programme

Logarithmic differentiation

19

The rules for differentiating a product or a quotient that we have revised are used when there are just two-factor functions, i.e. uv or $\dfrac{u}{v}$. Where there are more than two functions in any arrangement top or bottom, the derivative is best found by what is known as 'logarithmic differentiation'.

It all depends on the basic fact that $\dfrac{d}{dx}\left\{\ln x\right\} = \dfrac{1}{x}$ and that if x is replaced by a function F then $\dfrac{d}{dx}\left\{\ln F\right\} = \dfrac{1}{F}\cdot\dfrac{dF}{dx}$. Bearing that in mind, let us consider the case where $y = \dfrac{uv}{w}$, where u, v and w – and also y – are functions of x.

First take logs to the base e.

$\ln y = \ln u + \ln v - \ln w$

Now differentiate each side with respect to x, remembering that u, v, w and y are all functions of x. What do we get?

20

$$\boxed{\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} - \frac{1}{w}\cdot\frac{dw}{dx}}$$

So to get $\dfrac{dy}{dx}$ by itself, we merely have to multiply across by y. Note that when we do this, we put the grand function that y represents:

$$\frac{dy}{dx} = \frac{uv}{w}\left\{\frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} - \frac{1}{w}\cdot\frac{dw}{dx}\right\}$$

This is not a formula to memorize, but a *method* of working, since the actual terms on the right-hand side will depend on the functions you start with.

Let us do an example to make it quite clear.

If $y = \dfrac{x^2 \sin x}{\cos 2x}$, find $\dfrac{dy}{dx}$

The first step in the process is . . . .

21

To take logs of both sides

$$y = \frac{x^2 \sin x}{\cos 2x} \quad \therefore \quad \ln y = \ln(x^2) + \ln(\sin x) - \ln(\cos 2x)$$

Now differentiate both sides with respect to x, remembering that $\frac{d}{dx}(\ln F) = \frac{1}{F}\cdot\frac{dF}{dx}$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^2}.2x + \frac{1}{\sin x}.\cos x - \frac{1}{\cos 2x}.(-2\sin 2x)$$

$$= \frac{2}{x} + \cot x + 2\tan 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 \sin x}{\cos 2x}\left\{\frac{2}{x} + \cot x + 2\tan 2x\right\}$$

This is a pretty complicated result, but the original function was also somewhat involved!

Do this one on your own:

If $y = x^4 e^{3x} \tan x$, then $\frac{dy}{dx} = \ldots\ldots\ldots\ldots$

22

$$\frac{dy}{dx} = x^4 e^{3x} \tan x\left\{\frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}\right\}$$

Here is the working. Follow it through.

$$y = x^4 e^{3x} \tan x \quad \therefore \quad \ln y = \ln(x^4) + \ln(e^{3x}) + \ln(\tan x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^4}.4x^3 + \frac{1}{e^{3x}}.3e^{3x} + \frac{1}{\tan x}.\sec^2 x$$

$$= \frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}$$

$$\therefore \quad \frac{dy}{dx} = x^4 e^{3x} \tan x\left\{\frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}\right\}$$

There it is.

Always use the logarithmic differentiation method where there are more than two functions involved in a product or quotient (or both).

Here is just one more for you to do. Find $\frac{dy}{dx}$, given that

$$y = \frac{e^{4x}}{x^3 \cosh 2x}$$

23

$$\boxed{\frac{dy}{dx} = \frac{e^{4x}}{x^3 \cosh 2x}\left\{4 - \frac{3}{x} - 2\tanh 2x\right\}}$$

Working. Check yours.

$$y = \frac{e^{4x}}{x^3 \cosh 2x} \quad \therefore \quad \ln y = \ln(e^{4x}) - \ln(x^3) - \ln(\cosh 2x)$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{e^{4x}}.4e^{4x} - \frac{1}{x^3}.3x^2 - \frac{1}{\cosh 2x}.2\sinh 2x$$

$$= 4 - \frac{3}{x} - 2\tanh 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{4x}}{x^3 \cosh 2x}\left\{4 - \frac{3}{x} - 2\tanh 2x\right\}$$

Well now, before continuing with the rest of the Programme, here is a revision exercise on the work so far for you to deal with.

Move on for details

24

Revision exercise

Differentiate with respect to x:

1 (a) $\ln 4x$ (b) $\ln(\sin 3x)$

2 $e^{3x} \sin 4x$

3 $\dfrac{\sin 2x}{2x + 5}$

4 $\dfrac{(3x + 1)\cos 2x}{e^{2x}}$

5 $x^5 \sin 2x \cos 4x$

When you have finished them all (and not before) move on to Frame 25 to check your results

25

1 (a) $y = \ln 4x \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{4x}.4 = \dfrac{1}{x}$

(b) $y = \ln \sin 3x \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\sin 3x}.3\cos 3x$

$$= 3\cot 3x$$

2 $y = e^{3x} \sin 4x \quad \therefore \quad \dfrac{dy}{dx} = e^{3x} 4\cos 4x + 3e^{3x} \sin 4x$

$$= e^{3x}(4\cos 4x + 3\sin 4x)$$

3 $y = \dfrac{\sin 2x}{2x + 5} \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{(2x + 5)2\cos 2x - 2\sin 2x}{(2x + 5)^2}$

▶

4 $y = \dfrac{(3x+1)\cos 2x}{e^{2x}}$

$\therefore \quad \ln y = \ln(3x+1) + \ln(\cos 2x) - \ln(e^{2x})$

$\therefore \quad \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{3x+1}.3 + \dfrac{1}{\cos 2x}.(-2\sin 2x) - \dfrac{1}{e^{2x}}.2e^{2x}$

$$= \frac{3}{3x+1} - 2\tan 2x - 2$$

$$\frac{dy}{dx} = \frac{(3x+1)\cos 2x}{e^{2x}}\left\{\frac{3}{3x+1} - 2\tan 2x - 2\right\}$$

5 $y = x^5 \sin 2x \cos 4x$

$\therefore \quad \ln y = \ln(x^5) + \ln(\sin 2x) + \ln(\cos 4x)$

$\therefore \quad \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x^5}.5x^4 + \dfrac{2\cos 2x}{\sin 2x} + \dfrac{1}{\cos 4x}(-4\sin 4x)$

$$= \frac{5}{x} + 2\cot 2x - 4\tan 4x$$

$$\frac{dy}{dx} = x^5 \sin 2x \cos 4x\left\{\frac{5}{x} + 2\cot 2x - 4\tan 4x\right\}$$

So far so good. Now on to the next part of the Programme in Frame 26

Implicit functions

26

If $y = x^2 - 4x + 2$, y is completely defined in terms of x, and y is called an *explicit function* of x.

When the relationship between x and y is more involved, it may not be possible (or desirable) to separate y completely on the left-hand side, e.g. $xy + \sin y = 2$. In such a case as this, y is called an *implicit function* of x, because a relationship of the form $y = f(x)$ is implied in the given equation.

It may still be necessary to determine the derivatives of y with respect to x and in fact this is not at all difficult. All we have to remember is that y is a function of x, even if it is difficult to see what it is. In fact, this is really an extension of our 'function of a function' routine.

$x^2 + y^2 = 25$, as it stands is an example of an .... function.

27

$\boxed{\text{implicit}}$

Once again, all we have to remember is that y is a function of x. So, if $x^2+y^2=25$, let us find $\dfrac{dy}{dx}$

If we differentiate as it stands with respect to x, we get

$$2x+2y\frac{dy}{dx}=0$$

Note that we differentiate y^2 as a function squared, giving 'twice times the function, times the derivative of the function'. The rest is easy.

$$2x+2y\frac{dy}{dx}=0$$

$$\therefore\ y\frac{dy}{dx}=-x \qquad \therefore\ \frac{dy}{dx}=-\frac{x}{y}$$

As you will have noticed, with an implicit function the derivative may contain (and usually does) both x and ....

28

$\boxed{y}$

Let us look at some examples.

If $x^2+y^2-2x-6y+5=0$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $x=3$, $y=2$.

Differentiate as it stands with respect to x.

$$2x+2y\frac{dy}{dx}-2-6\frac{dy}{dx}=0$$

$$\therefore\ (2y-6)\frac{dy}{dx}=2-2x \qquad \therefore\ \frac{dy}{dx}=\frac{2-2x}{2y-6}=\frac{1-x}{y-3}$$

$$\therefore\ \text{at }(3,2)\qquad \frac{dy}{dx}=\frac{1-3}{2-3}=\frac{-2}{-1}=2$$

$$\text{Then}\qquad \frac{d^2y}{dx^2}=\frac{d}{dx}\left\{\frac{1-x}{y-3}\right\}=\frac{(y-3)(-1)-(1-x)\dfrac{dy}{dx}}{(y-3)^2}$$

$$=\frac{(3-y)-(1-x)\dfrac{dy}{dx}}{(y-3)^2}$$

$$\text{at }(3,2)\qquad \frac{d^2y}{dx^2}=\frac{(3-2)-(1-3)2}{(2-3)^2}=\frac{1-(-4)}{1}=5$$

$$\therefore\ \text{At }(3,2)\quad \frac{dy}{dx}=2,\ \frac{d^2y}{dx^2}=5$$

Now this one. If $x^2+2xy+3y^2=4$, find $\dfrac{dy}{dx}$

Away you go, but beware of the product term. When you come to $2xy$ treat this as $(2x)(y)$.

29

Here is the working:

$$x^2 + 2xy + 3y^2 = 4$$

$$2x + 2x\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = 0$$

$$\therefore \ (2x + 6y)\frac{dy}{dx} = -(2x + 2y)$$

$$\therefore \ \frac{dy}{dx} = -\frac{(2x + 2y)}{(2x + 6y)} = -\frac{(x + y)}{(x + 3y)}$$

And now, just one more:

If $x^3 + y^3 + 3xy^2 = 8$, find $\dfrac{dy}{dx}$

Turn to Frame 30 for the solution

30

Solution in detail:

$$x^3 + y^3 + 3xy^2 = 8$$

$$3x^2 + 3y^2\frac{dy}{dx} + 3x.2y\frac{dy}{dx} + 3y^2 = 0$$

$$\therefore \ (y^2 + 2xy)\frac{dy}{dx} = -(x^2 + y^2)$$

$$\therefore \ \frac{dy}{dx} = -\frac{(x^2 + y^2)}{(y^2 + 2xy)}$$

That is really all there is to it. All examples are tackled the same way. The key to it is simply that 'y is a function of x' and then apply the 'function of a function' routine.

Now on to the last section of this particular Programme, which starts in Frame 31

Parametric equations

31

In some cases, it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable, e.g. $y = \cos 2t$, $x = \sin t$. In this case, any value we give to t will produce a pair of values for x and y, which could if necessary be plotted and provide one point of the curve of $y = f(x)$.

The third variable, e.g. t, is called a *parameter*, and the two expressions for x and y *parametric equations*. We may still need to find the derivatives of the function with respect to x, so how do we go about it?

Let us take the case already quoted above. The parametric equations of a function are given as $y = \cos 2t$, $x = \sin t$. We are required to find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Move to the next frame to see how we go about it

32

$y = \cos 2t$, $x = \sin t$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

From $y = \cos 2t$, we can get $\dfrac{dy}{dt} = -2\sin 2t$

From $x = \sin t$, we can get $\dfrac{dx}{dt} = \cos t$

We can now use the fact that $\dfrac{dy}{dx} = \dfrac{dy}{dt}.\dfrac{dt}{dx}$

so that
$$\frac{dy}{dx} = -2\sin 2t.\frac{1}{\cos t} \qquad \left(\text{noting that } \frac{dt}{dx} = \frac{1}{dx/dt}\right)$$
$$= -4\sin t\cos t.\frac{1}{\cos t}$$
$$\therefore \quad \frac{dy}{dx} = -4\sin t$$

That was easy enough. Now how do we find the second derivative? We *cannot* get it by finding $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^2x}{dt^2}$ from the parametric equations and joining them together as we did for the first derivative. So what do we do?

On to the next frame and all will be revealed!

33

To find the second derivative, we must go back to the very meaning of $\dfrac{d^2y}{dx^2}$

i.e. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(-4\sin t\right)$

But we cannot differentiate a function of t directly with respect to x.

Therefore we say $\dfrac{d}{dx}\left(-4\sin t\right) = \dfrac{d}{dt}\left(-4\sin t\right).\dfrac{dt}{dx}$.

$$\therefore \quad \frac{d^2y}{dx^2} = -4\cos t.\frac{1}{\cos t} = -4$$
$$\therefore \quad \frac{d^2y}{dx^2} = -4$$

Let us work through another one. What about this? The parametric equations of a function are given as:

$y = 3\sin\theta - \sin^3\theta$, $x = \cos^3\theta$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

Move on to Frame 34

34

$$y = 3\sin\theta - \sin^3\theta \quad \therefore \quad \frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta(1-\sin^2\theta)$$

$$x = \cos^3\theta \quad \therefore \quad \frac{dx}{d\theta} = 3\cos^2\theta(-\sin\theta) = -3\cos^2\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= 3\cos\theta(1-\sin^2\theta)\cdot\frac{1}{-3\cos^2\theta\sin\theta} \qquad \left[\text{Remember: } \frac{d\theta}{dx} = \frac{1}{dx/d\theta}\right]$$

$$= \frac{3\cos^3\theta}{-3\cos^2\theta\sin\theta} \qquad \therefore \quad \frac{dy}{dx} = -\cot\theta$$

Also $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-\cot\theta) = \dfrac{d}{d\theta}(-\cot\theta)\dfrac{d\theta}{dx}$

$$= -(-\mathrm{cosec}^2\theta)\frac{1}{-3\cos^2\theta\sin\theta}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-1}{3\cos^2\theta\sin^3\theta}$$

Now here is one for you to do in just the same way.

If $x = \dfrac{2-3t}{1+t}$, $y = \dfrac{3+2t}{1+t}$, find $\dfrac{dy}{dx}$

When you have done it, move on to Frame 35

35

$$\boxed{\frac{dy}{dx} = \frac{1}{5}}$$

Because

$$x = \frac{2-3t}{1+t} \qquad \therefore \quad \frac{dx}{dt} = \frac{(1+t)(-3)-(2-3t)}{(1+t)^2}$$

$$y = \frac{3+2t}{1+t} \qquad \therefore \quad \frac{dy}{dt} = \frac{(1+t)(2)-(3+2t)}{(1+t)^2}$$

$$\frac{dx}{dt} = \frac{-3-3t-2+3t}{(1+t)^2} = \frac{-5}{(1+t)^2}$$

$$\frac{dy}{dt} = \frac{2+2t-3-2t}{(1+t)^2} = \frac{-1}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{-1}{(1+t)^2}\cdot\frac{(1+t)^2}{-5} = \frac{1}{5} \qquad \therefore \quad \frac{dy}{dx} = \frac{1}{5}$$

And now here is one more for you to do to finish up this part of the work. It is done in just the same way as the others.

If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

36

Here it is, set out like the previous examples.

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\therefore \quad \frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta) = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta) = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = a\theta\sin\theta.\frac{1}{a\theta\cos\theta} = \tan\theta$$

$$\frac{dy}{dx} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta) = \frac{d}{d\theta}(\tan\theta).\frac{d\theta}{dx}$$

$$= \sec^2\theta.\frac{1}{a\theta\cos\theta}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{a\theta\cos^3\theta}$$

You have now reached the end of this Programme on differentiation, much of which has been useful revision of what you have done before. This brings you now to the **Can You?** checklist and **Test exercise**, so move on to them and work through them carefully.

Next frame

Can You?

37

Checklist 7

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Differentiate by using a list of standard derivatives? 1 to 4
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Apply the chain rule? 5 to 11
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Apply the product and quotient rules? 12 to 18
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Perform logarithmic differentiation? 19 to 23
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Frames

- Differentiate implicit functions? — 26 to 30

Yes ☐ ☐ ☐ ☐ ☐ *No*

- Differentiate parametric equations? — 31 to 36

Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 7

38

Write out the solutions carefully. They are all quite straightforward.

1 Differentiate the following with respect to x:

(a) $\tan 2x$ (b) $(5x+3)^6$ (c) $\cosh^2 x$

(d) $\log_{10}(x^2 \quad 3x \quad 1)$ (e) $\ln\cos 3x$ (f) $\sin^3 4x$

(g) $e^{2x}\sin 3x$ (h) $\dfrac{x^4}{(x+1)^2}$ (i) $\dfrac{e^{4x}\sin x}{x\cos 2x}$

2 If $x^2+y^2-2x+2y=23$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point where $x=-2, y=3$.

3 Find an expression for $\dfrac{dy}{dx}$ when $x^3+y^3+4xy^2=5$.

4 If $x=3(1-\cos\theta)$ and $y=3(\theta-\sin\theta)$ find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in their simplest forms.

Further problems 7

39

1 Differentiate with respect to x:

(a) $\ln\left\{\dfrac{\cos x+\sin x}{\cos x-\sin x}\right\}$ (b) $\ln(\sec x+\tan x)$

(c) $\sin^4 x\cos^3 x$

2 Find $\dfrac{dy}{dx}$ when:

(a) $y=\dfrac{x\sin x}{1+\cos x}$ (b) $y=\ln\left\{\dfrac{1-x^2}{1+x^2}\right\}$

3 If y is a function of x, and $x=\dfrac{e^t}{e^t+1}$ show that $\dfrac{dy}{dt}=x(1-x)\dfrac{dy}{dx}$.

4 Find $\dfrac{dy}{dx}$ when $x^3+y^3-3xy^2=8$.

▶

5 Differentiate:

(a) $y = e^{\sin^2 5x}$ (b) $y = \ln\left\{\dfrac{\cosh x - 1}{\cosh x + 1}\right\}$ (c) $y = \ln\left\{e^x\left(\dfrac{x-2}{x+2}\right)^{3/4}\right\}$

6 Differentiate:

(a) $y = x^2\cos^2 x$ (b) $y = \ln\left\{x^2\sqrt{1-x^2}\right\}$ (c) $y = \dfrac{e^{2x}\ln x}{(x-1)^3}$

7 If $(x-y)^3 = A(x+y)$, prove that $(2x+y)\dfrac{dy}{dx} = x + 2y$.

8 If $x^2 - xy + y^2 = 7$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $x = 3$, $y = 2$.

9 If $x^2 + 2xy + 3y^2 = 1$, prove that $(x+3y)^3\dfrac{d^2y}{dx^2} + 2(x^2 + 2xy + 3y^2) = 0$.

10 If $x = \ln\tan\dfrac{\theta}{2}$ and $y = \tan\theta - \theta$, prove that $\dfrac{d^2y}{dx^2} = \tan^2\theta\sin\theta(\cos\theta + 2\sec\theta)$.

11 If $y = 3e^{2x}\cos(2x-3)$, verify that $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 8y = 0$.

12 The parametric equations of a curve are $x = \cos 2\theta$, $y = 1 + \sin 2\theta$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $\theta = \pi/6$. Find also the equation of the curve as a relationship between x and y.

13 If $y = \left\{x + \sqrt{1+x^2}\right\}^{3/2}$, show that $4(1+x^2)\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - 9y = 0$.

14 Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $x = a\cos^3\theta$, $y = a\sin^3\theta$.

15 If $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$, express $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of θ.

16 Show that $y = e^{-2mx}\sin 4mx$ is a solution of the equation

$$\frac{d^2y}{dx^2} + 4m\frac{dy}{dx} + 20m^2y = 0.$$

17 If $y = \sec x$, prove that $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + y^4$.

18 Prove that $x = Ae^{-kt}\sin pt$, satisfies the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + (p^2 + k^2)x = 0.$$

19 If $y = e^{-kt}(A\cosh qt + B\sinh qt)$ where A, B, q and k are constants, show that $\dfrac{d^2y}{dt^2} + 2k\dfrac{dy}{dt} + (k^2 - q^2)y = 0$.

20 If $\sinh y = \dfrac{4\sinh x - 3}{4 + 3\sinh x}$, show that $\dfrac{dy}{dx} = \dfrac{-5}{4 + 3\sinh x}$.

Differentiation applications 1

Frames 1 to 47

Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate the gradient of a straight line
- Recognize the relationship satisfied by two mutually perpendicular straight lines
- Derive the equations of a tangent and a normal to a curve
- Evaluate the curvature and radius of curvature at a point on a curve
- Locate the centre of curvature for a point on a curve

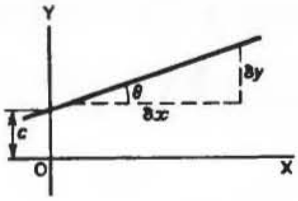
Equation of a straight line

1

The basic equation of a straight line is $y = mx + c$,

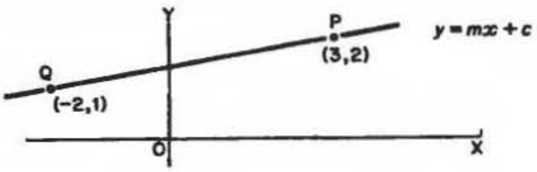
where $m = \text{gradient} = \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$

$c = $ intercept on real y-axis



Note that if the scales of x and y are identical, $\dfrac{dy}{dx} = \tan\theta$

e.g. To find the equation of the straight line passing through P (3, 2) and Q (−2, 1), we could argue thus:



Line passes through P, i.e. when $x = 3$, $y = 2$ $\therefore$ $2 = m.3 + c$.

Line passes through Q, i.e. when $x = -2$, $y = 1$ $\therefore$ $1 = m(-2) + c$.

So we obtain a pair of simultaneous equations from which the values of m and c can be found. Therefore the equation is ....

2

We find $m = 1/5$ and $c = 7/5$. Therefore the equation of the line is

$$y = \frac{x}{5} + \frac{7}{5}, \quad \text{i.e. } 5y = x + 7$$

Sometimes we are given the gradient, m, of a straight line passing through a given point (x_1, y_1) and we are required to find its equation. In that case, it is more convenient to use the form:

$$y - y_1 = m(x - x_1)$$

For example, the equation of the line passing through the point (5, 3) with gradient 2 is simply .... which simplifies to ....

Move on to the next frame

3

$$\boxed{\begin{array}{c} y - 3 = 2(x - 5) \\ \text{i.e. } y - 3 = 2x - 10 \qquad \therefore\ y = 2x - 7 \end{array}}$$

Similarly, the equation of the line through the point $(-2, -1)$ and having a gradient $\frac{1}{2}$ is

$$y - (-1) = \frac{1}{2}\left\{x - (-2)\right\} \quad \therefore\ y + 1 = \frac{1}{2}(x + 2)$$

$$2y + 2 = x + 2 \qquad \therefore\ y = \frac{x}{2}$$

So, in the same way, the line passing through $(2, -3)$ and having gradient (-2) is ....

4

$$\boxed{y = 1 - 2x}$$

Because

$$y - (-3) = -2(x - 2)$$

$$\therefore\ y + 3 = -2x + 4 \qquad \therefore\ y = 1 - 2x$$

Right. So in general terms, the equation of the line passing through the point (x_1, y_1) with gradient m is ....

Move on to Frame 5

5

$\boxed{y - y_1 = m(x - x_1)}$ It is well worth remembering.

So for one last time:

If a point P has coordinates (4, 3) and the gradient m of a straight line through P is 2, then the equation of the line is thus:

$$\begin{aligned} y - 3 &= 2(x - 4) \\ &= 2x - 8 \\ \therefore\ y &= 2x - 5 \end{aligned}$$

The equation of the line through P, perpendicular to the line we have just considered, will have a gradient m_1, such that $mm_1 = -1$

i.e. $m_1 = -\frac{1}{m}$. And since $m = 2$, then $m_1 = -\frac{1}{2}$. This line passes through (4, 3) and its equation is therefore:

$$\begin{aligned} y - 3 &= -\frac{1}{2}(x - 4) \\ &= -x/2 + 2 \\ y &= -\frac{x}{2} + 5, \quad 2y = 10 - x \end{aligned}$$

6

If m and m_1 represent the gradients of two lines perpendicular to each other, then $mm_1 = -1$ or $m_1 = -\dfrac{1}{m}$

Consider the two straight lines:

$2y = 4x - 5$ and $6y = 2 - 3x$

If we convert each of these to the form $y = mx + c$, we get:

(a) $y = 2x - \dfrac{5}{2}$ and (b) $y = -\dfrac{1}{2}x + \dfrac{1}{3}$

So in (a) the gradient $m = 2$ and in (b) the gradient $m_1 = -\dfrac{1}{2}$

We notice that, in this case, $m_1 = -\dfrac{1}{m}$ or that $mm_1 = -1$

Therefore we know that the two given lines are at right-angles to each other.

Which of these represents a pair of lines perpendicular to each other:

(a) $y = 3x - 5$ and $3y = x + 2$
(b) $2y = x - 5$ and $y = 6 - x$
(c) $y - 3x - 2 = 0$ and $3y + x + 9 = 0$
(d) $5y - x = 4$ and $2y + 10x + 3 = 0$

7

(c) and (d)

Because

If we convert each to the form $y = mx + c$, we get:

(a) $y = 3x - 5$ and $y = \dfrac{x}{3} + \dfrac{2}{3}$

$m = 3;\ m_1 = \dfrac{1}{3}\qquad \therefore\ mm_1 \neq -1$ Not perpendicular.

(b) $y = \dfrac{x}{2} - \dfrac{5}{2}$ and $y = -x + 6$

$m = \dfrac{1}{2};\ m_1 = -1\qquad \therefore\ mm_1 \neq -1$ Not perpendicular.

(c) $y = 3x + 2$ and $y = -\dfrac{x}{3} - 3$

$m = 3;\ m_1 = -\dfrac{1}{3}\qquad \therefore\ mm_1 = -1$ Perpendicular.

(d) $y = \dfrac{x}{5} + \dfrac{4}{5}$ and $y = -5x - \dfrac{3}{2}$

$m = \dfrac{1}{5};\ m_1 = -5\qquad \therefore\ mm_1 = -1$ Perpendicular.

Do you agree with these?

8

Remember that if $y = mx + c$ and $y = m_1x + c_1$ are perpendicular to each other, then:

$$mm_1 = -1, \text{ i.e. } m_1 = -\frac{1}{m}$$

Here is one further example:

A line AB passes through the point P (3, −2) with gradient $-\frac{1}{2}$. Find its equation and also the equation of the line CD through P perpendicular to AB.

When you have finished, check your results with those in Frame 9

9

Equation of AB: $\quad y - (-2) = -\frac{1}{2}(x - 3)$

$$\therefore\ y + 2 = -\frac{x}{2} + \frac{3}{2}$$

$$\therefore\ y = -\frac{x}{2} - \frac{1}{2}$$

$$\therefore\ 2y + x + 1 = 0 \text{ or } y = -\frac{x}{2} - \frac{1}{2}$$

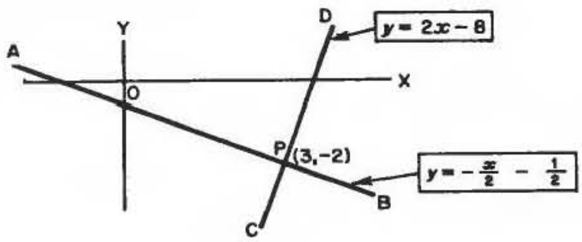
Equation of CD: $\quad \text{gradient } m_1 = -\frac{1}{m} = -\frac{1}{-\frac{1}{2}} = 2$

$$y - (-2) = 2(x - 3)$$

$$y + 2 = 2x - 6$$

$$y = 2x - 8$$

So we have:



$$mm_1 = \ldots\ldots\ldots\ldots$$

10

$$\boxed{mm_1 = -1}$$

And now, just one more to do on your own.

The point P (3, 4) is a point on the line $y = 5x - 11$.

Find the equation of the line through P which is perpendicular to the given line.

That should not take long. When you have finished it, move on to the next frame

11

$$\boxed{5y + x = 23}$$

Because

gradient of the given line, $y = 5x - 11$ is 5

gradient of required line $= -\dfrac{1}{5}$

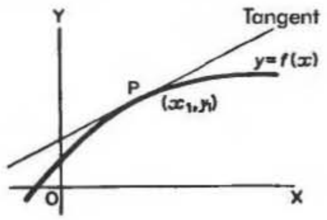
The line passes through P, i.e. when $x = 3$, $y = 4$.

$$y - 4 = -\frac{1}{5}(x - 3)$$

$$5y - 20 = -x + 3 \qquad \therefore\; 5y + x = 23$$

Tangents and normals to a curve at a given point

12



The gradient of a curve, $y = f(x)$, at a point P on the curve is given by the gradient of the tangent at P. It is also given by the value of $\dfrac{dy}{dx}$ at the point P, which we can calculate, knowing the equation of the curve. Thus we can calculate the gradient of the tangent to the curve at any point P.

What else do we know about the tangent which will help us to determine its equation?

13

We know that the tangent passes through P, i.e. when $x = x_1$, $y = y_1$

Correct. This is sufficient information for us to find the equation of the tangent. Let us do an example.

Find the equation of the tangent to the curve $y = 2x^3 + 3x^2 - 2x - 3$ at the point P, $x = 1$, $y = 0$.

$$\frac{dy}{dx} = 6x^2 + 6x - 2$$

Gradient of tangent $= \left\{\dfrac{dy}{dx}\right\}_{x=1} = 6 + 6 - 2 = 10$, i.e. $m = 10$

Passes through P, i.e. $x = 1$, $y = 0$.

$y - y_1 = m(x - x_1)$ gives $y - 0 = 10(x - 1)$

Therefore the tangent is $y = 10x - 10$

▶

We could also, if required, find the equation of the normal at P which is defined as the line through P perpendicular to the tangent at P. We know, for example, that the gradient of the normal is ....

14

$$\boxed{\text{Gradient of normal} = \frac{-1}{\text{Gradient of tangent}} = -\frac{1}{10}}$$

The normal also passes through P, i.e. when $x = 1$, $y = 0$.

$\therefore$ Equation of normal is $y - 0 = -\dfrac{1}{10}(x - 1)$

$10y = -x + 1 \qquad \therefore \; 10y + x = 1$

That was very easy. Do this one just to get your hand in:

Find the equations of the tangent and normal to the curve

$$y = x^3 - 2x^2 + 3x - 1$$

at the point (2, 5).

Off you go. Do it in just the same way.

When you have got the results, move on to Frame 15

15

$$\boxed{\text{Tangent: } y = 7x - 9 \qquad \text{Normal: } 7y + x = 37}$$

Here are the details:

$y = x^3 - 2x^2 + 3x - 1$

$\therefore \; \dfrac{dy}{dx} = 3x^2 - 4x + 3 \qquad \therefore$ At P (2, 5), $\dfrac{dy}{dx} = 12 - 8 + 3 = 7$

Tangent passes through (2, 5), i.e. $x = 2$, $y = 5$

$y - 5 = 7(x - 2) \qquad$ Tangent is $y = 7x - 9$

For normal, gradient $= \dfrac{-1}{\text{gradient of tangent}} = -\dfrac{1}{7}$

Normal passes through P (2, 5)

$\therefore \; y - 5 = -\dfrac{1}{7}(x - 2)$

$7y - 35 = -x + 2$

Normal is $7y + x = 37$

You will perhaps remember doing all this long ago.

Anyway, on to Frame 16

16

The equation of the curve may, of course, be presented as an implicit function or as a pair of parametric equations. But this will not worry you for you already know how to differentiate functions in these two forms. Let us have an example or two.

Find the equations of the tangent and normal to the curve

$x^2 + y^2 + 3xy - 11 = 0$ at the point $x = 1$, $y = 2$.

First of all we must find $\frac{dy}{dx}$ at (1, 2). So differentiate right away.

$$2x + 2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$(2y + 3x)\frac{dy}{dx} = -(2x + 3y)$$

$$\frac{dy}{dx} = -\frac{2x + 3y}{2y + 3x}$$

Therefore, at $x = 1$, $y = 2$,

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots$$

17

$$\boxed{\frac{dy}{dx} = -\frac{8}{7}}$$

Because

$$\frac{dy}{dx} = -\frac{2+6}{4+3} = -\frac{8}{7}$$

Now we proceed as for the previous cases.

Tangent passes through (1, 2) $\therefore\ y - 2 = -\frac{8}{7}(x - 1)$

$$7y - 14 = -8x + 8$$

$\therefore$ Tangent is $7y + 8x = 22$

Now to find the equation of the normal.

$$\text{Gradient } = \frac{-1}{\text{Gradient of tangent}} = \frac{7}{8}$$

Normal passes through (1, 2) $\therefore\ y - 2 = \frac{7}{8}(x - 1)$

$$8y - 16 = 7x - 7$$

$\therefore$ Normal is $8y = 7x + 9$ That's that!

Now try this one:

Find the equations of the tangent and normal to the curve

$$x^3 + x^2y + y^3 - 7 = 0$$

at the point $x = 2$, $y = 3$.

18

Tangent: $31y + 24x = 141$	Normal: $24y = 31x + 10$

Here is the working:

$$x^3 + x^2y + y^3 - 7 = 0$$

$$3x^2 + x^2\frac{dy}{dx} + 2xy + 3y^2\frac{dy}{dx} = 0$$

$$(x^2 + 3y^2)\frac{dy}{dx} = -(3x^2 + 2xy) \qquad \therefore \frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2 + 3y^2}$$

$$\therefore \text{ At } (2, 3) \qquad \frac{dy}{dx} = -\frac{12 + 12}{4 + 27} = -\frac{24}{31}$$

(a) Tangent passes through (2, 3) $\therefore\ y - 3 = -\frac{24}{31}(x - 2)$

$31y - 93 = -24x + 48 \qquad \therefore\ 31y + 24x = 141$

(b) Normal gradient $= \frac{31}{24}$. Passes through (2, 3) $\therefore\ y - 3 = \frac{31}{24}(x - 2)$

$24y - 72 = 31x - 62 \qquad \therefore\ 24y = 31x + 10$

Now on to the next frame for another example

19

Now what about this one?

The parametric equations of a curve are $x = \frac{3t}{1+t}$, $y = \frac{t^2}{1+t}$

Find the equations of the tangent and normal at the point for which $t = 2$.

First find the value of $\frac{dy}{dx}$ when $t = 2$.

$$x = \frac{3t}{1+t} \quad \therefore \frac{dx}{dt} = \frac{(1+t)3 - 3t}{(1+t)^2} = \frac{3 + 3t - 3t}{(1+t)^2} = \frac{3}{(1+t)^2}$$

$$y = \frac{t^2}{1+t} \quad \therefore \frac{dy}{dt} = \frac{(1+t)2t - t^2}{(1+t)^2} = \frac{2t + 2t^2 - t^2}{(1+t)^2} = \frac{2t + t^2}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{2t + t^2}{(1+t)^2}\cdot\frac{(1+t)^2}{3} = \frac{2t + t^2}{3} \qquad \therefore \text{ At } t = 2,\ \frac{dy}{dx} = \frac{8}{3}$$

To get the equation of the tangent, we must know the x- and y-values of a point through which it passes. At P:

$$x = \frac{3t}{1+t} = \frac{6}{1+2} = \frac{6}{3} = 2, \quad y = \frac{t^2}{1+t} = \frac{4}{3}$$

Continued in Frame 20

20

So the tangent has a gradient of $\frac{8}{3}$ and passes through $\left(2, \frac{4}{3}\right)$

$\therefore$ Its equation is $y - \frac{4}{3} = \frac{8}{3}(x - 2)$

$3y - 4 = 8x - 16 \quad \therefore \; 3y = 8x - 12$ (Tangent)

For the normal, gradient $= \dfrac{-1}{\text{gradient of tangent}} = -\dfrac{3}{8}$

Also passes through $\left(2, \frac{4}{3}\right) \qquad \therefore \; y - \frac{4}{3} = -\frac{3}{8}(x - 2)$

$24y - 32 = -9x + 18 \qquad \therefore \; 24y + 9x = 50$ (Normal)

Now you do this one. When you are satisfied with your result, check it with the results in Frame 21. Here it is:

If $y = \cos 2t$ and $x = \sin t$, find the equations of the tangent and normal to the curve at $t = \frac{\pi}{6}$.

21

$$\boxed{\text{Tangent: } 2y + 4x = 3 \qquad \text{Normal: } 4y = 2x + 1}$$

Working:

$y = \cos 2t \qquad \therefore \; \dfrac{dy}{dt} = -2\sin 2t = -4\sin t \cos t$

$x = \sin t \qquad \therefore \; \dfrac{dx}{dt} = \cos t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{-4\sin t \cos t}{\cos t} = -4\sin t$

At $t = \frac{\pi}{6}$, $\qquad \dfrac{dy}{dx} = -4\sin\frac{\pi}{6} = -4\left(\frac{1}{2}\right) = -2$

$\therefore$ gradient of tangent $= -2$

Passes through $\qquad x = \sin\frac{\pi}{6} = 0{\cdot}5$; $y = \cos\frac{\pi}{3} = 0{\cdot}5$

$\therefore$ Tangent is $y - \frac{1}{2} = -2\left(x - \frac{1}{2}\right) \qquad \therefore \; 2y - 1 = -4x + 2$

$\therefore \; 2y + 4x = 3$ (Tangent)

Gradient of normal $= \frac{1}{2}$. Line passes through $(0{\cdot}5, 0{\cdot}5)$

Equation is $\quad y - \frac{1}{2} = \frac{1}{2}\left(x - \frac{1}{2}\right)$

$\therefore \; 4y - 2 = 2x - 1$

$\therefore \; 4y = 2x + 1$ (Normal)

22

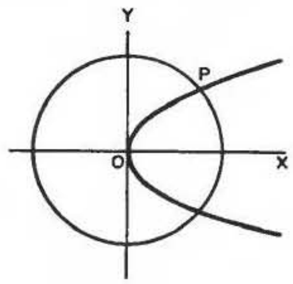
Before we leave this part of the Programme, let us revise the fact that we can easily find the angle between two intersecting curves.

Since the gradient of a curve at (x_1, y_1) is given by the value of $\frac{dy}{dx}$ at that point, and $\frac{dy}{dx} = \tan\theta$, where θ is the angle of slope, then we can use these facts to determine the angle between the curves at their point of intersection. One example will be sufficient.

Find the angle between $y^2 = 8x$ and $x^2 + y^2 = 16$ at their point of intersection for which y is positive.

First find the point of intersection:

i.e. solve $y^2 = 8x$ and $x^2 + y^2 = 16$



We have

$$x^2 + 8x = 16 \quad \therefore\; x^2 + 8x - 16 = 0$$

$$x = \frac{-8 \pm \sqrt{64+64}}{2} = \frac{-8 \pm \sqrt{128}}{2}$$

$$= \frac{-8 \pm 11{\cdot}314}{2} = \frac{3{\cdot}314}{2} \text{ or } \frac{-19{\cdot}314}{2}$$

$x = 1{\cdot}657$ or $[-9{\cdot}657]$ Not a real point of intersection

When $x = 1{\cdot}657$, $y^2 = 8(1{\cdot}657) = 13{\cdot}256$, $y = 3{\cdot}641$.

Coordinates of P are $x = 1{\cdot}657$, $y = 3{\cdot}641$.

Now we have to find $\frac{dy}{dx}$ for each of the two curves. Do that.

23

(a) $y^2 = 8x \quad \therefore\; 2y\frac{dy}{dx} = 8 \quad \therefore\; \frac{dy}{dx} = \frac{4}{y} = \frac{4}{3{\cdot}641} = \frac{1}{0{\cdot}910} = 1{\cdot}099$

$\tan\theta_1 = 1{\cdot}099 \quad \therefore\; \theta_1 = 47^\circ 42'$

(b) Similarly for $x^2 + y^2 = 16$:

$2x + 2y\frac{dy}{dx} = 0 \quad \therefore\; \frac{dy}{dx} = -\frac{x}{y} = -\frac{1{\cdot}657}{3{\cdot}641} = -0{\cdot}4551$

$\tan\theta_2 = -0{\cdot}4551 \quad \therefore\; \theta_2 = -24^\circ 28'$

Finally $\theta = \theta_1 - \theta_2 = 47^\circ 42' - (-24^\circ 28')$

$= 47^\circ 42' + 24^\circ 28'$

$= 72^\circ 10'$

That just about covers all there is to know about finding tangents and normals to a curve. We now look at another application of differentiation.

Curvature

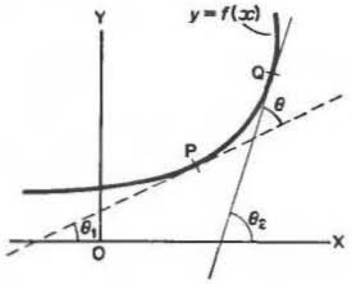
24

The value of $\frac{dy}{dx}$ at any point on a curve denotes the gradient of the curve at that point. Curvature is concerned with how quickly the curve is changing direction in the neighbourhood of that point.

Let us see in the next few frames what it is all about.

25

Let us first consider the change in direction of a curve $y = f(x)$ between the points P and Q as shown. The direction of a curve is measured by the gradient of the tangent.



$$\text{Gradient at P} = \tan\theta_1 = \left\{\frac{dy}{dx}\right\}_P$$

$$\text{Gradient at Q} = \tan\theta_2 = \left\{\frac{dy}{dx}\right\}_Q$$

These can be calculated, knowing the equation of the curve.

From the values of $\tan\theta_1$ and $\tan\theta_2$, the angles θ_1 and θ_2 can be found using a calculator. Then from the diagram, $\theta = \theta_2 - \theta_1$.

If we are concerned with how fast the curve is bending, we must consider not only the change in direction from P to Q, but also the length of .... which provides this change in direction.

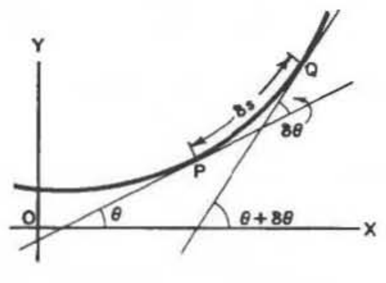
26

The arc PQ

That is, we must know not only the change of direction, but also how far along the curve we must go to obtain this change in direction.

Now let us consider the two points, P and Q, near to each other, so that PQ is a small arc ($= \delta s$). The change in direction will not be great, so that if θ is the slope at P, then the angle of slope at Q can be put as $\theta + \delta\theta$.

The change in direction from P to Q is therefore $\delta\theta$.



The length of arc from P to Q is δs.

The average rate of change of direction with arc from P to Q is

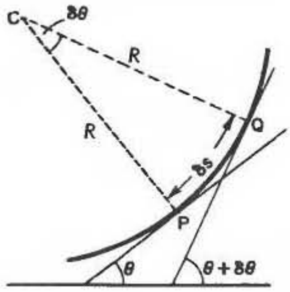
$$\frac{\text{the change in direction from P to Q}}{\text{the length of arc from P to Q}} = \frac{\delta\theta}{\delta s}$$

▶

This could be called the average curvature from P to Q. If Q now moves down towards P, i.e. $\delta s \to 0$, we finally get $\dfrac{d\theta}{ds}$, which is the *curvature* at P. It tells us how quickly the curve is bending in the immediate neighbourhood of P.

27

In practice, it is difficult to find $\dfrac{d\theta}{ds}$ since we should need a relationship between θ and s, and usually all we have is the equation of the curve, $y = f(x)$ and the coordinates of P. So we must find some other way round it.



Let the normals at P and Q meet at C. Since P and Q are close, CP ≈ QC (= R say) and the arc PQ can be thought of as a small arc of a circle of radius R. Note that PCQ = $\delta\theta$ (for if the tangent turns through $\delta\theta$, the radius at right-angles to it will also turn through the same angle).

You remember that the arc of a circle of radius r which subtends an angle θ radians at the centre is given by arc $= r\theta$. So, in the diagram above, arc PQ $= \delta s =$ ....

28

$$\boxed{\text{arc PQ} = \delta s = R\delta\theta}$$

$$\delta s = R\delta\theta \qquad \therefore \quad \frac{\delta\theta}{\delta s} = \frac{1}{R}$$

If $\delta s \to 0$, this becomes $\dfrac{d\theta}{ds} = \dfrac{1}{R}$ which is the curvature at P.

That is, we can state the curvature at a point, in terms of the radius R of the circle we have considered. This is called the *radius of curvature*, and the point C the *centre of curvature*.

So we have now found that we can obtain the curvature $\dfrac{d\theta}{ds}$ if we have some way of finding the radius of curvature R.

If R is large, is the curvature large or small?

If you think 'large', move on to Frame 29.

If you think 'small' go on to Frame 30.

29

Your answer was: 'If R is large, the curvature is large'.

This is not so. For the curvature $= \dfrac{d\theta}{ds}$ and we have just shown that $\dfrac{d\theta}{ds} = \dfrac{1}{R}$. R is the denominator, so that a large value for R gives a small value for the fraction $\dfrac{1}{R}$ and hence a small value for the curvature.

You can see it this way. If you walk round a circle with a large radius R, then the curve is relatively a gentle one, i.e. small value of curvature, but if R is small, the curve is more abrupt.

So once again, if R is large, the curvature is ....

30

$\boxed{\text{If } R \text{ is large, the curvature is small}}$

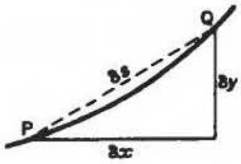
Correct, since the curvature $\dfrac{d\theta}{ds} = \dfrac{1}{R}$

In practice, we often indicate the curvature in terms of the radius of curvature R, since this is something we can appreciate.

Let us consider our two points P and Q again. Since δs is very small, there is little difference between the arc PQ and the chord PQ, or between the direction of the chord and that of the tangent.

So, when

$$\delta s \to 0, \quad \frac{dy}{dx} = \tan\theta \text{ and } \frac{dx}{ds} = \cos\theta$$



$\dfrac{dy}{dx} = \tan\theta$. Differentiate with respect to s.

Then: $$\frac{d}{ds}\left\{\frac{dy}{dx}\right\} = \frac{d}{ds}\left\{\tan\theta\right\}$$

$$\frac{d}{dx}\left\{\frac{dy}{dx}\right\}.\frac{dx}{ds} = \frac{d}{d\theta}\left\{\tan\theta\right\}.\frac{d\theta}{ds}$$

$$\therefore \quad \frac{d^2y}{dx^2}\cos\theta = \sec^2\theta\frac{d\theta}{ds}$$

$$\sec^3\theta\frac{d\theta}{ds} = \frac{d^2y}{dx^2}$$

Now: $\sec^3\theta = (\sec^2\theta)^{3/2} = (1+\tan^2\theta)^{3/2} = \left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}$

$$\therefore \quad \frac{d\theta}{ds} = \frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} \qquad \therefore \; R = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}$$

Now we have got somewhere. For knowing the equation $y = f(x)$ of the curve, we can calculate the first and second derivatives at the point P and substitute these values in the formula for R.

This is an important result. Copy it down and remember it. You may never be asked to prove it, but you will certainly be expected to know it and to apply it.

So now for some examples. Move on to Frame 31

31

Example 1

Find the radius of curvature for the hyperbola $xy = 4$ at the point $x = 2$, $y = 2$.

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

So all we need to find are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(2, 2)$

$$xy = 4 \qquad \therefore\ y = \frac{4}{x} = 4x^{-1} \qquad \therefore\ \frac{dy}{dx} = -4x^{-2} = \frac{-4}{x^2}$$

$$\text{and} \qquad \frac{d^2y}{dx^2} = 8x^{-3} = \frac{8}{x^3}$$

$$\text{At } (2, 2) \qquad \frac{dy}{dx} = -\frac{4}{4} = -1; \qquad \frac{d^2y}{dx^2} = \frac{8}{8} = 1$$

$$\therefore\ R = \frac{\left\{1 + (-1)^2\right\}^{3/2}}{1} = \frac{\{1 + 1\}^{3/2}}{1} = (2)^{3/2} = 2\sqrt{2}$$

$$\therefore\ R = 2\sqrt{2} = 2{\cdot}828 \text{ units}$$

There we are. Another example on Frame 32

32

Example 2

If $y = x + 3x^2 - x^3$, find R at $x = 0$.

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = 1 + 6x - 3x^2 \qquad \therefore\ \text{At } x = 0,\ \frac{dy}{dx} = 1 \qquad \therefore\ \left(\frac{dy}{dx}\right)^2 = 1$$

$$\frac{d^2y}{dx^2} = 6 - 6x \qquad \therefore\ \text{At } x = 0,\ \frac{d^2y}{dx^2} = 6$$

$$R = \frac{\{1 + 1\}^{3/2}}{6} = \frac{2^{3/2}}{6} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$$

$$\therefore\ R = 0{\cdot}471 \text{ units}$$

Now you do this one:

Find the radius of curvature of the curve $y^2 = \frac{x^3}{4}$ at the point $\left(1, \frac{1}{2}\right)$

When you have finished, check with the solution in Frame 33

33

$$\boxed{R = 5{\cdot}21 \text{ units}}$$

Here is the solution in full:

$$y^2 = \frac{x^3}{4} \qquad \therefore\ 2y\frac{dy}{dx} = \frac{3x^2}{4} \qquad \therefore\ \frac{dy}{dx} = \frac{3x^2}{8y}$$

$$\therefore\ \text{At } \left(1, \frac{1}{2}\right),\ \frac{dy}{dx} = \frac{3}{4} \qquad \therefore\ \left(\frac{dy}{dx}\right)^2 = \frac{9}{16}$$

$$\frac{dy}{dx} = \frac{3x^2}{8y} \qquad \therefore\ \frac{d^2y}{dx^2} = \frac{8y(6x) - 3x^2.8\dfrac{dy}{dx}}{64y^2}$$

$$\therefore\ \text{At } \left(1, \frac{1}{2}\right),\ \frac{d^2y}{dx^2} = \frac{24 - 24.\frac{3}{4}}{16} = \frac{24 - 18}{16} = \frac{3}{8}$$

$$R = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left\{1 + \dfrac{9}{16}\right\}^{3/2}}{\dfrac{3}{8}} = \frac{\left\{\dfrac{25}{16}\right\}^{3/2}}{\dfrac{3}{8}} = \frac{8}{3}.\frac{125}{64} = \frac{125}{24} = 5\frac{5}{24}$$

$$\therefore\ R = 5{\cdot}21 \text{ units}$$

34

Of course, the equation of the curve could be an implicit function, as in the previous example, or a pair of parametric equations.

e.g. If $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$, find R when $\theta = 60^\circ = \dfrac{\pi}{3}$ radians.

$$\left.\begin{aligned} x &= \theta - \sin\theta \quad \therefore\ \frac{dx}{d\theta} = 1 - \cos\theta \\ y &= 1 - \cos\theta \quad \therefore\ \frac{dy}{d\theta} = \sin\theta \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$$

$$\therefore\ \frac{dy}{dx} = \sin\theta.\frac{1}{1 - \cos\theta} = \frac{\sin\theta}{1 - \cos\theta}$$

$$\text{At } \theta = 60^\circ,\ \sin\theta = \frac{\sqrt{3}}{2},\ \cos\theta = \frac{1}{2},\ \frac{dy}{dx} = \frac{\sqrt{3}}{1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{\sin\theta}{1 - \cos\theta}\right\} = \frac{d}{d\theta}\left\{\frac{\sin\theta}{1 - \cos\theta}\right\}.\frac{d\theta}{dx}$$

$$= \frac{(1 - \cos\theta)\cos\theta - \sin\theta.\sin\theta}{(1 - \cos\theta)^2}.\frac{1}{1 - \cos\theta}$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1 - \cos\theta)^3} = \frac{\cos\theta - 1}{(1 - \cos\theta)^3} = \frac{-1}{(1 - \cos\theta)^2}$$

$$\therefore\ \text{At } \theta = 60^\circ, \qquad \frac{d^2y}{dx^2} = \frac{-1}{(1 - \frac{1}{2})^2} = \frac{-1}{\frac{1}{4}} = -4$$

$$\therefore\ R = \frac{\{1 + 3\}^{3/2}}{-4} = \frac{2^3}{-4} = \frac{8}{-4} = -2 \qquad \therefore\ R = -2 \text{ units}$$

35

You notice in this last example that the value of R is negative. This merely indicates which way the curve is bending. Since R is a physical length, then for all practical purposes, R is taken as 2 units long.

If the value of R is to be used in further calculations however, it is usually necessary to maintain the negative sign. You will see an example of this later.

Here is one for you to do in just the same way as before:

Find the radius of curvature of the curve $x = 2\cos^3\theta$, $y = 2\sin^3\theta$, at the point for which $\theta = \frac{\pi}{4} = 45^\circ$.

Work through it and then go to Frame 36 to check your work

36

$$\boxed{R = 3 \text{ units}}$$

Because

$$x = 2\cos^3\theta \qquad \therefore \quad \frac{dx}{d\theta} = 6\cos^2\theta(-\sin\theta) = -6\sin\theta\cos^2\theta$$

$$y = 2\sin^3\theta \qquad \therefore \quad \frac{dy}{d\theta} = 6\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{6\sin^2\theta\cos\theta}{-6\sin\theta\cos^2\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$\text{At } \theta = 45^\circ, \quad \frac{dy}{dx} = -1 \qquad \therefore \quad \left(\frac{dy}{dx}\right)^2 = 1$$

$$\text{Also} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{-\tan\theta\right\} = \frac{d}{d\theta}\left\{-\tan\theta\right\}\frac{d\theta}{dx} = \frac{-\sec^2\theta}{-6\sin\theta\cos^2\theta}$$

$$= \frac{1}{6\sin\theta\cos^4\theta}$$

$$\therefore \quad \text{At } \theta = 45^\circ, \ \frac{d^2y}{dx^2} = \frac{1}{6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{4}\right)} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$$

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+1\right\}^{3/2}}{\frac{2\sqrt{2}}{3}} = \frac{3}{2\sqrt{2}}2^{3/2}$$

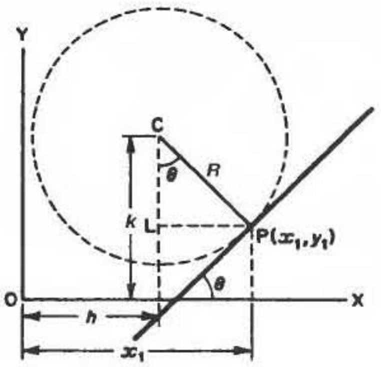
$$= \frac{3 \times 2\sqrt{2}}{2\sqrt{2}} = 3$$

$$R = 3 \text{ units}$$

37

Centre of curvature

To get a complete picture, we need to know also the position of the centre of the circle of curvature for the point P (x_1, y_1).



If the centre C is the point (h, k), we can see from the diagram that:

$$h = x_1 - \mathrm{LP} = x_1 - R\sin\theta$$
$$k = y_1 + \mathrm{LC} = y_1 + R\cos\theta$$

That is, $\left\{\begin{array}{l} h = x_1 - R\sin\theta \\ k = y_1 + R\cos\theta \end{array}\right\}$

where x_1 and y_1 are the coordinates of P, R is the radius of curvature at P and θ is the angle of slope at P,

i.e. $\tan\theta = \left\{\dfrac{dy}{dx}\right\}_P$

38

As an example, find the radius of curvature and the coordinates of the centre of curvature of the curve $y = \dfrac{11-4x}{3-x}$ at the point (2, 3).

$$\frac{dy}{dx} = \frac{(3-x)(-4)-(11-4x)(-1)}{(3-x)^2} = \frac{-12+4x+11-4x}{(3-x)^2} = \frac{-1}{(3-x)^2}$$

$$\therefore \text{ At } x = 2, \ \frac{dy}{dx} = \frac{-1}{1} = -1 \qquad \therefore \left(\frac{dy}{dx}\right)^2 = 1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{-(3-x)^{-2}\right\} = 2(3-x)^{-3}(-1) = \frac{-2}{(3-x)^3}$$

$$\therefore \text{ At } x = 2, \ \frac{d^2y}{dx^2} = \frac{-2}{1} = -2$$

$$R = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\{1+1\}^{3/2}}{-2} = \frac{2\sqrt{2}}{-2} = -\sqrt{2}$$

$$R = -\sqrt{2}$$

Now before we find the centre of curvature (h, k) we must find the angle of slope θ from the fact that $\tan\theta = \dfrac{dy}{dx}$ at P.

i.e. $\tan\theta = -1 \quad \therefore \theta = -45^\circ \quad$ (θ measured between $\pm 90^\circ$)

$\therefore \ \sin\theta = \ldots\ldots\ldots\ldots$ and $\cos\theta = \ldots\ldots\ldots\ldots$

39

$$\boxed{\sin\theta = -\frac{1}{\sqrt{2}} \qquad \cos\theta = \frac{1}{\sqrt{2}}}$$

So we have:

$x_1 = 2, \quad y_1 = 3$

$R = -\sqrt{2}$

$\sin\theta = -\frac{1}{\sqrt{2}}, \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\therefore\; h = x_1 - R\sin\theta = 2 - \left(-\sqrt{2}\right)\left(-\frac{1}{\sqrt{2}}\right) = 2 - 1 = 1, \;\; h = 1$

$k = y_1 + R\cos\theta = 3 + \left(-\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right) = 3 - 1 = 2, \;\; k = 2$

$\therefore$ centre of curvature C is the point $(1, 2)$

Note: If, by chance, the calculated value of R is negative, the minus sign must be included when we substitute for R in the expressions for h and k.

Next frame for a final example

40

Find the radius of curvature and the centre of curvature for the curve $y = \sin^2\theta$, $x = 2\cos\theta$, at the point for which $\theta = \frac{\pi}{3}$.

Before we rush off and deal with this one, let us heed an important *warning*. You will remember that the centre of curvature (h, k) is given by:

$$\left.\begin{aligned} h &= x_1 - R\sin\theta \\ k &= y_1 + R\cos\theta \end{aligned}\right\} \quad \text{and in these expressions}$$

θ is the angle of slope of the curve at the point being considered

i.e. $\tan\theta = \left\{\frac{\mathrm{d}y}{\mathrm{d}x}\right\}_{\mathrm{P}}$

Now, in the problem stated above, θ is a parameter and not the angle of slope at any particular point. In fact, if we proceed with our usual notation, we shall be using θ to stand for two completely different things and that can be troublesome, to say the least.

So the safest thing to do is this. *Where you have to find the centre of curvature of a curve given in parametric equations involving θ, change the symbol of the parameter to something other than θ.* Then you will be safe. The trouble occurs only when we find C, not when we are finding R only.

41

So, in this case, we will rewrite the problem thus:

Find the radius of curvature and the centre of curvature for the curve $y = \sin^2 t$, $x = 2\cos t$, at the point for which $t = \frac{\pi}{3}$.

Start off by finding the radius of curvature only. Then check your result so far with the solution given in the next frame before setting out to find the centre of curvature.

42

$$\boxed{R = -2{\cdot}795, \text{ i.e. } 2{\cdot}795 \text{ units}}$$

Here is the working:

$$y = \sin^2 t \qquad \therefore \ \frac{dy}{dt} = 2\sin t\cos t$$

$$x = 2\cos t \qquad \therefore \ \frac{dx}{dt} = -2\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{2\sin t\cos t}{-2\sin t} = -\cos t$$

$$\text{At } t = 60^\circ, \quad \frac{dy}{dx} = -\cos 60^\circ = -\frac{1}{2} \qquad \therefore \ \frac{dy}{dx} = -\frac{1}{2}$$

$$\text{Also } \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{-\cos t\right\} = \frac{d}{dt}\left\{-\cos t\right\}.\frac{dt}{dx} = \frac{\sin t}{-2\sin t} = -\frac{1}{2}$$

$$\therefore \ \frac{d^2y}{dx^2} = -\frac{1}{2}$$

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1 + \frac{1}{4}\right\}^{3/2}}{-\frac{1}{2}} = -2\left\{\frac{5}{4}\right\}^{3/2}$$

$$= \frac{-10\sqrt{5}}{8} = \frac{-5\sqrt{5}}{4} = \frac{-5}{4}(2{\cdot}2361)$$

$$= \frac{-11{\cdot}1805}{4} = -2{\cdot}7951$$

$$R = -2{\cdot}795$$

All correct so far? Move on to the next frame then

43

Now to find the centre of curvature (h, k):

$$h = x_1 - R\sin\theta$$

$$k = y_1 + R\cos\theta$$

$$\text{where} \quad \tan\theta = \frac{dy}{dx} = -\frac{1}{2} \qquad \therefore \ \theta = -26^\circ 34' \quad (\theta \text{ between } \pm 90^\circ)$$

$$\therefore \ \sin(-26^\circ 34') = -0{\cdot}4472; \quad \cos(-26^\circ 34') = 0{\cdot}8944$$

$$\text{Also} \qquad x_1 = 2\cos 60^\circ = 2.\frac{1}{2} = 1$$

$$y_1 = \sin^2 60^\circ = \left\{\frac{\sqrt{3}}{2}\right\}^2 = \frac{3}{4}$$

and you have already proved that $R = -2{\cdot}795$.

What then are the coordinates of the centre of curvature?

Calculate them and when you have finished, move on to the next frame

44

$$\boxed{h = -0{\cdot}25; \qquad k = -1{\cdot}75}$$

Because $h = 1 - (-2{\cdot}795)(-0{\cdot}4472)$

$= 1 - 1{\cdot}250$

$\therefore\ h = -0{\cdot}25$

and $k = 0{\cdot}75 + (-2{\cdot}795)(0{\cdot}8944)$

$= 0{\cdot}75 - 2{\cdot}50$

$\therefore\ k = -1{\cdot}75$

Therefore, the centre of curvature is the point $(-0{\cdot}25, -1{\cdot}75)$

This brings us to the end of this particular Programme. If you have followed it carefully and carried out the exercises set, you must know quite a lot about the topics we have covered. So move on now and look at the **Can You?** checklist before you work through the **Test exercise**.

Can You?

Checklist 8

45

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Evaluate the gradient of a straight line? 1 to 5
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognize the relationship satisfied by two mutually perpendicular straight lines? 6 to 11
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Derive the equations of a tangent and a normal to a curve? 12 to 23
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate the curvature and radius of curvature at a point on a curve? 24 to 36
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Locate the centre of curvature for a point on a curve? 37 to 44
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 8

46

The questions are all straightforward.

1 Find the angle between the curves $x^2 + y^2 = 4$ and $5x^2 + y^2 = 5$ at their point of intersection for which x and y are positive.

2 Find the equations of the tangent and normal to the curve $y^2 = 11 - \dfrac{10}{4-x}$ at the point (6, 4).

3 The parametric equations of a function are $x = 2\cos^3\theta$, $y = 2\sin^3\theta$. Find the equation of the normal at the point for which $\theta = \dfrac{\pi}{4} = 45°$.

4 If $x = 1 + \sin 2\theta$, $y = 1 + \cos\theta + \cos 2\theta$, find the equation of the tangent at $\theta = 60°$.

5 Find the radius of curvature and the coordinates of the centre of curvature at the point $x = 4$ on the curve whose equation is $y = x^2 + 5\ln x - 24$.

6 Given that $x = 1 + \sin\theta$, $y = \sin\theta - \frac{1}{2}\cos 2\theta$, show that $\dfrac{d^2y}{dx^2} = 2$.

Find the radius of curvature and the centre of curvature for the point on this curve where $\theta = 30°$.

Now you are ready for the next Programme

Further problems 8

47

1 Find the equation of the normal to the curve $y = \dfrac{2x}{x^2+1}$ at the point (3, 0·6) and the equation of the tangent at the origin.

2 Find the equations of the tangent and normal to the curve $4x^3 + 4xy + y^2 = 4$ at (0, 2), and find the coordinates of a further point of intersection of the tangent and the curve.

3 Obtain the equations of the tangent and normal to the ellipse $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$ at the point $(13\cos\theta, 5\sin\theta)$. If the tangent and normal meet the x-axis at the points T and N respectively, show that ON.OT is constant, O being the origin of coordinates.

4 If $x^2y + xy^2 - x^3 - y^3 + 16 = 0$, find $\dfrac{dy}{dx}$ in its simplest form. Hence find the equation of the normal to the curve at the point (1, 3).

▶

5 Find the radius of curvature of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ at the point (x_1, y_1).

6 If $2x^2 + y^2 - 6y - 9x = 0$, determine the equation of the normal to the curve at the point (1, 7).

7 Show that the equation of the tangent to the curve $x = 2a\cos^3 t$, $y = a\sin^3 t$, at any point P $\left(0 \leq t \leq \frac{\pi}{2}\right)$ is $x\sin t + 2y\cos t - 2a\sin t\cos t = 0$.
If the tangent at P cuts the y-axis at Q, determine the area of the triangle POQ.

8 Find the equation of the normal at the point $x = a\cos\theta$, $y = b\sin\theta$, of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The normal at P on the ellipse meets the major axis of the ellipse at N. Show that the locus of the mid-point of PN is an ellipse and state the lengths of its principal axes.

9 For the point where the curve $y = \frac{x - x^2}{1 + x^2}$ passes through the origin, determine:

(a) the equations of the tangent and normal to the curve
(b) the radius of curvature
(c) the coordinates of the centre of curvature.

10 In each of the following cases, find the radius of curvature and the coordinates of the centre of curvature for the point stated.

(a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at (0, 4)
(b) $y^2 = 4x - x^2 - 3$ at $x = 2{\cdot}5$
(c) $y = 2\tan\theta$, $x = 3\sec\theta$ at $\theta = 45°$.

11 Find the radius of curvature at the point (1, 1) on the curve $x^3 - 2xy + y^3 = 0$.

12 If $3ay^2 = x(x - a)^2$ with $a > 0$, prove that the radius of curvature at the point $(3a, 2a)$ is $\frac{50a}{3}$.

13 If $x = 2\theta - \sin 2\theta$ and $y = 1 - \cos 2\theta$, show that $\frac{dy}{dx} = \cot\theta$ and that $\frac{d^2y}{dx^2} = \frac{-1}{4\sin^4\theta}$. If ρ is the radius of curvature at any point on the curve, show that $\rho^2 = 8y$.

14 Find the radius of curvature of the curve $2x^2 + y^2 - 6y - 9x = 0$ at the point $(1, 7)$.

15 Prove that the centre of curvature (h, k) at the point P $(at^2, 2at)$ on the parabola $y^2 = 4ax$ has coordinates $h = 2a + 3at^2$, $k = -2at^3$.

▶

16 If ρ is the radius of curvature at any point P on the parabola $x^2 = 4ay$, and S is the point $(0, a)$, show that $\rho = 2\sqrt{(\mathrm{SP})^3/\mathrm{SO}}$, where O is the origin of coordinates.

17 The parametric equations of a curve are $x = \cos t + t\sin t$, $y = \sin t - t\cos t$. Determine an expression for the radius of curvature (ρ) and for the coordinates (h, k) of the centre of curvature in terms of t.

18 Find the radius of curvature and the coordinates of the centre of curvature of the curve $y = 3\ln x$, at the point where it meets the x-axis.

19 Show that the numerical value of the radius of curvature at the point (x_1, y_1) on the parabola $y^2 = 4ax$ is $\dfrac{2(a + x_1)^{3/2}}{a^{1/2}}$. If C is the centre of curvature at the origin O and S is the point $(a, 0)$, show that OC = 2(OS).

20 The equation of a curve is $4y^2 = x^2(2 - x^2)$:

(a) Determine the equations of the tangents at the origin.

(b) Show that the angle between these tangents is $\tan^{-1}(2\sqrt{2})$.

(c) Find the radius of curvature at the point $(1, 1/2)$.

Programme 9

Differentiation applications 2

Frames 1 to 38

Learning outcomes

When you have completed this Programme you will be able to:

- Differentiate the inverse trigonometric functions
- Differentiate the inverse hyperbolic functions
- Identify and locate a maximum and a minimum
- Identify and locate a point of inflexion

Differentiation of inverse trigonometric functions

1

$\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ depend, of course, on the values assigned to x. They are therefore functions of x and we may well be required to find their derivatives. So let us deal with them in turn.

(1) Let $y = \sin^{-1} x$. We have to find $\dfrac{dy}{dx}$

First of all, write this inverse statement as a direct statement:

$$y = \sin^{-1} x \qquad \therefore\ x = \sin y$$

Now we can differentiate this with respect to y and obtain $\dfrac{dx}{dy}$

$$\frac{dx}{dy} = \cos y \qquad \therefore\ \frac{dy}{dx} = \dots\dots\dots\dots$$

2

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos y}}$$

Now we express $\cos y$ in terms of x, thus:

We know that $\cos^2 y + \sin^2 y = 1$

$\therefore\ \cos^2 y = 1 - \sin^2 y = 1 - x^2 \qquad$ (since $x = \sin y$)

$$\therefore\ \cos y = \sqrt{1 - x^2}$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\left\{\sin^{-1} x\right\} = \frac{1}{\sqrt{1 - x^2}}$$

(2) Now you can determine $\dfrac{d}{dx}\{\cos^{-1} x\}$ in exactly the same way.

Go through the same steps and finally check your result with that in Frame 3

3

$$\boxed{\frac{d}{dx}\left\{\cos^{-1}x\right\}=\frac{-1}{\sqrt{1-x^2}}}$$

Here is the working:

$$\text{Let}\quad y=\cos^{-1}x \qquad \therefore\ x=\cos y$$

$$\therefore\ \frac{dx}{dy}=-\sin y \qquad \therefore\ \frac{dy}{dx}=\frac{-1}{\sin y}$$

$$\cos^2 y+\sin^2 y=1 \qquad \therefore\ \sin^2 y=1-\cos^2 y=1-x^2$$

$$\sin y=\sqrt{1-x^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}} \qquad \therefore\ \frac{d}{dx}\left\{\cos^{-1}x\right\}=\frac{-1}{\sqrt{1-x^2}}$$

So we have two very similar results:

$$\left.\begin{array}{ll}(1) & \dfrac{d}{dx}\{\sin^{-1}x\}=\dfrac{1}{\sqrt{1-x^2}}\\ (2) & \dfrac{d}{dx}\{\cos^{-1}x\}=\dfrac{-1}{\sqrt{1-x^2}}\end{array}\right\}\quad \text{Different only in sign}$$

(3) Now you find the derivative of $\tan^{-1}x$. The working is slightly different, but the general method the same. See what you get and then move to Frame 4 where the detailed working is set out.

4

$$\boxed{\frac{d}{dx}\left\{\tan^{-1}x\right\}=\frac{1}{1+x^2}}$$

Working: $\quad$ Let $y=\tan^{-1}x \qquad \therefore\ x=\tan y$

$$\frac{dx}{dy}=\sec^2 y=1+\tan^2 y=1+x^2$$

$$\frac{dx}{dy}=1+x^2 \qquad \therefore\ \frac{dy}{dx}=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left\{\tan^{-1}x\right\}=\frac{1}{1+x^2}$$

Let us collect these three results together. Here they are:

$$\frac{d}{dx}\left\{\sin^{-1}x\right\}=\frac{1}{\sqrt{1-x^2}} \qquad (1)$$

$$\frac{d}{dx}\left\{\cos^{-1}x\right\}=\frac{-1}{\sqrt{1-x^2}} \qquad (2)$$

$$\frac{d}{dx}\left\{\tan^{-1}x\right\}=\frac{1}{1+x^2} \qquad (3)$$

Copy these results into your record book. You will need to remember them.

On to the next frame

5

Of course, these derivatives can occur in all the usual combinations, e.g. products, quotients, etc.

Example 1

Find $\frac{dy}{dx}$, given that $y = (1 - x^2)\sin^{-1} x$

Here we have a product

$$\therefore \quad \frac{dy}{dx} = (1 - x^2)\frac{1}{\sqrt{1 - x^2}} + \sin^{-1} x.(-2x)$$
$$= \sqrt{1 - x^2} - 2x.\sin^{-1} x$$

Example 2

If $y = \tan^{-1}(2x - 1)$, find $\frac{dy}{dx}$

This time, it is a function of a function

$$\frac{dy}{dx} = \frac{1}{1 + (2x - 1)^2}.2 = \frac{2}{1 + 4x^2 - 4x + 1}$$
$$= \frac{2}{2 + 4x^2 - 4x} = \frac{1}{2x^2 - 2x + 1}$$

and so on.

There you are. Now here is a short exercise. Do all the questions.

6

Revision exercise

Differentiate with respect to x:

1. $y = \sin^{-1} 5x$
2. $y = \cos^{-1} 3x$
3. $y = \tan^{-1} 2x$
4. $y = \sin^{-1}(x^2)$
5. $y = x^2.\sin^{-1}\left(\frac{x}{2}\right)$

When you have finished them all, check your results with those in Frame 7

7

1 $y = \sin^{-1} 5x$ $\quad \therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - (5x)^2}}.5 = \frac{5}{\sqrt{1 - 25x^2}}$

2 $y = \cos^{-1} 3x$ $\quad \therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1 - (3x)^2}}.3 = \frac{-3}{\sqrt{1 - 9x^2}}$

3 $y = \tan^{-1} 2x$ $\quad \therefore \quad \frac{dy}{dx} = \frac{1}{1 + (2x)^2}.2 = \frac{2}{1 + 4x^2}$

4 $y = \sin^{-1}(x^2)$ $\quad \therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^2)^2}}.2x = \frac{2x}{\sqrt{1 - x^4}}$

▶

5 $y = x^2 . \sin^{-1}\left(\frac{x}{2}\right)$ $\quad \therefore \frac{dy}{dx} = x^2 \frac{1}{\sqrt{\left\{1 - \left(\frac{x}{2}\right)^2\right\}}} . \frac{1}{2} + 2x . \sin^{-1}\left(\frac{x}{2}\right)$

$$= \frac{x^2}{2\sqrt{\left\{1 - \frac{x^2}{4}\right\}}} + 2x . \sin\left(\frac{x}{2}\right)$$

$$= \frac{x^2}{\sqrt{4 - x^2}} + 2x . \sin^{-1}\left(\frac{x}{2}\right)$$

Right, now on to the next frame

Derivatives of inverse hyperbolic functions

In just the same way that we have inverse trig functions, so we have inverse hyperbolic functions and we would not be unduly surprised if their derivatives bore some resemblance to those of the inverse trig functions. **8**

Anyway, let us see what we get. The method is very much as before.

(4) $y = \sinh^{-1} x$ $\quad$ To find $\frac{dy}{dx}$

First express the inverse statement as a direct statement:

$$y = \sinh^{-1} x \qquad \therefore x = \sinh y \qquad \therefore \frac{dx}{dy} = \cosh y \qquad \therefore \frac{dy}{dx} = \frac{1}{\cosh y}$$

We now need to express $\cosh y$ in terms of x.

We know that $\cosh^2 y - \sinh^2 y = 1$ $\qquad \therefore \cosh^2 y = \sinh^2 y + 1 = x^2 + 1$

$$\cosh y = \sqrt{x^2 + 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}} \qquad \therefore \frac{d}{dx}\left\{\sinh^{-1} x\right\} = \frac{1}{\sqrt{x^2 + 1}}$$

Let us obtain similar results for $\cosh^{-1} x$ and $\tanh^{-1} x$ and then we will take a look at them.

So on to the next frame

9

We have just established $\dfrac{d}{dx}\left\{\sinh^{-1} x\right\} = \dfrac{1}{\sqrt{x^2+1}}$

(5) $y = \cosh^{-1} x \quad \therefore \; x = \cosh y$

$\therefore \; \dfrac{dx}{dy} = \sinh y \quad \therefore \; \dfrac{dy}{dx} = \dfrac{1}{\sinh y}$

Now $\cosh^2 y - \sinh^2 y = 1 \quad \therefore \; \sinh^2 y = \cosh^2 y - 1 = x^2 - 1$

$\therefore \; \sinh y = \sqrt{x^2 - 1}$

$\therefore \; \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2-1}} \quad \therefore \; \dfrac{d}{dx}\left\{\cosh^{-1} x\right\} = \dfrac{1}{\sqrt{x^2-1}}$

Now you can deal with the remaining one.

(6) If $y = \tanh^{-1} x$, $\dfrac{dy}{dx} = \dots\dots\dots\dots$

Tackle it in much the same way as we did for $\tan^{-1} x$, remembering this time, however, that $\text{sech}^2 x = 1 - \tanh^2 x$. You will find that useful.

When you have finished, move to Frame 10

10

$$\boxed{\frac{dy}{dx} = \frac{1}{1-x^2}}$$

Because

$y = \tanh^{-1} x \quad \therefore \; x = \tanh y$

$\therefore \; \dfrac{dx}{dy} = \text{sech}^2 y = 1 - \tanh^2 y = 1 - x^2 \qquad \therefore \; \dfrac{dy}{dx} = \dfrac{1}{1-x^2}$

$\dfrac{d}{dx}\left\{\tanh^{-1} x\right\} = \dfrac{1}{1-x^2}$

Now here are the results, all together, so that we can compare them:

$$\frac{d}{dx}\left\{\sinh^{-1} x\right\} = \frac{1}{\sqrt{x^2+1}} \qquad (4)$$

$$\frac{d}{dx}\left\{\cosh^{-1} x\right\} = \frac{1}{\sqrt{x^2-1}} \qquad (5)$$

$$\frac{d}{dx}\left\{\tanh^{-1} x\right\} = \frac{1}{1-x^2} \qquad (6)$$

Make a note of these in your record book. You will need to remember these results.

Now on to Frame 11

11

Here are some examples, using the previous results

Example 1

$$y = \cosh^{-1}\left\{3 - 2x\right\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{(3-2x)^2 - 1}}.(-2) = \frac{-2}{\sqrt{9 - 12x + 4x^2 - 1}}$$

$$= \frac{-2}{\sqrt{8 - 12x + 4x^2}} = \frac{-2}{2\sqrt{x^2 - 3x + 2}} = \frac{-1}{\sqrt{x^2 - 3x + 2}}$$

Example 2

$$y = \tanh^{-1}\left(\frac{3x}{4}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 - \left(\frac{3x}{4}\right)^2}.\frac{3}{4} = \frac{1}{1 - \frac{9x^2}{16}}.\frac{3}{4}$$

$$= \frac{16}{16 - 9x^2}.\frac{3}{4} = \frac{12}{16 - 9x^2}$$

Example 3

$$y = \sinh^{-1}\{\tan x\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x + 1}}.\sec^2 x = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \sec x$$

12

Now here are a few exercises for you to do.

Differentiate:

1 $y = \sinh^{-1} 3x$

2 $y = \cosh^{-1}\left(\frac{5x}{2}\right)$

3 $y = \tanh^{-1}(\tan x)$

4 $y = \sinh^{-1}\left\{\sqrt{x^2 - 1}\right\}$

5 $y = \cosh^{-1}(e^{2x})$

Finish them all. Then move on to Frame 13 for the results

13

1 $y = \sinh^{-1} 3x \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{(3x)^2 + 1}}.3 = \dfrac{3}{\sqrt{9x^2 + 1}}$

2 $y = \cosh^{-1}\left(\dfrac{5x}{2}\right) \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{\left(\dfrac{5x}{2}\right)^2 - 1}}.\dfrac{5}{2} = \dfrac{5}{2\sqrt{\dfrac{25x^2}{1} - 4}}$

$$= \frac{5}{2\sqrt{\dfrac{25x^2 - 4}{4}}} = \frac{5}{\sqrt{25x^2 - 4}}$$

3 $y = \tanh^{-1}(\tan x) \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{1 - \tan^2 x}.\sec^2 x = \dfrac{\sec^2 x}{1 - \tan^2 x}$

4 $y = \sinh^{-1}\left\{\sqrt{x^2 - 1}\right\}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1 + 1}}.\frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x) = \frac{1}{\sqrt{x^2 - 1}}$$

5 $y = \cosh^{-1}(e^{2x}) \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{(e^{2x})^2 - 1}}.2e^{2x} = \dfrac{2e^{2x}}{\sqrt{e^{4x} - 1}}$

All correct?

On then to Frame 14

14

Before we leave these inverse trig and hyperbolic functions, let us look at them all together.

Inverse trig functions		*Inverse hyperbolic functions*	
y	$\dfrac{dy}{dx}$	y	$\dfrac{dy}{dx}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1 - x^2}}$	$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2 + 1}}$
$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1 - x^2}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2 - 1}}$
$\tan^{-1} x$	$\dfrac{1}{1 + x^2}$	$\tanh^{-1} x$	$\dfrac{1}{1 - x^2}$

It would be a good idea to copy down this combined table, so that you compare and use the results. Do that: it will help you to remember them and to distinguish clearly between them.

Before you do a revision exercise, cover up the table you have just copied and see if you can complete the following correctly: 15

1 If $y = \sin^{-1} x$ $\quad \dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

2 If $y = \cos^{-1} x$ $\quad \dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

3 If $y = \tan^{-1} x$ $\quad \dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

4 If $y = \sinh^{-1} x$ $\quad \dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

5 If $y = \cosh^{-1} x$ $\quad \dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

6 If $y = \tanh^{-1} x$ $\quad \dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

Now check your results with your table and make a special point of brushing up any of which you are not really sure.

Revision exercise 16

Differentiate the following with respect to x:

1 $\tan^{-1}(\sinh x)$

2 $\sinh^{-1}(\tan x)$

3 $\cosh^{-1}(\sec x)$

4 $\tanh^{-1}(\sin x)$

5 $\sin^{-1}\left(\dfrac{x}{a}\right)$

Take care with these; we have mixed them up to some extent.

When you have finished them all – and you are sure you have done what was required – check your results with those in Frame 17

17

1 $y = \tan^{-1}(\sinh x)$ $\qquad \dfrac{d}{dx}\left\{\tan^{-1} x\right\} = \dfrac{1}{1 + x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1 + \sinh^2 x} . \cosh x = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$$

2 $y = \sinh^{-1}(\tan x)$ $\qquad \dfrac{d}{dx}\left\{\sinh^{-1} x\right\} = \dfrac{1}{\sqrt{x^2 + 1}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x + 1}} . \sec^2 x = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \sec x$$

3 $y = \cosh^{-1}(\sec x)$ $\qquad \dfrac{d}{dx}\left\{\cosh^{-1} x\right\} = \dfrac{1}{\sqrt{x^2 - 1}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{\sec^2 x - 1}} . \sec x . \tan x = \frac{\sec x . \tan x}{\sqrt{\tan^2 x}} = \sec x$$

▶

4 $y = \tanh^{-1}(\sin x)$ $\qquad \dfrac{d}{dx}\left\{\tanh^{-1} x\right\} = \dfrac{1}{1-x^2}$

$$\therefore \ \frac{dy}{dx} = \frac{1}{1-\sin^2 x}.\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$

5 $y = \sin^{-1}\left\{\dfrac{x}{a}\right\}$ $\qquad \dfrac{d}{dx}\left\{\sin^{-1} x\right\} = \dfrac{1}{\sqrt{1-x^2}}$

$$\therefore \ \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{1}{a} = \frac{1}{a}.\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}$$

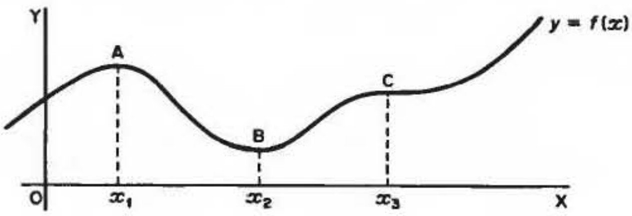
$$= \frac{1}{a}.\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} = \frac{1}{\sqrt{a^2-x^2}}$$

If you have got those all correct, or nearly all correct, you now know quite a lot about the derivatives of inverse trig and hyperbolic functions.

You are now ready to move on to the next topic of this Programme, so off you go to Frame 18

Maximum and minimum values

18



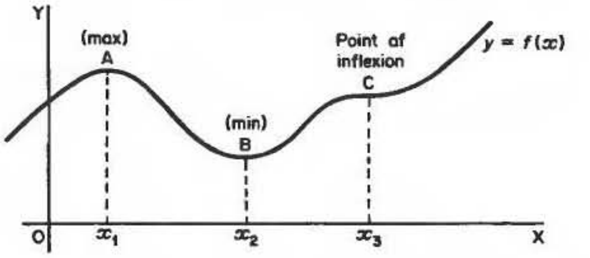
You are already familiar with the basic techniques for finding maximum and minimum values of a function. You have done this kind of operation many times in the past, but just to refresh your memory, let us consider some function, $y = f(x)$ whose graph is shown above.

At the point A, i.e. at $x = x_1$, a maximum value of y occurs since at A, the y-value is greater than the y-values on either side of it and close to it.

Similarly, at B, y is a ...., since the y-value at the point B is less than the y-values on either side of it and close to it.

19

At B, y is a *minimum* value



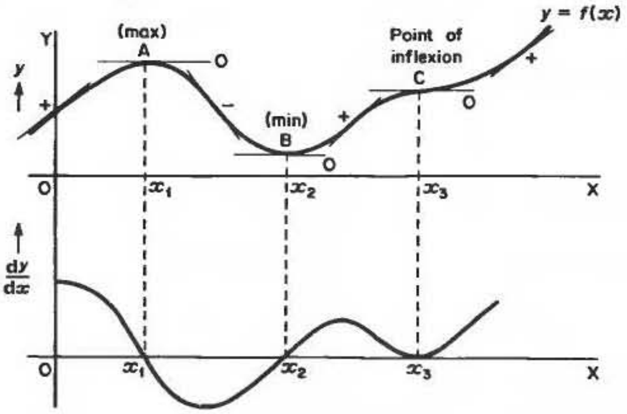
The point C is worth a second consideration. It looks like 'half a max and half a min'. The curve flattens out at C, but instead of dipping down, it then goes on with an increasingly positive gradient. Such a point is an example of a *point of inflexion*, i.e. it is essentially a form of S-bend.

Points A, B and C are called *stationary points* on the graph, or *stationary values of y*, and while you know how to find the positions of A and B, you may know considerably less about points of inflexion. We shall be taking a special look at these.

On to Frame 20

20

If we consider the gradient of the graph as we travel left to right, we can draw a graph to show how this gradient varies. We have no actual values for the gradient, but we can see whether it is positive or negative, more or less steep. The graph we obtain is the first derived curve of the function and we are really plotting the values of $\frac{dy}{dx}$ against values of x.



We see that at $x = x_1, x_2, x_3$ (corresponding to our three stationary points), the graph of $\frac{dy}{dx}$ is at the x-axis and at no other points.

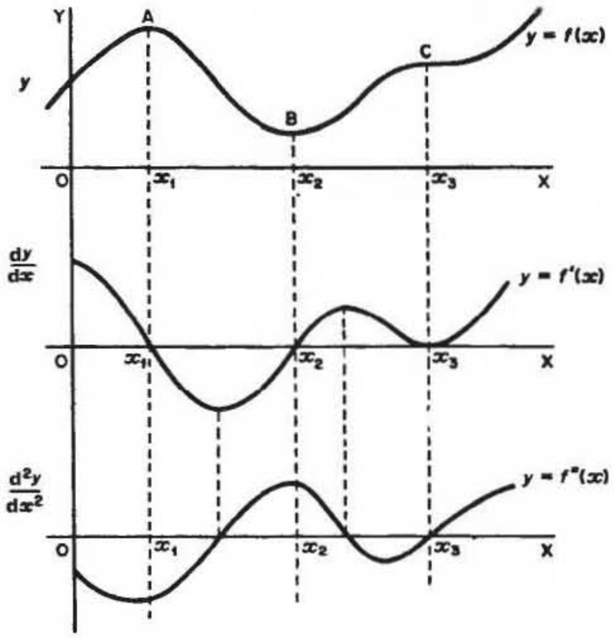
Therefore, we obtain the first rule, which is that for stationary points, $\frac{dy}{dx} = \ldots\ldots\ldots\ldots$

Move on to Frame 21

21

$$\boxed{\text{For stationary points, A, B, C: } \frac{dy}{dx} = 0}$$

If we now trace the gradient of the *first derived curve* and plot this against x, we obtain the *second derived curve*, which shows values of $\frac{d^2y}{dx^2}$ against x.



From the first derived curve, we see that for stationary points:

$\frac{dy}{dx} = 0$

For the second derived curve, we see that:

for maximum y,	$\frac{d^2y}{dx^2}$ is negative
for minimum y,	$\frac{d^2y}{dx^2}$ is positive
for point of inflexion	$\frac{d^2y}{dx^2}$ is zero

Copy the diagram into your record book. It summarizes all the facts on max and min values so far.

22

From the results we have just established, we can now determine:

(a) the values of x at which stationary points occur, by differentiating the function and then solving the equation $\frac{dy}{dx} = 0$

(b) the corresponding values of y at these points by merely substituting the x-values found, in $y = f(x)$

(c) the type of each stationary point (max, min, or point of inflexion) by testing in the expression for $\frac{d^2y}{dx^2}$

With this information, we can go a long way towards drawing a sketch of the curve. So let us apply these results to a straightforward example in the next frame.

23

Find the stationary points on the graph of the function $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 5$. Distinguish between them and sketch the graph of the function.

There are, of course, two stages:

(a) Stationary points are given by $\frac{dy}{dx} = 0$

(b) The type of each stationary point is determined by substituting the roots of the equation $\frac{dy}{dx} = 0$ in the expression for $\frac{d^2y}{dx^2}$

If $\frac{d^2y}{dx^2}$ is negative, then y is a maximum.

If $\frac{d^2y}{dx^2}$ is positive, then y is a minimum.

If $\frac{d^2y}{dx^2}$ is zero, then y may be a point of inflexion.

We shall need both the first and second derivatives, so make sure you are ready. If $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 5$, then $\frac{dy}{dx} = \ldots\ldots\ldots\ldots$ and $\frac{d^2y}{dx^2} = \ldots\ldots\ldots\ldots$

24

$$\boxed{\frac{dy}{dx} = x^2 - x - 2; \qquad \frac{d^2y}{dx^2} = 2x - 1}$$

(a) Stationary points occur at $\dfrac{dy}{dx} = 0$

$\therefore\ x^2 - x - 2 = 0 \qquad \therefore\ (x-2)(x+1) = 0 \qquad \therefore\ x = 2$ and $x = -1$

i.e. stationary points occur at $x = 2$ and $x = -1$.

(b) To determine the type of each stationary point, substitute $x = 2$ and then $x = -1$ in the expression for $\dfrac{d^2y}{dx^2}$

At $x = 2$, $\dfrac{d^2y}{dx^2} = 4 - 1 = 3$, i.e. positive $\qquad \therefore\ x = 2$ gives y_{min}.

At $x = -1$, $\dfrac{d^2y}{dx^2} = -2 - 1$, i.e. negative $\qquad \therefore\ x = -1$ gives y_{max}

Substituting in $y = f(x)$ gives $x = 2$, $y_{min} = 1\frac{2}{3}$ and $x = -1$, $y_{max} = 6\frac{1}{6}$.

Also, we can see at a glance from the function, that when $x = 0$, $y = 5$.

You can now sketch the graph of the function. Do it

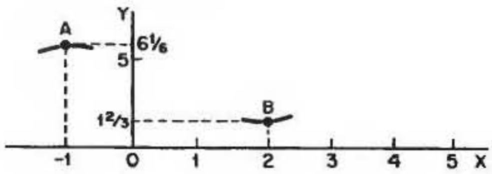
25

We know that:

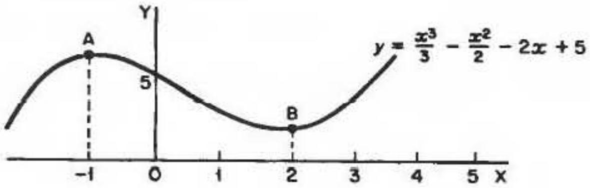
(a) at $x = -1$, $y_{max} = 6\frac{1}{6}$

(b) at $x = 2$, $y_{min} = 1\frac{2}{3}$

(c) at $x = 0$, $y = 5$



Joining up with a smooth curve gives:



There is no point of inflexion, like the point C, on this particular graph.

All that was just by way of refreshing your memory on work you have done before. Now let us take a wider look at points of inflexion.

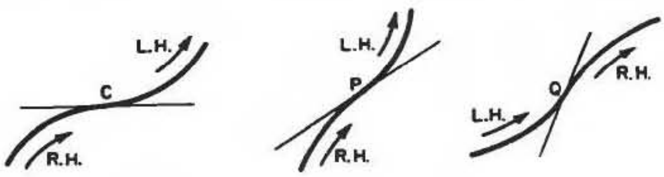
Move on

Points of inflexion

26

The point C that we considered on our first diagram was rather a special kind of point of inflexion. In general, it is not necessary for the curve at a point of inflexion to have zero gradient.

A *point of inflexion* (P-of-I) is defined simply as a point on a curve at which the *direction of bending* changes, i.e. from a right-hand bend (R.H.) to a left-hand bend (L.H.), or from a left-hand bend to a right-hand bend.



The point C we considered is, of course, a P-of-I, but it is not *essential* at a P-of-I for the gradient to be zero. Points P and Q are perfectly good points of inflexion and in fact in these cases the gradient is:

$\left\{\begin{array}{c}\text{positive}\\ \text{negative}\\ \text{zero}\end{array}\right\}$ Which?

27

At the points of inflexion, P and Q, the gradient is in fact

$\boxed{\text{positive}}$

Correct. The gradient can of course be positive, negative or zero in any one case, but there is no restriction on its sign.

A point of inflexion, then, is simply a point on a curve at which there is a change in the .... of ....

28

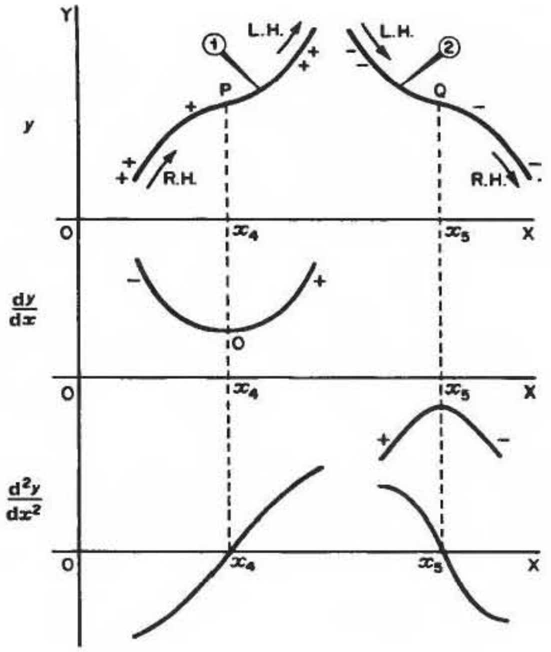
Point of inflexion: a point at which there is a change in the

$\boxed{\text{direction of bending}}$

If the gradient at a P-of-I is not zero, it will not appear in our usual max and min routine, for $\frac{dy}{dx}$ will not be zero. How, then, are we going to find where such points of inflexion occur?

▶

Let us sketch the graphs of the gradients as we did before:



P and Q are points of inflexion.

In curve 1, the gradient is always positive, ++ indicating a greater positive gradient than +.

Similarly in curve 2, the gradient is always negative.

In curve 1, $\frac{dy}{dx}$ reaches a minimum value but not zero.

In curve 2, $\frac{dy}{dx}$ reaches a maximum value but not zero.

For both points of inflexion, i.e. at $x = x_4$ and $x = x_5$, $\frac{d^2y}{dx^2} = 0$

We see that where points of inflexion occur, $\frac{d^2y}{dx^2} = 0$

So, is this the clue we have been seeking? If so, it simply means that to find the points of inflexion we differentiate the function of the curve twice and solve the equation $\frac{d^2y}{dx^2} = 0$

That sounds easy enough! But move on to the next frame to see what is involved

29

We have just found that

$$\boxed{\text{where points of inflexion occur, } \frac{d^2y}{dx^2} = 0}$$

This is perfectly true. Unfortunately, this is not the whole of the story, for it is also possible for $\frac{d^2y}{dx^2}$ to be zero at points other than points of inflexion!

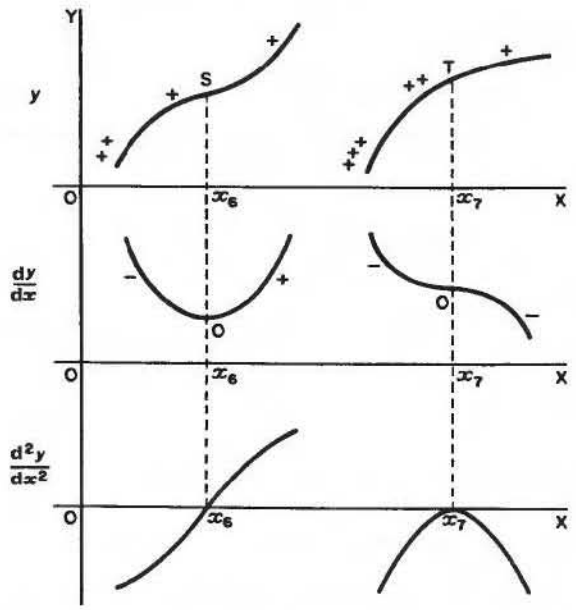
So if we solve $\frac{d^2y}{dx^2} = 0$, we cannot as yet be sure whether the solution $x = a$ gives a point of inflexion or not. How can we decide?

Let us consider just one more set of graphs. This should clear the matter up.

30

Let S be a true point of inflexion and T a point on $y = f(x)$ as shown. Clearly, T is not a point of inflexion.

The first derived curves could well look like this.



Notice the difference between the two second derived curves.

Although $\frac{d^2y}{dx^2}$ is zero for each (at $x = x_6$ and $x = x_7$), how do they differ?

When you have discovered the difference, move on to Frame 31

31

In the case of the real P-of-I, the graph of $\dfrac{d^2y}{dx^2}$ crosses the x-axis.
In the case of no P-of-I, the graph of $\dfrac{d^2y}{dx^2}$ only touches the x-axis and $\dfrac{d^2y}{dx^2}$ does not change sign.

This is the clue we have been after, and gives us our final rule:

For a point of inflexion, $\dfrac{d^2y}{dx^2}=0$ *and there is a change of sign of* $\dfrac{d^2y}{dx^2}$ *as we go through the point.*

(In the phoney case, there is no change of sign.)

So, to find where points of inflexion occur:

(a) we differentiate $y=f(x)$ twice to get $\dfrac{d^2y}{dx^2}$

(b) we solve the equation $\dfrac{d^2y}{dx^2}=0$

(c) we test to see whether or not a change of sign occurs in $\dfrac{d^2y}{dx^2}$ as we go through this value of x.

For points of inflexion, then, $\dfrac{d^2y}{dx^2}=0$, with .... of ....

32

For a P-of-I, $\dfrac{d^2y}{dx^2}=0$ with $\boxed{\text{change of sign}}$

This last phrase is all-important.

Example 1

Find the points of inflexion, if any, on the graph of the function:

$$y=\frac{x^3}{3}-\frac{x^2}{2}-2x+5$$

(a) *Differentiate twice.* $\dfrac{dy}{dx}=x^2-x-2,\ \dfrac{d^2y}{dx^2}=2x-1$

For P-of-I, $\dfrac{d^2y}{dx^2}=0$, with change of sign. $\therefore\ 2x-1=0\quad\therefore\ x=\dfrac{1}{2}$

If there is a P-of-I, it occurs at $x=\dfrac{1}{2}$.

(b) *Test for change of sign.* We take a point just before $x=\dfrac{1}{2}$, i.e. $x=\dfrac{1}{2}-a$, and a point just after $x=\dfrac{1}{2}$, i.e. $x=\dfrac{1}{2}+a$, where a is a small positive quantity, and investigate the sign of $\dfrac{d^2y}{dx^2}$ at these two values of x.

Move on

33

$$\frac{d^2y}{dx^2} = 2x - 1$$

(a) At $x = \frac{1}{2} - a$, $\frac{d^2y}{dx^2} = 2\left(\frac{1}{2} - a\right) - 1 = 1 - 2a - 1$

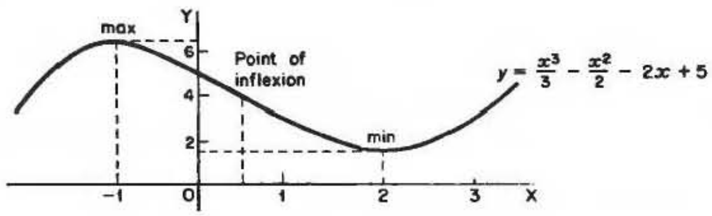
$= -2a$ (negative)

(b) At $x = \frac{1}{2} + a$, $\frac{d^2y}{dx^2} = 2\left(\frac{1}{2} + a\right) - 1 = 1 + 2a - 1$

$= 2a$ (positive)

There *is* a change in sign of $\frac{d^2y}{dx^2}$ as we go through $x = \frac{1}{2}$

$\therefore$ There *is* a point of inflexion at $x = \frac{1}{2}$



If you look at the sketch graph of this function which you have already drawn, you will see the point of inflexion where the right-hand curve changes to the left-hand curve.

34

Example 2

Find the points of inflexion on the graph of the function:

$y = 3x^5 - 5x^4 + x + 4$

First, differentiate twice and solve the equation $\frac{d^2y}{dx^2} = 0$. This will give the values of x at which there are possibly points of inflexion. We cannot be sure until we have then tested for a change of sign in $\frac{d^2y}{dx^2}$. We will do that in due course.

So start off by finding an expression for $\frac{d^2y}{dx^2}$ and solving the equation $\frac{d^2y}{dx^2} = 0$.

When you have done that, move on to the next frame

35

We have $y = 3x^5 - 5x^4 + x + 4$

$$\therefore \quad \frac{dy}{dx} = 15x^4 - 20x^3 + 1$$

$$\therefore \quad \frac{d^2y}{dx^2} = 60x^3 - 60x^2 = 60x^2(x-1)$$

For P-of-I, $\frac{d^2y}{dx^2} = 0$, with change of sign.

$\therefore \quad 60x^2(x-1) = 0 \qquad \therefore \quad x = 0$ or $x = 1$

If there is a point of inflexion, it occurs at $x = 0$, $x = 1$, or both. Now comes the test for a change of sign. For each of the two values of x we have found, i.e. $x = 0$ and $x = 1$, take points on either side of it, differing from it by a very small amount a, where $0 < a < 1$.

(a) For $x = 0$

$$\left.\begin{aligned} \text{At } x = -a, \quad \frac{d^2y}{dx^2} &= 60(-a)^2(-a-1) \\ &= (+)(+)(-) = \text{negative} \\ \text{At } x = +a, \quad \frac{d^2y}{dx^2} &= 60(+a)^2(a-1) \\ &= (+)(+)(-) = \text{negative} \end{aligned}\right\} \begin{array}{l} \text{No sign change.} \\ \text{No P-of-I.} \end{array}$$

(b) For $x = 1$

$$\left.\begin{aligned} \text{At } x = 1-a, \quad \frac{d^2y}{dx^2} &= 60(1-a)^2(1-a-1) \\ &= (+)(+)(-) = \text{negative} \\ \text{At } x = 1+a, \quad \frac{d^2y}{dx^2} &= 60(1+a)^2(1+a-1) \\ &= (+)(+)(+) = \text{positive} \end{aligned}\right\} \begin{array}{l} \text{Change in sign.} \\ \therefore \ \text{P-of-I.} \end{array}$$

Therefore, the only point of inflexion occurs when $x = 1$, i.e. at the point

$x = 1, y = 3$

That is just about all there is to it. The functions with which we have to deal differ, of course, from problem to problem, but the method remains the same.

Now go on to the next frame and complete the **Can You?** *checklist and* **Test exercise**

Can You?

Checklist 9

36

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Differentiate the inverse trigonometric functions? 1 to 7
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Differentiate the inverse hyperbolic functions? 8 to 17
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Identify and locate a maximum and a minimum? 18 to 25
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Identify and locate a point of inflexion? 26 to 35
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 9

37

The questions are all very straightforward and should not cause you any anxiety.

1 Evaluate:

(a) $\cos^{-1}(-0{\cdot}6428)$ (b) $\tan^{-1}(-0{\cdot}7536)$

2 Differentiate with respect to x:

(a) $y = \sin^{-1}(3x+2)$ (d) $y = \cosh^{-1}(1-3x)$

(b) $y = \dfrac{\cos^{-1} x}{x}$ (e) $y = \sinh^{-1}(\cos x)$

(c) $y = x^2 \tan^{-1}\left(\dfrac{x}{2}\right)$ (f) $y = \tanh^{-1} 5x$

3 Find the stationary values of y and the points of inflexion on the graph of each of the following functions, and in each case, draw a sketch graph of the function:

(a) $y = x^3 - 6x^2 + 9x + 6$

(b) $y = x + \dfrac{1}{x}$

(c) $y = xe^{-x}$

Well done. You are now ready for the next Programme

Further problems 9

38

1 Differentiate:

(a) $\tan^{-1}\left\{\dfrac{1+\tan x}{1-\tan x}\right\}$ (b) $x\sqrt{1-x^2}-\sin^{-1}\sqrt{1-x^2}$

2 If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that for $0 \le x < 1$:

(a) $(1-x^2)\dfrac{dy}{dx}=xy+1$ (b) $(1-x^2)\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}=y$

3 Find $\dfrac{dy}{dx}$ when:

(a) $y=\tan^{-1}\left\{\dfrac{4\sqrt{x}}{1-4x}\right\}$ (b) $y=\tanh^{-1}\left\{\dfrac{2x}{1+x^2}\right\}$

4 Find the coordinates of the point of inflexion on the curves:

(a) $y=(x-2)^2(x-7)$
(b) $y=4x^3+3x^2-18x-9$

5 Find the values of x for which the function $y=f(x)$, defined by $y(3x-2)=(3x-1)^2$ has maximum and minimum values and distinguish between them. Sketch the graph of the function.

6 Find the values of x at which maximum and minimum values of y and points of inflexion occur on the curve $y=12\ln x+x^2-10x$.

7 If $4x^2+8xy+9y^2-8x-24y+4=0$, show that when $\dfrac{dy}{dx}=0$, $x+y=1$ and $\dfrac{d^2y}{dx^2}=\dfrac{4}{8-5y}$. Hence find the maximum and minimum values of y.

8 Determine the smallest positive value of x at which a point of inflexion occurs on the graph of $y=3e^{2x}\cos(2x-3)$.

9 If $y^3=6xy-x^3-1$, prove that $\dfrac{dy}{dx}=\dfrac{2y-x^2}{y^2-2x}$ and that the maximum value of y occurs where $x^3=8+2\sqrt{14}$ and the minimum value where $x^3=8-2\sqrt{14}$.

10 For the curve $y=e^{-x}\sin x$, express $\dfrac{dy}{dx}$ in the form $Ae^{-x}\cos(x+a)$ and show that the points of inflexion occur at $x=\dfrac{\pi}{2}+k\pi$ for any integral value of k.

11 Find the stationary points and points of inflexion on the following curves, and, in each case, sketch the graph:

(a) $y=2x^3-5x^2+4x-1$ (b) $y=\dfrac{x(x-1)}{x-2}$
(c) $y=x+\sin x$ (take x and y scales as multiples of π)

12 Find the values of x at which points of inflexion occur on the following curves:

(a) $y = e^{-x^2}$ (b) $y = e^{-2x}(2x^2 + 2x + 1)$

(c) $y = x^4 - 10x^2 + 7x + 4$

13 The signalling range (x) of a submarine cable is proportional to $r^2 \ln\left(\frac{1}{r}\right)$, where r is the ratio of the radii of the conductor and cable. Find the value of r for maximum range.

14 The power transmitted by a belt drive is proportional to $Tv - \frac{wv^3}{g}$, where v = speed of the belt, T = tension on the driving side and ω = weight per unit length of belt. Find the speed at which the transmitted power is a maximum.

15 A right circular cone has a given curved surface A. Show that, when its volume is a maximum, the ratio of the height to the base radius is $\sqrt{2} : 1$.

16 The motion of a particle performing damped vibrations is given by $y = e^{-t}\sin 2t$, y being the displacement from its mean position at time t. Show that y is a maximum when $t = \frac{1}{2}\tan^{-1}(2)$ and determine this maximum displacement to three significant figures.

17 The cross-section of an open channel is a trapezium with base 6 cm and sloping sides each 10 cm wide. Calculate the width across the open top so that the cross-sectional area of the channel shall be a maximum.

18 The velocity (v) of a piston is related to the angular velocity (ω) of the crank by the relationship $v = \omega r\left\{\sin\theta + \frac{r}{2\ell}\sin 2\theta\right\}$ where r = length of crank and ℓ = length of connecting rod. Find the first positive value of θ for which v is a maximum, for the case when $\ell = 4r$.

19 A right circular cone of base radius r has a total surface area S and volume V. Prove that $9V^2 = r^2(S^2 - 2\pi r^2 S)$. If S is constant, prove that the vertical angle (θ) of the cone for maximum volume is given by $\theta = 2\sin^{-1}\left(\frac{1}{3}\right)$.

20 Show that the equation $4\frac{d^2x}{dt^2} + 4\mu\frac{dx}{dt} + \mu^2 x = 0$ is satisfied by $x = (At + B)e^{-\mu t/2}$, where A and B are arbitrary constants. If $x = 0$ and $\frac{dx}{dt} = C$ when $t = 0$, find A and B and show that the maximum value of x is $\frac{2C}{\mu e}$ and that this occurs when $t = \frac{2}{\mu}$.

Partial differentiation 1

Frames 1 to 40

Learning outcomes

When you have completed this Programme you will be able to:

- Find the first partial derivatives of a function of two real variables
- Find second-order partial derivatives of a function of two real variables
- Calculate errors using partial differentiation

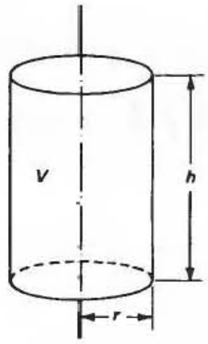
Partial differentiation

1

The volume V of a cylinder of radius r and height h is given by

$V = \pi r^2 h$

i.e. V depends on two quantities, the values of r and h.



If we keep r constant and increase the height h, the volume V will increase. In these circumstances, we can consider the derivative of V with respect to h – but only if r is kept constant.

i.e. $\left[\dfrac{dV}{dh}\right]_{r \text{ constant}}$ is written $\dfrac{\partial V}{\partial h}$

Notice the new type of 'delta'. We already know the meaning of $\dfrac{\delta y}{\delta x}$ and $\dfrac{dy}{dx}$. Now we have a new one, $\dfrac{\partial V}{\partial h}$. $\dfrac{\partial V}{\partial h}$ is called the *partial derivative* of V with respect to h and implies that for our present purpose, the value of r is considered as being kept ....

2

$\boxed{\text{constant}}$

$V = \pi r^2 h$. To find $\dfrac{\partial V}{\partial h}$, we differentiate the given expression, taking all symbols except V and h as being constant $\therefore \dfrac{\partial V}{\partial h} = \pi r^2.1 = \pi r^2$

Of course, we could have considered h as being kept constant, in which case, a change in r would also produce a change in V. We can therefore talk about $\dfrac{\partial V}{\partial r}$ which simply means that we now differentiate $V = \pi r^2 h$ with respect to r, taking all symbols except V and r as being constant for the time being.

$\therefore \dfrac{\partial V}{\partial r} = \pi 2rh = 2\pi rh$

In the statement $V = \pi r^2 h$, V is expressed as a function of two variables, r and h. It therefore has two partial derivatives, one with respect to .... and one with respect to ....

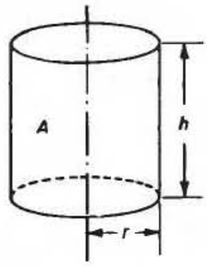
3

One with respect to r; one with respect to h

Another example:

Let us consider the area of the curved surface of the cylinder $A = 2\pi rh$

A is a function of r and h, so we can find $\frac{\partial A}{\partial r}$ and $\frac{\partial A}{\partial h}$



To find $\frac{\partial A}{\partial r}$ we differentiate the expression for A with respect to r, keeping all other symbols constant.

To find $\frac{\partial A}{\partial h}$ we differentiate the expression for A with respect to h, keeping all other symbols constant.

So, if $A = 2\pi rh$, then $\frac{\partial A}{\partial r} =$ .... and $\frac{\partial A}{\partial h} =$ ....

4

$$\frac{\partial A}{\partial r} = 2\pi h \text{ and } \frac{\partial A}{\partial h} = 2\pi r$$

Of course, we are not restricted to the mensuration of the cylinder. The same will happen with any function which is a function of two independent variables. For example, consider $z = x^2y^3$.

Here z is a function of x and y. We can therefore find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(a) To find $\frac{\partial z}{\partial x}$, differentiate with respect to x, regarding y as a constant.

$$\therefore \quad \frac{\partial z}{\partial x} = 2xy^3$$

(b) To find $\frac{\partial z}{\partial y}$, differentiate with respect to y, regarding x as a constant.

$$\frac{\partial z}{\partial y} = x^2 3y^2 = 3x^2y^2$$

Partial differentiation is easy! For we regard every independent variable, except the one with respect to which we are differentiating, as being for the time being ....

5

$\boxed{\text{constant}}$

Here are some examples. 'With respect to' is abbreviated to w.r.t.

Example 1

$u = x^2 + xy + y^2$

(a) To find $\frac{\partial u}{\partial x}$, we regard y as being constant.

Partial diff w.r.t. x of $x^2 = 2x$
Partial diff w.r.t. x of $xy = y$ (y is a constant factor)
Partial diff w.r.t. x of $y^2 = 0$ (y^2 is a constant term)

$$\frac{\partial u}{\partial x} = 2x + y$$

(b) To find $\frac{\partial u}{\partial y}$, we regard x as being constant.

Partial diff w.r.t. y of $x^2 = 0$ (x^2 is a constant term)
Partial diff w.r.t. y of $xy = x$ (x is a constant factor)
Partial diff w.r.t. y of $y^2 = 2y$

$$\frac{\partial u}{\partial y} = x + 2y$$

Another example in Frame 6

6

Example 2

$z = x^3 + y^3 - 2x^2y$

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 4xy = 3x^2 - 4xy$$
$$\frac{\partial z}{\partial y} = 0 + 3y^2 - 2x^2 = 3y^2 - 2x^2$$

And it is all just as easy as that.

Example 3

$z = (2x - y)(x + 3y)$

This is a product, and the usual product rule applies except that we keep y constant when finding $\frac{\partial z}{\partial x}$, and x constant when finding $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = (2x - y)(1 + 0) + (x + 3y)(2 - 0) = 2x - y + 2x + 6y = 4x + 5y$$
$$\frac{\partial z}{\partial y} = (2x - y)(0 + 3) + (x + 3y)(0 - 1) = 6x - 3y - x - 3y = 5x - 6y$$

▶

Here is one for you to do.

If $z = (4x - 2y)(3x + 5y)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

Find the results and then move on to Frame 7

7

$$\boxed{\frac{\partial z}{\partial x} = 24x + 14y; \qquad \frac{\partial z}{\partial y} = 14x - 20y}$$

Because $z = (4x - 2y)(3x + 5y)$, i.e. product

$$\begin{aligned}
\therefore \quad \frac{\partial z}{\partial x} &= (4x - 2y)(3 + 0) + (3x + 5y)(4 - 0) \\
&= 12x - 6y + 12x + 20y = 24x + 14y \\
\frac{\partial z}{\partial y} &= (4x - 2y)(0 + 5) + (3x + 5y)(0 - 2) \\
&= 20x - 10y - 6x - 10y = 14x - 20y
\end{aligned}$$

There we are. Now what about this one?

Example 4

If $z = \dfrac{2x - y}{x + y}$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

Applying the quotient rule, we have:

$$\frac{\partial z}{\partial x} = \frac{(x + y)(2 - 0) - (2x - y)(1 + 0)}{(x + y)^2} = \frac{3y}{(x + y)^2}$$

and $$\frac{\partial z}{\partial y} = \frac{(x + y)(0 - 1) - (2x - y)(0 + 1)}{(x + y)^2} = \frac{-3x}{(x + y)^2}$$

That was not difficult. Now you do this one:

If $z = \dfrac{5x + y}{x - 2y}$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

When you have finished, on to the next frame

8

$$\boxed{\frac{\partial z}{\partial x}=\frac{-11y}{(x-2y)^2};\quad \frac{\partial z}{\partial y}=\frac{11x}{(x-2y)^2}}$$

Here is the working:

(a) To find $\frac{\partial z}{\partial x}$, we regard y as being constant.

$$\therefore\ \frac{\partial z}{\partial x}=\frac{(x-2y)(5+0)-(5x+y)(1-0)}{(x-2y)^2}$$
$$=\frac{5x-10y-5x-y}{(x-2y)^2}=\frac{-11y}{(x-2y)^2}$$

(b) To find $\frac{\partial z}{\partial y}$, we regard x as being constant.

$$\therefore\ \frac{\partial z}{\partial y}=\frac{(x-2y)(0+1)-(5x+y)(0-2)}{(x-2y)^2}$$
$$=\frac{x-2y+10x+2y}{(x-2y)^2}=\frac{11y}{(x-2y)^2}$$

In practice, we do not write down the zeros that occur in the working, but this is how we think.

Let us do one more example, so move on to the next frame

9

Example 5

If $z=\sin(3x+2y)$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Here we have what is clearly a 'function of a function'. So we apply the usual procedure, except to remember that when we are finding:

(a) $\frac{\partial z}{\partial x}$, we treat y as constant, and

(b) $\frac{\partial z}{\partial y}$, we treat x as constant.

Here goes then.

$$\frac{\partial z}{\partial x}=\cos(3x+2y)\times\frac{\partial}{\partial x}(3x+2y)=\cos(3x+2y)\times 3=3\cos(3x+2y)$$
$$\frac{\partial z}{\partial y}=\cos(3x+2y)\times\frac{\partial}{\partial y}(3x+2y)=\cos(3x+2y)\times 2=2\cos(3x+2y)$$

There it is. So in partial differentiation, we can apply all the ordinary rules of normal differentiation, except that we regard the independent variables other than the one we are using, as being for the time being ....

10

constant

Fine. Now here is a short exercise for you to do by way of revision.

In each of the following cases, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

1 $z = 4x^2 + 3xy + 5y^2$ **3** $z = \tan(3x + 4y)$

2 $z = (3x + 2y)(4x - 5y)$ **4** $z = \dfrac{\sin(3x + 2y)}{xy}$

Finish them all, then move on to Frame 11 for the results

11

Here are the answers:

1 $z = 4x^2 + 3xy + 5y^2$ $\quad \dfrac{\partial z}{\partial x} = 8x + 3y$ $\quad \dfrac{\partial z}{\partial y} = 3x + 10y$

2 $z = (3x + 2y)(4x - 5y)$ $\quad \dfrac{\partial z}{\partial x} = 24x - 7y$ $\quad \dfrac{\partial z}{\partial y} = -7x - 20y$

3 $z = \tan(3x + 4y)$ $\quad \dfrac{\partial z}{\partial x} = 3\sec^2(3x + 4y)$ $\quad \dfrac{\partial z}{\partial y} = 4\sec^2(3x + 4y)$

4 $z = \dfrac{\sin(3x + 2y)}{xy}$

$$\frac{\partial z}{\partial x} = \frac{3x\cos(3x + 2y) - \sin(3x + 2y)}{x^2y} \qquad \frac{\partial z}{\partial y} = \frac{2y\cos(3x + 2y) - \sin(3x + 2y)}{xy^2}$$

If you have got *all* the answers correct, turn straight on to Frame 15. If you have not got all these answers, or are at all uncertain, move to Frame 12.

12

Let us work through these examples in detail.

1 $z = 4x^2 + 3xy + 5y^2$

To find $\dfrac{\partial z}{\partial x}$, regard y as a constant:

$\therefore \dfrac{\partial z}{\partial x} = 8x + 3y + 0$, i.e. $8x + 3y$ $\qquad \therefore \dfrac{\partial z}{\partial x} = 8x + 3y$

Similarly, regarding x as constant:

$\dfrac{\partial z}{\partial y} = 0 + 3x + 10y$, i.e. $3x + 10y$ $\qquad \therefore \dfrac{\partial z}{\partial y} = 3x + 10y$

2 $z = (3x + 2y)(4x - 5y)$ Product rule

$$\frac{\partial z}{\partial x} = (3x + 2y)(4) + (4x - 5y)(3)$$
$$= 12x + 8y + 12x - 15y = 24x - 7y$$
$$\frac{\partial z}{\partial y} = (3x + 2y)(-5) + (4x - 5y)(2)$$
$$= -15x - 10y + 8x - 10y = -7x - 20y$$

Move on for the solutions to **3** *and* **4**

13

3 $z = \tan(3x + 4y)$

$$\frac{\partial z}{\partial x} = \sec^2(3x + 4y)(3) = 3\sec^2(3x + 4y)$$

$$\frac{\partial z}{\partial y} = \sec^2(3x + 4y)(4) = 4\sec^2(3x + 4y)$$

4 $z = \dfrac{\sin(3x + 2y)}{xy}$

$$\frac{\partial z}{\partial x} = \frac{xy\cos(3x + 2y)(3) - \sin(3x + 2y)(y)}{x^2y^2}$$

$$= \frac{3x\cos(3x + 2y) - \sin(3x + 2y)}{x^2y}$$

Now have another go at finding $\dfrac{\partial z}{\partial y}$ in the same way.

Then check it with Frame 14

14

Here it is:

$$z = \frac{\sin(3x + 2y)}{xy}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{xy\cos(3x + 2y).(2) - \sin(3x + 2y).(x)}{x^2y^2}$$

$$= \frac{2y\cos(3x + 2y) - \sin(3x + 2y)}{xy^2}$$

That should have cleared up any troubles. This business of partial differentiation is perfectly straightforward. All you have to remember is that for the time being, all the independent variables except the one you are using are kept constant – and behave like constant factors or constant terms according to their positions.

On you go now to Frame 15 and continue the Programme

15

Right. Now let us move on a step.

Consider $z = 3x^2 + 4xy - 5y^2$

Then $\dfrac{\partial z}{\partial x} = 6x + 4y$ and $\dfrac{\partial z}{\partial y} = 4x - 10y$

The expression $\dfrac{\partial z}{\partial x} = 6x + 4y$ is itself a function of x and y. We could therefore find its partial derivatives with respect to x or to y.

▶

(a) If we differentiate it partially w.r.t. x, we get:

$\dfrac{\partial}{\partial x}\left\{\dfrac{\partial z}{\partial x}\right\}$ and this is written $\dfrac{\partial^2 z}{\partial x^2}$ (much like an ordinary second derivative, but with the partial ∂)

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(6x + 4y) = 6$$

This is called the second partial derivative of z with respect to x.

(b) If we differentiate partially w.r.t. y, we get:

$\dfrac{\partial}{\partial y}\left\{\dfrac{\partial z}{\partial x}\right\}$ and this is written $\dfrac{\partial^2 z}{\partial y.\partial x}$

Note that the operation now being performed is given by the left-hand of the two symbols in the denominator.

$$\frac{\partial^2 z}{\partial y.\partial x} = \frac{\partial}{\partial y}\left\{\frac{\partial z}{\partial x}\right\} = \frac{\partial}{\partial y}\{6x + 4y\} = 4$$

16

So we have this:

$$z = 3x^2 + 4xy - 5y^2$$

$$\frac{\partial z}{\partial x} = 6x + 4y \qquad \frac{\partial z}{\partial y} = 4x - 10y$$

$$\frac{\partial^2 z}{\partial x^2} = 6$$

$$\frac{\partial^2 z}{\partial y.\partial x} = 4$$

Of course, we could carry out similar steps with the expression for $\dfrac{\partial z}{\partial y}$ on the right. This would give us:

$$\frac{\partial^2 z}{\partial y^2} = -10$$

$$\frac{\partial^2 z}{\partial x.\partial y} = 4$$

Note that $\dfrac{\partial^2 z}{\partial y.\partial x}$ means $\dfrac{\partial}{\partial y}\left\{\dfrac{\partial z}{\partial x}\right\}$ so $\dfrac{\partial^2 z}{\partial x.\partial y}$ means ....

17

$$\boxed{\frac{\partial^2 z}{\partial x.\partial y} \text{ means } \frac{\partial}{\partial x}\left\{\frac{\partial z}{\partial y}\right\}}$$

Collecting our previous results together then, we have:

$z = 3x^2 + 4xy - 5y^2$

$$\frac{\partial z}{\partial x} = 6x + 4y \qquad \frac{\partial z}{\partial y} = 4x - 10y$$

$$\frac{\partial^2 z}{\partial x^2} = 6 \qquad \frac{\partial^2 z}{\partial y^2} = -10$$

$$\frac{\partial^2 z}{\partial y.\partial x} = 4 \qquad \frac{\partial^2 z}{\partial x.\partial y} = 4$$

We see in this case, that $\dfrac{\partial^2 z}{\partial y.\partial x} = \dfrac{\partial^2 z}{\partial x.\partial y}$. There are then, *two* first derivatives and *four* second derivatives, though the last two seem to have the same value.

Here is one for you to do.

If $z = 5x^3 + 3x^2y + 4y^3$, find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}, \dfrac{\partial^2 z}{\partial x.\partial y}$ and $\dfrac{\partial^2 z}{\partial y.\partial x}$

When you have completed all that, move to Frame 18

18

Here are the results:

$z = 5x^3 + 3x^2y + 4y^3$

$$\frac{\partial z}{\partial x} = 15x^2 + 6xy \qquad \frac{\partial z}{\partial y} = 3x^2 + 12y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 30x + 6y \qquad \frac{\partial^2 z}{\partial y^2} = 24y$$

$$\frac{\partial^2 z}{\partial y.\partial x} = 6x \qquad \frac{\partial^2 z}{\partial x.\partial y} = 6x$$

Again in this example also, we see that $\dfrac{\partial^2 z}{\partial y.\partial x} = \dfrac{\partial^2 z}{\partial x.\partial y}$. Now do this one.

It looks more complicated, but it is done in just the same way. Do not rush at it; take your time and all will be well. Here it is. Find all the first and second partial derivatives of $z = x\cos y - y\cos x$.

Then to Frame 19

19

Check your results with these.

$$z = x\cos y - y\cos x$$

When differentiating w.r.t. x, y is constant (and therefore $\cos y$ also).
When differentiating w.r.t. y, x is constant (and therefore $\cos x$ also).

So we get:

$$\frac{\partial z}{\partial x} = \cos y + y.\sin x \qquad \frac{\partial z}{\partial y} = -x.\sin y - \cos x$$

$$\frac{\partial^2 z}{\partial x^2} = y.\cos x \qquad \frac{\partial^2 z}{\partial y^2} = -x.\cos y$$

$$\frac{\partial^2 z}{\partial y.\partial x} = -\sin y + \sin x \qquad \frac{\partial^2 z}{\partial x.\partial y} = -\sin y + \sin x$$

And again, $\dfrac{\partial^2 z}{\partial y.\partial x} = \dfrac{\partial^2 z}{\partial x.\partial y}$

In fact this will always be so for the functions you are likely to meet, so that there are really *three* different second partial derivatives (and not four). In practice, if you have found $\dfrac{\partial^2 z}{\partial y.\partial x}$ it is a useful check to find $\dfrac{\partial^2 z}{\partial x.\partial y}$ separately. They should give the same result, of course.

20

What about this one?

If $V = \ln(x^2 + y^2)$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

This merely entails finding the two second partial derivatives and substituting them in the left-hand side of the statement. So here goes:

$$V = \ln(x^2 + y^2)$$

$$\frac{\partial V}{\partial x} = \frac{1}{(x^2 + y^2)}2x$$

$$= \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{(x^2 + y^2)2 - 2x.2x}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} \qquad \text{(a)}$$

Now you find $\dfrac{\partial^2 V}{\partial y^2}$ in the same way and hence prove the given identity.

When you are ready, move on to Frame 21

21

We had found that $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{2y^2 - 2x^2}{(x^2 + y^2)^2}$

So making a fresh start from $V = \ln(x^2 + y^2)$, we get:

$$\frac{\partial V}{\partial y} = \frac{1}{(x^2 + y^2)}.2y = \frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{(x^2 + y^2)2 - 2y.2y}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4y^2}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \qquad \text{(b)}$$

Substituting now the two results in the identity, gives:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$$

Now on to Frame 22

22

Here is another kind of example that you should see.

Example 1

If $V = f(x^2 + y^2)$, show that $x\dfrac{\partial V}{\partial y} - y\dfrac{\partial V}{\partial x} = 0$

Here we are told that V is a function of $(x^2 + y^2)$ but the precise nature of the function is not given. However, we can treat this as a 'function of a function' and write $f'(x^2 + y^2)$ to represent the derivative of the function w.r.t. its own combined variable $(x^2 + y^2)$.

$$\therefore \quad \frac{\partial V}{\partial x} = f'(x^2 + y^2) \times \frac{\partial}{\partial x}(x^2 + y^2) = f'(x^2 + y^2).2x$$

$$\frac{\partial V}{\partial y} = f'(x^2 + y^2).\frac{\partial}{\partial y}(x^2 + y^2) = f'(x^2 + y^2).2y$$

$$\therefore \quad x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x} = x.f'(x^2 + y^2).2y - y.f'(x^2 + y^2).2x$$

$$= 2xy.f'(x^2 + y^2) - 2xy.f'(x^2 + y^2)$$

$$= 0$$

Let us have another one of that kind in the next frame

23

Example 2

If $z = f\left\{\frac{y}{x}\right\}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Much the same as before:

$$\frac{\partial z}{\partial x} = f'\left\{\frac{y}{x}\right\}.\frac{\partial}{\partial x}\left\{\frac{y}{x}\right\} = f'\left\{\frac{y}{x}\right\}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2}f'\left\{\frac{y}{x}\right\}$$

$$\frac{\partial z}{\partial y} = f'\left\{\frac{y}{x}\right\}.\frac{\partial}{\partial y}\left\{\frac{y}{x}\right\} = f'\left\{\frac{y}{x}\right\}.\frac{1}{x} = \frac{1}{x}f'\left\{\frac{y}{x}\right\}$$

$$\therefore\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(-\frac{y}{x^2}\right)f'\left\{\frac{y}{x}\right\} + y\frac{1}{x}f'\left\{\frac{y}{x}\right\}$$

$$= -\frac{y}{x}f'\left\{\frac{y}{x}\right\} + \frac{y}{x}f'\left\{\frac{y}{x}\right\}$$

$$= 0$$

And one for you, just to get your hand in:

If $V = f(ax + by)$, show that $b\frac{\partial V}{\partial x} - a\frac{\partial V}{\partial y} = 0$

When you have done it, check your working against that in Frame 24

24

Here is the working; this is how it goes.

$$V = f(ax + by)$$

$$\therefore\ \frac{\partial V}{\partial x} = f'(ax + by).\frac{\partial}{\partial x}(ax + by)$$

$$= f'(ax + by).a = a.f'(ax + by) \qquad \text{(a)}$$

$$\frac{\partial z}{\partial y} = f'(ax + by).\frac{\partial}{\partial y}(ax + by)$$

$$= f'(ax + by).b = b.f'(ax + by) \qquad \text{(b)}$$

$$\therefore\ b\frac{\partial V}{\partial x} - a\frac{\partial V}{\partial y} = ab.f'(ax + by) - ab.f'(ax + by)$$

$$= 0$$

Move on to Frame 25

25

So to sum up so far.

Partial differentiation is easy, no matter how complicated the expression to be differentiated may seem.

To differentiate partially w.r.t. x, all independent variables other than x are constant for the time being.

To differentiate partially w.r.t. y, all independent variables other than y are constant for the time being.

So that, if z is a function of x and y, i.e. if $z = f(x, y)$, we can find:

$$\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} \quad \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial y.\partial x} \quad \frac{\partial^2 z}{\partial x.\partial y} \qquad \text{And also:} \qquad \frac{\partial^2 z}{\partial y.\partial x} = \frac{\partial^2 z}{\partial x.\partial y}$$

Now for a revision exercise

26

Revision exercise

1 Find all first and second partial derivatives for each of the following functions:

(a) $z - 3x^2 + 2xy + 4y^2$

(b) $z = \sin xy$

(c) $z = \dfrac{x+y}{x-y}$

2 If $z = \ln(e^x + e^y)$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.

3 If $z = x.f(xy)$, express $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$ in its simplest form.

When you have finished, check with the solutions in Frame 27

27

1 (a) $z = 3x^2 + 2xy + 4y^2$

$$\frac{\partial z}{\partial x} = 6x + 2y \qquad \frac{\partial z}{\partial y} = 2x + 8y$$

$$\frac{\partial^2 z}{\partial x^2} = 6 \qquad \frac{\partial^2 z}{\partial y^2} = 8$$

$$\frac{\partial^2 z}{\partial y.\partial x} = 2 \qquad \frac{\partial^2 z}{\partial x.\partial y} = 2$$

▶

(b) $z = \sin xy$

$$\frac{\partial z}{\partial x} = y\cos xy \qquad \frac{\partial z}{\partial y} = x\cos xy$$

$$\frac{\partial^2 z}{\partial x^2} = -y^2\sin xy \qquad \frac{\partial^2 z}{\partial y^2} = -x^2\sin xy$$

$$\frac{\partial^2 z}{\partial y.\partial x} = y(-x\sin xy) + \cos xy \qquad \frac{\partial^2 z}{\partial x.\partial y} = x(-y\sin xy) + \cos xy$$

$$= \cos xy - xy\sin xy \qquad = \cos xy - xy\sin xy$$

(c) $z = \dfrac{x+y}{x-y}$

$$\frac{\partial z}{\partial x} = \frac{(x-y)1-(x+y)1}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x-y)1-(x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = (-2y)\frac{(-2)}{(x-y)^3} = \frac{4y}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = 2x\frac{(-2)}{(x-y)^3}(-1) = \frac{4x}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial y.\partial x} = \frac{(x-y)^2(-2)-(-2y)2(x-y)(-1)}{(x-y)^4}$$

$$= \frac{-2(x-y)^2-4y(x-y)}{(x-y)^4}$$

$$= \frac{-2}{(x-y)^2} - \frac{4y}{(x-y)^3}$$

$$= \frac{-2x+2y-4y}{(x-y)^3} = \frac{-2x-2y}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial x.\partial y} = \frac{(x-y)^2(2)-2x.2(x-y)1}{(x-y)^4}$$

$$= \frac{2(x-y)^2-4x(x-y)}{(x-y)^4}$$

$$= \frac{2}{(x-y)^2} - \frac{4x}{(x-y)^3}$$

$$= \frac{2x-2y-4x}{(x-y)^3} = \frac{-2x-2y}{(x-y)^3}$$

▶

2 $z = \ln(e^x + e^y)$

$$\frac{\partial z}{\partial x} = \frac{1}{e^x + e^y}.e^x \qquad \frac{\partial z}{\partial y} = \frac{1}{e^x + e^y}.e^y$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{e^x}{e^x + e^y} + \frac{e^y}{e^x + e^y}$$

$$= \frac{e^x + e^y}{e^x + e^y} = 1$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

3 $z = x.f(xy)$

$$\frac{\partial z}{\partial x} = x.f'(xy).y + f(xy)$$

$$\frac{\partial z}{\partial y} = x.f'(xy).x$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x^2yf'(xy) + xf(xy) - x^2yf'(xy)$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xf(xy) = z$$

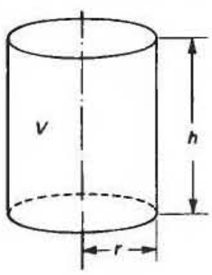
That was a pretty good revision test. Do not be unduly worried if you made a slip or two in your working. Try to avoid doing so, of course, but you are doing fine. Now on to the next part of the Programme.

So far we have been concerned with the technique of partial differentiation. Now let us look at one of its applications.

So move on to Frame 28

Small increments

28



If we return to the volume of the cylinder with which we started this Programme, we have once again that $V = \pi r^2 h$. We have seen that we can find $\frac{\partial V}{\partial r}$ with h constant, and $\frac{\partial V}{\partial h}$ with r constant.

$$\frac{\partial V}{\partial r} = 2\pi rh; \quad \frac{\partial V}{\partial h} = \pi r^2$$

Now let us see what we get if r and h both change simultaneously.

If r becomes $r + \delta r$, and h becomes $h + \delta h$, let V become $V + \delta V$. Then the new volume is given by:

$$\begin{aligned} V + \delta V &= \pi(r + \delta r)^2(h + \delta h) \\ &= \pi(r^2 + 2r\delta r + [\delta r]^2)(h + \delta h) \\ &= \pi(r^2 h + 2rh\delta r + h[\delta r]^2 + r^2\delta h + 2r\delta r\delta h + [\delta r]^2\delta h) \end{aligned}$$

Subtract $V = \pi r^2 h$ from each side, giving:

$$\delta V = \pi(2rh\delta r + h[\delta r]^2 + r^2\delta h + 2r\delta r\delta h + [\delta r]^2\delta h)$$

$$\approx \pi(2rh\delta r + r^2\delta h)$$ since δr and δh are small and all the remaining terms are of a higher degree of smallness.

Therefore

$$\delta V \approx 2\pi rh\delta r + \pi r^2\delta h, \quad \text{that is:}$$

$$\delta V \approx \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h$$

Let us now do a numerical example to see how it all works out.

On to Frame 29

29

A cylinder has dimensions $r = 5$ cm, $h = 10$ cm. Find the approximate increase in volume when r increases by 0·2 cm and h decreases by 0·1 cm. Well now

$$V = \pi r^2 h \text{ so } \frac{\partial V}{\partial r} = 2\pi rh \qquad \frac{\partial V}{\partial h} = \pi r^2$$

In this case, when $r = 5$ cm, $h = 10$ cm so

$$\frac{\partial V}{\partial r} = 2\pi 5.10 = 100\pi \qquad \frac{\partial V}{\partial h} = \pi r^2 = \pi 5^2 = 25\pi$$

$\delta r = 0{\cdot}2$ and $\delta h = -0{\cdot}1$ (minus because h is decreasing)

$$\therefore\ \delta V \approx \frac{\partial V}{\partial r}.\delta r + \frac{\partial V}{\partial h}.\delta h$$

$$\begin{aligned} \delta V &= 100\pi(0{\cdot}2) + 25\pi(-0{\cdot}1) \\ &= 20\pi - 2{\cdot}5\pi = 17{\cdot}5\pi \end{aligned}$$

$$\therefore\ \delta V \approx 54{\cdot}98 \text{ cm}^3$$

i.e. the volume increases by $54{\cdot}98$ cm^3

Just like that!

30

This kind of result applies not only to the volume of the cylinder, but to any function of two independent variables. Here is an example:

If z is a function of x and y, i.e. $z = f(x, y)$ and if x and y increase by small amounts δx and δy, the increase δz will also be relatively small. If we expand δz in powers of δx and δy, we get:

$\delta z = A\delta x + B\delta y +$ higher powers of δx and δy,

where A and B are functions of x and y.

If y remains constant, so that $\delta y = 0$, then:

$$\delta z = A\delta x + \text{higher powers of } \delta x$$

$$\therefore \quad \frac{\delta z}{\delta x} = A. \text{ So that if } \delta x \to 0, \text{ this becomes } A = \frac{\partial z}{\partial x}$$

Similarly, if x remains constant, making $\delta y \to 0$ gives $B = \dfrac{\partial z}{\partial y}$

$$\therefore \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \text{ higher powers of very small quantities which can be ignored}$$

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

31

So, if $z = f(x, y)$

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

This is the key to all the forthcoming applications and will be quoted over and over again.

The result is quite general and a similar result applies for a function of three independent variables. For example:

$$\text{If } z = f(x, y, w)$$

$$\text{then } \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \frac{\partial z}{\partial w}\delta w$$

If we remember the rule for a function of two independent variables, we can easily extend it when necessary.

Here it is once again:

$$\text{If } z = f(x, y) \text{ then } \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

Copy this result into your record book in a prominent position, such as it deserves!

32

Now for a couple of examples

Example 1

If $I = \frac{V}{R}$, and $V = 250$ volts and $R = 50$ ohms, find the change in I resulting from an increase of 1 volt in V and an increase of 0·5 ohm in R.

$$I = f(V, R) \qquad \therefore \ \delta I = \frac{\partial I}{\partial V}\delta V + \frac{\partial I}{\partial R}\delta R$$

$$\frac{\partial I}{\partial V} = \frac{1}{R} \text{ and } \frac{\partial I}{\partial R} = -\frac{V}{R^2}$$

$$\therefore \ \delta I = \frac{1}{R}\delta V - \frac{V}{R^2}\delta R$$

So when $R = 50$, $V = 250$, $\delta V = 1$ and $\delta R = 0{\cdot}5$:

$$\begin{aligned}\delta I &= \frac{1}{50}(1) - \frac{250}{2500}(0{\cdot}5)\\ &= \frac{1}{50} - \frac{1}{20}\\ &= 0{\cdot}02 - 0{\cdot}05 = -0{\cdot}03\end{aligned}$$

i.e. I decreases by 0·03 amperes

33

Here is another example.

Example 2

If $y = \frac{ws^3}{d^4}$, find the percentage increase in y when w increases by 2 per cent, s decreases by 3 per cent and d increases by 1 per cent.

Notice that, in this case, y is a function of three variables, w, s and d. The formula therefore becomes:

$$\delta y = \frac{\partial y}{\partial w}\delta w + \frac{\partial y}{\partial s}\delta s + \frac{\partial y}{\partial d}\delta d$$

We have

$$\frac{\partial y}{\partial w} = \frac{s^3}{d^4}; \quad \frac{\partial y}{\partial s} = \frac{3ws^2}{d^4}; \quad \frac{\partial y}{\partial d} = -\frac{4ws^3}{d^5}$$

$$\therefore \ \delta y = \frac{s^3}{d^4}\delta w + \frac{3ws^2}{d^4}\delta s + \frac{-4ws^3}{d^5}\delta d$$

Now then, what are the values of δw, δs and δd?

Is it true to say that $\delta w = \frac{2}{100}$; $\delta s = \frac{-3}{100}$; $\delta d = \frac{1}{100}$?

If not, why not?

Next frame

34

No. It is not correct

Because δw is not $\frac{2}{100}$ of a unit, but 2 per cent of w, i.e. $\delta w = \frac{2}{100}$ of $w = \frac{2w}{100}$

Similarly, $\delta s = \frac{-3}{100}$ of $s = \frac{-3s}{100}$ and $\delta d = \frac{d}{100}$. Now that we have cleared that point up, we can continue with the problem.

$$\delta y = \frac{s^3}{d^4}\left(\frac{2w}{100}\right) + \frac{3ws^2}{d^4}\left(\frac{-3s}{100}\right) - \frac{4ws^3}{d^5}\left(\frac{d}{100}\right)$$

$$= \frac{ws^3}{d^4}\left(\frac{2}{100}\right) - \frac{ws^3}{d^4}\left(\frac{9}{100}\right) - \frac{ws^3}{d^4}\left(\frac{4}{100}\right)$$

$$= \frac{ws^3}{d^4}\left\{\frac{2}{100} - \frac{9}{100} - \frac{4}{100}\right\}$$

$$= y\left\{-\frac{11}{100}\right\} = -11 \text{ per cent of } y$$

i.e. y decreases by 11 per cent

Remember that where the increment of w is given as 2 per cent, it is *not* $\frac{2}{100}$ of a unit, but $\frac{2}{100}$ of w, and the symbol w must be included.

Move on to Frame 35

35

Now here is an exercise for you to do.

$P = w^2hd$. If errors of up to 1 per cent (plus or minus) are possible in the measured values of w, h and d, find the maximum possible percentage error in the calculated values of P.

This is very much like the previous example, so you will be able to deal with it without any trouble. Work it right through and then go on to Frame 36 and check your result.

36

$P = w^2hd. \quad \therefore \ \delta P = \frac{\partial P}{\partial w}.\delta w + \frac{\partial P}{\partial h}.\delta h + \frac{\partial P}{\partial d}.\delta d$

$\frac{\partial P}{\partial w} = 2whd; \quad \frac{\partial P}{\partial h} = w^2d; \quad \frac{\partial P}{\partial d} = w^2h$

$\delta P = 2whd.\delta w + w^2d.\delta h + w^2h.\delta d$

Now $\quad \delta w = \pm\frac{w}{100}; \quad \delta h = \pm\frac{h}{100}, \quad \delta d = \pm\frac{d}{100}$

$$\delta P = 2whd\left(\pm\frac{w}{100}\right) + w^2d\left(\pm\frac{h}{100}\right) + w^2h\left(\pm\frac{d}{100}\right)$$

$$= \pm\frac{2w^2hd}{100} \pm \frac{w^2dh}{100} \pm \frac{w^2hd}{100}$$

The greatest possible error in P will occur when the signs are chosen so that they are all of the same kind, i.e. all plus or minus. If they were mixed, they would tend to cancel each other out.

$$\therefore\ \delta P = \pm w^2 hd\left\{\frac{2}{100}+\frac{1}{100}+\frac{1}{100}\right\} = \pm P\left(\frac{4}{100}\right)$$

$\therefore$ Maximum possible error in P is 4 per cent of P

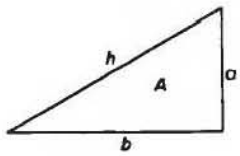
Finally, here is one last example for you to do. Work right through it and then check your results with those in Frame 37.

The two sides forming the right-angle of a right-angled triangle are denoted by a and b. The hypotenuse is h. If there are possible errors of $\pm 0{\cdot}5$ per cent in measuring a and b, find the maximum possible error in calculating (a) the area of the triangle and (b) the length of h.

37

(a) $\delta A = 1$ per cent of A
(b) $\delta h = 0{\cdot}5$ per cent of h

Here is the working in detail:



(a) $A = \dfrac{a.b}{2} \qquad \delta A = \dfrac{\partial A}{\partial a}.\delta a + \dfrac{\partial A}{\partial b}.\delta b$

$$\frac{\partial A}{\partial a} = \frac{b}{2};\quad \frac{\partial A}{\partial b} = \frac{a}{2};\quad \delta a = \pm\frac{a}{200};\quad \delta b = \pm\frac{b}{200}$$

$$\delta A = \frac{b}{2}\left(\pm\frac{a}{200}\right) + \frac{a}{2}\left(\pm\frac{b}{200}\right)$$

$$= \pm\frac{a.b}{2}\left[\frac{1}{200}+\frac{1}{200}\right] = \pm A.\frac{1}{100}$$

$\therefore\ \delta A = 1$ per cent of A

(b) $h = \sqrt{a^2+b^2} = (a^2+b^2)^{\frac{1}{2}}$

$$\delta h = \frac{\partial h}{\partial a}\delta a + \frac{\partial h}{\partial b}\delta b$$

$$\frac{\partial h}{\partial a} = \frac{1}{2}(a^2+b^2)^{-\frac{1}{2}}(2a) = \frac{a}{\sqrt{a^2+b^2}}$$

$$\frac{\partial h}{\partial b} = \frac{1}{2}(a^2+b^2)^{-\frac{1}{2}}(2b) = \frac{b}{\sqrt{a^2+b^2}}$$

Also $\quad \delta a = \pm\dfrac{a}{200};\quad \delta b = \pm\dfrac{b}{200}$

$$\therefore\ \delta h = \frac{a}{\sqrt{a^2+b^2}}\left(\pm\frac{a}{200}\right) + \frac{b}{\sqrt{a^2+b^2}}\left(\pm\frac{b}{200}\right)$$

$$= \pm\frac{1}{200}\frac{a^2+b^2}{\sqrt{a^2+b^2}}$$

$$= \pm\frac{1}{200}\sqrt{a^2+b^2} = \pm\frac{1}{200}(h)$$

$\therefore\ \delta h = 0{\cdot}5$ per cent of h

That brings us to the end of this particular Programme. We shall meet partial differentiation again in the next Programme when we shall consider some more of its applications. But for the time being, there remain only the **Can You?** checklist and the **Test exercise**.

So on now to Frames 38 and 39

Can You?

38

Checklist 10

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Find the first partial derivatives of a function of two real variables? 1 to 14
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Find second-order partial derivatives of a function of two real variables? 15 to 25
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Calculate errors using partial differentiation? 28 to 37
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 10

39

Take your time over the questions; do them carefully.

1 Find all first and second partial derivatives of the following:

(a) $z = 4x^3 - 5xy^2 + 3y^3$

(b) $z = \cos(2x + 3y)$

(c) $z = e^{x^2 - y^2}$

(d) $z = x^2 \sin(2x + 3y)$

2 (a) If $V = x^2 + y^2 + z^2$, express in its simplest form $x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} + z\dfrac{\partial V}{\partial z}$.

(b) If $z = f(x + ay) + F(x - ay)$, find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ and hence prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \cdot \dfrac{\partial^2 z}{\partial x^2}$.

▶

3 The power P dissipated in a resistor is given by $P = \dfrac{E^2}{R}$.
If $E = 200$ volts and $R = 8$ ohms, find the change in P resulting from a drop of 5 volts in E and an increase of 0·2 ohm in R.

4 If $\theta = kHLV^{-\frac{1}{2}}$, where k is a constant, and there are possible errors of ± 1 per cent in measuring H, L and V, find the maximum possible error in the calculated value of θ.

That's it

Further problems 10

40

1 If $z = \dfrac{1}{x^2 + y^2 - 1}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -2z(1 + z)$.

2 Prove that, if $V = \ln(x^2 + y^2)$, then $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$.

3 If $z = \sin(3x + 2y)$, verify that $3\dfrac{\partial^2 z}{\partial y^2} - 2\dfrac{\partial^2 z}{\partial x^2} = 6z$.

4 If $u = \dfrac{x + y + z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

5 Show that the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$, is satisfied by

$$z = \ln\sqrt{x^2 + y^2} + \frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)$$

6 If $z = e^x(x\cos y - y\sin y)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

7 If $u = (1 + x)\sinh(5x - 2y)$, verify that $4\dfrac{\partial^2 u}{\partial x^2} + 20\dfrac{\partial^2 u}{\partial x.\partial y} + 25\dfrac{\partial^2 u}{\partial y^2} = 0$.

8 If $z = f\left(\dfrac{y}{x}\right)$, show that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x.\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0$.

9 If $z = (x + y).f\left(\dfrac{y}{x}\right)$, where f is an arbitrary function, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

10 In the formula $D = \dfrac{Eh^3}{12(1 - \nu^2)}$, h is given as $0{\cdot}1 \pm 0{\cdot}002$ and ν as $0{\cdot}3 \pm 0{\cdot}02$. Express the approximate maximum error in D in terms of E.

11 The formula $z = \dfrac{a^2}{x^2 + y^2 - a^2}$ is used to calculate z from observed values of x and y. If x and y have the same percentage error p, show that the percentage error in z is approximately $-2p(1 + z)$.

12 In a balanced bridge circuit, $R_1 = R_2R_3/R_4$. If R_2, R_3, R_4 have known tolerances of $\pm x$ per cent, $\pm y$ per cent, $\pm z$ per cent respectively, determine the maximum percentage error in R_1, expressed in terms of x, y and z.

13 The deflection y at the centre of a circular plate suspended at the edge and uniformly loaded is given by $y = \dfrac{kwd^4}{t^3}$, where w = total load, d = diameter of plate, t = thickness and k is a constant.

Calculate the approximate percentage change in y if w is increased by 3 per cent, d is decreased by $2\frac{1}{2}$ per cent and t is increased by 4 per cent.

14 The coefficient of rigidity (n) of a wire of length (L) and uniform diameter (d) is given by $n = \dfrac{AL}{d^4}$, where A is a constant. If errors of $\pm 0{\cdot}25$ per cent and ± 1 per cent are possible in measuring L and d respectively, determine the maximum percentage error in the calculated value of n.

15 If $k/k_0 = (T/T_0)^n.p/760$, show that the change in k due to small changes of a per cent in T and b per cent in p is approximately $(na + b)$ per cent.

16 The deflection y at the centre of a rod is known to be given by $y = \dfrac{kwl^3}{d^4}$, where k is a constant. If w increases by 2 per cent, l by 3 per cent, and d decreases by 2 per cent, find the percentage increase in y.

17 The displacement y of a point on a vibrating stretched string, at a distance x from one end, at time t, is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2.\frac{\partial^2 y}{\partial x^2}$$

Show that one solution of this equation is $y = A\sin\dfrac{px}{c}.\sin(pt + a)$, where A, p, c and a are constants.

18 If $y = A\sin(px + a)\cos(qt + b)$, find the error in y due to small errors δx and δt in x and t respectively.

19 Show that $\phi = Ae^{-kt/2}\sin pt\cos qx$, satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\left\{\frac{\partial^2\phi}{\partial t^2} + k\frac{\partial\phi}{\partial t}\right\}, \text{ provided that } p^2 = c^2q^2 - \frac{k^2}{4}.$$

20 Show that (a) the equation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$ is satisfied by $V = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, and that (b) the equation $\dfrac{\partial^2 V}{\partial x^2} + + \dfrac{\partial^2 V}{\partial y^2} = 0$ is satisfied by $V = \tan^{-1}\left(\dfrac{y}{x}\right)$.

Partial differentiation 2

Frames 1 to 32

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the first- and second-order partial derivatives of a function of two real variables
- Apply partial differentiation to rate-of-change problems
- Apply partial differentiation to change-of-variable problems

Partial differentiation

1

In the first part of the Programme on partial differentiation, we established a result which, we said, would be the foundation of most of the applications of partial differentiation to follow.

You surely remember it: it went like this:

If z is a function of two independent variables, x and y, i.e. if $z = f(x, y)$, then

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

We were able to use it, just as it stands, to work out certain problems on small increments, errors and tolerances. It is also the key to much of the work of this Programme, so copy it down into your record book, thus:

If $z = f(x, y)$ then $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$

2

If $z = f(x, y)$, then $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$

In this expression, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are the partial derivatives of z with respect to x and y respectively, and you will remember that to find:

(a) $\frac{\partial z}{\partial x}$, we differentiate the function z, with respect to x, keeping all independent variables other than x, for the time being, ...

(b) $\frac{\partial z}{\partial y}$, we differentiate the function z with respect to y, keeping all independent variables other than y, for the time being, ...

3

constant (in both cases)

An example, just to remind you:

If $\quad z = x^3 + 4x^2y - 3y^3$

then $\quad \frac{\partial z}{\partial x} = 3x^2 + 8xy - 0 \qquad$ (y is constant)

and $\quad \frac{\partial z}{\partial y} = 0 + 4x^2 - 9y^2 \qquad$ (x is constant)

In practice, of course, we do not write down the zero terms.

Before we tackle any further applications, we must be expert at finding partial derivatives, so with the reminder above, have a go at this one.

(1) If $z = \tan(x^2 - y^2)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

When you have finished it, check with the next frame

4

$$\boxed{\frac{\partial z}{\partial x} = 2x\sec^2(x^2 - y^2); \quad \frac{\partial z}{\partial y} = -2y\sec^2(x^2 - y^2)}$$

Because $\quad z = \tan(x^2 - y^2)$

$$\therefore \quad \frac{\partial z}{\partial x} = \sec^2(x^2 - y^2) \times \frac{\partial}{\partial x}(x^2 - y^2)$$
$$= \sec^2(x^2 - y^2)(2x) = 2x\sec^2(x^2 - y^2)$$

and
$$\frac{\partial z}{\partial y} = \sec^2(x^2 - y^2) \times \frac{\partial}{\partial y}(x^2 - y^2)$$
$$= \sec^2(x^2 - y^2)(-2y) = -2y\sec^2(x^2 - y^2)$$

That was easy enough. Now do this one:

(2) If $z = e^{2x-3y}$, find $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}, \dfrac{\partial^2 z}{\partial x.\partial y}$

Finish them all. Then move on to Frame 5 and check your results

5

Here are the results in detail:

$z = e^{2x-3y}$
$$\therefore \quad \frac{\partial z}{\partial x} = e^{2x-3y}.2 = 2.e^{2x-3y}$$
$$\frac{\partial z}{\partial y} = e^{2x-3y}(-3) = -3.e^{2x-3y}$$
$$\frac{\partial^2 z}{\partial x^2} = 2.e^{2x-3y}.2 = 4.e^{2x-3y}$$
$$\frac{\partial^2 z}{\partial y^2} = -3.e^{2x-3y}(-3) = 9.e^{2x-3y}$$
$$\frac{\partial^2 z}{\partial x.\partial y} = -3.e^{2x-3y}.2 = -6.e^{2x-3y}$$

All correct?

You remember, too, that in the 'mixed' second partial derivative, the order of differentiating does not matter. So in this case, since

$$\frac{\partial^2 z}{\partial x.\partial y} = -6.e^{2x-3y}, \text{ then } \frac{\partial^2 z}{\partial y.\partial x} = \ldots\ldots\ldots\ldots$$

6

$$\boxed{\frac{\partial^2 z}{\partial x.\partial y} = \frac{\partial^2 z}{\partial y.\partial x} = -6.e^{2x-3y}}$$

Well now, before we move on to new work, see what you make of these. Find all the first and second partial derivatives of the following:

(a) $z = x\sin y$
(b) $z = (x + y)\ln(xy)$

When you have found all the derivatives, check your work with the solutions in the next frame

7

Here they are. Check your results carefully.

(a) $z = x\sin y$

$$\therefore \quad \frac{\partial z}{\partial x} = \sin y \qquad \frac{\partial z}{\partial y} = x\cos y$$

$$\frac{\partial^2 z}{\partial x^2} = 0 \qquad \frac{\partial^2 z}{\partial y^2} = -x\sin y$$

$$\frac{\partial^2 z}{\partial y.\partial x} = \cos y \qquad \frac{\partial^2 z}{\partial x.\partial y} = \cos y$$

(b) $z = (x+y)\ln(xy)$

$$\therefore \quad \frac{\partial z}{\partial x} = (x+y)\frac{1}{xy}.y + \ln(xy) = \frac{(x+y)}{x} + \ln(xy)$$

$$\frac{\partial z}{\partial y} = (x+y)\frac{1}{xy}.x + \ln(xy) = \frac{(x+y)}{y} + \ln(xy)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \frac{x-(x+y)}{x^2} + \frac{1}{xy}.y = \frac{\cancel{x}-\cancel{x}-y}{x^2} + \frac{1}{x}$$

$$= \frac{x-y}{x^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{y-(x+y)}{y^2} + \frac{1}{xy}.x = \frac{\cancel{y}-x-\cancel{y}}{y^2} + \frac{1}{y}$$

$$= \frac{y-x}{y^2}$$

$$\frac{\partial^2 z}{\partial y.\partial x} = \frac{1}{x} + \frac{1}{xy}.x = \frac{1}{x} + \frac{1}{y}$$

$$= \frac{y+x}{xy}$$

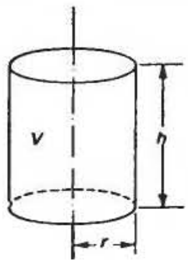
$$\frac{\partial^2 z}{\partial x.\partial y} = \frac{1}{y} + \frac{1}{xy}.y = \frac{1}{y} + \frac{1}{x}$$

$$= \frac{x+y}{xy}$$

Well now, that was just by way of warming up with work you have done before. Let us now move on to the next section of this Programme.

Rate-of-change problems

8



Let us consider a cylinder of radius r and height h as before. Then the volume is given by

$$V = \pi r^2 h$$

$$\therefore \quad \frac{\partial V}{\partial r} = 2\pi r h \text{ and } \frac{\partial V}{\partial h} = \pi r^2$$

Since V is a function of r and h, we also know that

$$\delta V = \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h$$

(Here it is, popping up again!)

Now divide both sides by δt: $\dfrac{\delta V}{\delta t} = \dfrac{\partial V}{\partial r}\cdot\dfrac{\delta r}{\delta t} + \dfrac{\partial V}{\partial h}\cdot\dfrac{\delta h}{\delta t}$

Then if $\delta t \to 0$, $\dfrac{\delta V}{\delta t} \to \dfrac{\mathrm{d}V}{\mathrm{d}t}$, $\dfrac{\delta r}{\delta t} \to \dfrac{\mathrm{d}r}{\mathrm{d}t}$, $\dfrac{\delta h}{\delta t} \to \dfrac{\mathrm{d}h}{\mathrm{d}t}$, but the partial derivatives, which do not contain δt, will remain unchanged.

So our result now becomes $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \ldots\ldots\ldots\ldots$

9

$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial r}\cdot\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\cdot\frac{\mathrm{d}h}{\mathrm{d}t}}$$

This result is really the key to problems of the kind we are about to consider. If we know the rate at which r and h are changing, we can now find the corresponding rate of change of V. Like this:

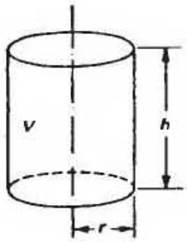
Example 1

The radius of a cylinder increases at the rate of 0·2 cm/s while the height decreases at the rate of 0·5 cm/s. Find the rate at which the volume is changing at the instant when $r = 8$ cm and $h = 12$ cm.

Warning: The first inclination is to draw a diagram and to put in the given values for its dimensions, i.e. $r = 8$ cm, $h = 12$ cm. This we *must NOT do*, for the radius and height are changing and the given values are instantaneous values only. Therefore on the diagram we keep the symbols r and h to indicate that they are variables.

10

Here it is then:



$$V = \pi r^2 h$$

$$\delta V = \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h$$

$$\therefore \quad \frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$\frac{\partial V}{\partial r} = 2\pi rh; \quad \frac{\partial V}{\partial h} = \pi r^2$$

$$\therefore \quad \frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$$

Now at the instant we are considering:

$r = 8$, $h = 12$, $\frac{dr}{dt} = 0{\cdot}2$, $\frac{dh}{dt} = -0{\cdot}5$ (minus since h is decreasing)

So you can now substitute these values in the last statement and finish off the calculation, giving:

$\frac{dV}{dt} = \ldots\ldots\ldots\ldots$

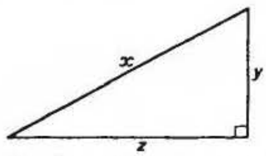
11

$$\boxed{\frac{dV}{dt} = 20{\cdot}1 \text{ cm}^3/\text{s}}$$

$$\begin{aligned} \text{Because} \quad \frac{dV}{dt} &= 2\pi rh.\frac{dr}{dt} + \pi r^2\frac{dh}{dt} \\ &= 2\pi 8.12.(0{\cdot}2) + \pi 64(-0{\cdot}5) \\ &= 38{\cdot}4\pi - 32\pi \\ &= 6{\cdot}4\pi = 20{\cdot}1 \text{ cm}^3/\text{s} \end{aligned}$$

Now another one.

Example 2



In the right-angled triangle shown, x is increasing at 2 cm/s while y is decreasing at 3 cm/s. Calculate the rate at which z is changing when $x = 5$ cm and $y = 3$ cm.

The first thing to do, of course, is to express z in terms of x and y. That is not difficult.

$z = \ldots\ldots\ldots\ldots$

12

$$\boxed{z = \sqrt{x^2 - y^2}}$$

$$z = \sqrt{x^2 - y^2} = (x^2 - y^2)^{\frac{1}{2}}$$

$$\therefore\ \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y \qquad \text{(The key to the whole business)}$$

$$\therefore\ \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

In this case $\dfrac{\partial z}{\partial x} = \dfrac{1}{2}(x^2 - y^2)^{-\frac{1}{2}}(2x) = \dfrac{x}{\sqrt{x^2 - y^2}}$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2 - y^2)^{-\frac{1}{2}}(-2y) = \frac{-y}{\sqrt{x^2 - y^2}}$$

$$\frac{dz}{dt} = \frac{x}{\sqrt{x^2 - y^2}}\cdot\frac{dx}{dt} - \frac{y}{\sqrt{x^2 - y^2}}\cdot\frac{dy}{dt}$$

So far so good. Now for the numerical values:

$$x = 5,\ y = 3,\ \frac{dx}{dt} = 2,\ \frac{dy}{dt} = -3$$

$$\frac{dz}{dt} = \ldots\ldots\ldots\ldots$$

Finish it off, then move to Frame 13

13

$$\boxed{\frac{dz}{dt} = 4{\cdot}75\text{ cm/s}}$$

Because we have $\dfrac{dz}{dt} = \dfrac{5}{\sqrt{5^2 - 3^2}}(2) - \dfrac{3}{\sqrt{5^2 - 3^2}}(-3)$

$$= \frac{5(2)}{4} + \frac{3(3)}{4} = \frac{10}{4} + \frac{9}{4} = \frac{19}{4} = 4{\cdot}75\text{ cm/s}$$

$\therefore$ Side z increases at the rate of 4·75 cm/s

Now here is

Example 3

The total surface area S of a cone of base radius r and perpendicular height h is given by

$$S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$$

If r and h are each increasing at the rate of 0·25 cm/s, find the rate at which S is increasing at the instant when $r = 3$ cm and $h = 4$ cm.

Do that one entirely on your own. Take your time: there is no need to hurry. Be quite sure that each step you write down is correct.

Then move to Frame 14 and check your result

14

Here is the solution in detail:

$$S = \pi r^2 + \pi r\sqrt{r^2 + h^2} = \pi r^2 + \pi r(r^2 + h^2)^{\frac{1}{2}}$$

$$\delta S = \frac{\partial S}{\partial r}.\delta r + \frac{\partial S}{\partial h}.\delta h \quad \therefore \quad \frac{dS}{dt} = \frac{\partial S}{\partial r}.\frac{dr}{dt} + \frac{\partial S}{\partial h}.\frac{dh}{dt}$$

(1) $$\frac{\partial S}{\partial r} = 2\pi r + \pi r.\frac{1}{2}(r^2 + h^2)^{-\frac{1}{2}}(2r) + \pi(r^2 + h^2)^{\frac{1}{2}}$$

$$= 2\pi r + \frac{\pi r^2}{\sqrt{r^2 + h^2}} + \pi\sqrt{r^2 + h^2}$$

When $r = 3$ and $h = 4$:

$$\frac{\partial S}{\partial r} = 2\pi 3 + \frac{\pi 9}{5} + \pi 5 = 11\pi + \frac{9\pi}{5} = \frac{64\pi}{5}$$

(2) $$\frac{\partial S}{\partial h} = \pi r\frac{1}{2}(r^2 + h^2)^{-\frac{1}{2}}(2h) = \frac{\pi rh}{\sqrt{r^2 + h^2}}$$

$$= \frac{\pi 3.4}{5} = \frac{12\pi}{5}$$

Also we are given that $\frac{dr}{dt} = 0{\cdot}25$ and $\frac{dh}{dt} = 0{\cdot}25$

$$\therefore \quad \frac{dS}{dt} = \frac{64\pi}{5}.\frac{1}{4} + \frac{12\pi}{5}.\frac{1}{4}$$

$$= \frac{16\pi}{5} + \frac{3\pi}{5} = \frac{19\pi}{5}$$

$$= 3{\cdot}8\pi = 11{\cdot}94 \text{ cm}^2/\text{s}$$

15

So there we are. Rate-of-change problems are all very much the same. What you must remember is simply this:

(a) The basic statement

$$\text{If } z = f(x, y) \text{ then } \delta z = \frac{\partial z}{\partial x}.\delta x + \frac{\partial z}{\partial y}.\delta y \qquad \text{(a)}$$

(b) Divide this result by δt and make $\delta t \to 0$. This converts the result into the form for rate-of-change problems:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt} \qquad \text{(b)}$$

The second result follows directly from the first. Make a note of both of these in your record book for future reference.

Then for the next part of the work, move on to Frame 16

16

Partial differentiation can also be used with advantage in finding *derivatives of implicit functions*.

For example, suppose we are required to find an expression for $\frac{dy}{dx}$ when we are given that $x^2 + 2xy + y^3 = 0$.

We can set about it in this way:

Let z stand for the function of x and y, i.e. $z = x^2 + 2xy + y^3$. Again we use the basic relationship $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$.

If we divide both sides by δx, we get:

$$\frac{\delta z}{\delta x} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{\delta y}{\delta x}$$

Now, if $\delta x \to 0$, $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$

If we now find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we shall be quite a way towards finding $\frac{dy}{dx}$ (which you see at the end of the expression).

In this particular example, where $z = x^2 + 2xy + y^3$, $\frac{\partial z}{\partial x} = \dots\dots\dots\dots$ and $\frac{\partial z}{\partial y} = \dots\dots\dots\dots$

17

$$\boxed{\frac{\partial z}{\partial x} = 2x + 2y; \quad \frac{\partial z}{\partial y} = 2x + 3y^2}$$

Substituting these in our previous result gives us:

$$\frac{dz}{dx} = (2x + 2y) + (2x + 3y^2)\frac{dy}{dx}$$

If only we knew $\frac{dz}{dx}$, we could rearrange this result and obtain an expression for $\frac{dy}{dx}$. So where can we find out something about $\frac{dz}{dx}$?

Refer back to the beginning of the problem. We have used z to stand for $x^2 + 2xy + y^3$ and we were told initially that $x^2 + 2xy + y^3 = 0$. Therefore $z = 0$, i.e. z is a constant (in this case zero) and hence $\frac{dz}{dx} = 0$.

$$\therefore \; 0 = (2x + 2y) + (2x + 3y^2)\frac{dy}{dx}$$

From this we can find $\frac{dy}{dx}$. So finish it off.

$$\frac{dy}{dx} = \dots\dots\dots\dots$$

On to Frame 18

18

$$\boxed{\frac{dy}{dx} = -\frac{2x+2y}{2x+3y^2}}$$

This is almost a routine that always works. In general, we have:

If $f(x, y) = 0$, find $\dfrac{dy}{dx}$

Let $z = f(x, y)$ then $\delta z = \dfrac{\partial z}{\partial x}\delta x + \dfrac{\partial z}{\partial y}\delta y$. Divide by δx and make $\delta x \to 0$, in which case:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}.\frac{dy}{dx}$$

But $z = 0$ (constant) $\therefore\ \dfrac{dz}{dx} = 0 \quad \therefore\ 0 = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}.\dfrac{dy}{dx}$

giving $\dfrac{dy}{dx} = -\dfrac{\partial z}{\partial x}\Big/\dfrac{\partial z}{\partial y}$

The easiest form to remember is the one that comes direct from the basic result:

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

Divide by δx, etc.

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}.\frac{dy}{dx} \quad \left\{\frac{dz}{dx} = 0\right\}$$

Make a note of this result.

19

Now for some examples.

Example 1

If $e^{xy} + x + y = 1$, evaluate $\dfrac{dy}{dx}$ at (0, 0). The function can be written $e^{xy} + x + y - 1 = 0$.

$$\text{Let } z = e^{xy} + x + y - 1 \qquad \delta z = \frac{\partial z}{\partial x}.\delta x + \frac{\partial z}{\partial y}.\delta y \qquad \therefore\ \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}.\frac{dy}{dx}$$

$$\frac{\partial z}{\partial x} = e^{xy}.y + 1; \quad \frac{\partial z}{\partial y} = e^{xy}.x + 1 \qquad \therefore\ \frac{dz}{dx} = (y.e^{xy} + 1) + (x.e^{xy} + 1)\frac{dy}{dx}$$

$$\text{But } z = 0 \qquad \therefore\ \frac{dz}{dx} = 0 \qquad \therefore\ \frac{dy}{dx} = -\left\{\frac{y.e^{xy} + 1}{x.e^{xy} + 1}\right\}$$

$$\text{At } x = 0,\ y = 0,\ \frac{dy}{dx} = -\frac{1}{1} = -1 \qquad \therefore\ \frac{dy}{dx} = -1$$

All very easy so long as you can find partial derivatives correctly.

On to Frame 20

20

Now here is:

Example 2

If $xy + \sin y = 2$, find $\dfrac{dy}{dx}$

Let $z = xy + \sin y - 2 = 0$

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{\partial z}{\partial x} = y; \quad \frac{\partial z}{\partial y} = x + \cos y$$

$$\therefore \quad \frac{dz}{dx} = y + (x + \cos y)\frac{dy}{dx}$$

But $z = 0 \quad \therefore \quad \dfrac{dz}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{-y}{x + \cos y}$$

Here is one for you to do:

Example 3

Find an expression for $\dfrac{dy}{dx}$ when $x\tan y = y\sin x$. Do it all on your own.

Then check your working with that in Frame 21

21

$$\boxed{\frac{dy}{dx} = -\frac{\tan y - y\cos x}{x\sec^2 y - \sin x}}$$

Did you get that? If so, go straight on to Frame 22. If not, here is the working below. Follow it through and see where you have gone astray!

$x\tan y = y\sin x \qquad \therefore \; x\tan y - y\sin x = 0$

Let $z = x\tan y - y\sin x = 0$

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{\partial z}{\partial x} = \tan y - y\cos x; \quad \frac{\partial z}{\partial y} = x\sec^2 y - \sin x$$

$$\therefore \quad \frac{dz}{dx} = (\tan y - y\cos x) + (x\sec^2 y - \sin x)\frac{dy}{dx}$$

But $z = 0 \quad \therefore \quad \dfrac{dz}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\tan y - y\cos x}{x\sec^2 y - \sin x}$$

On now to Frame 22

22

Right. Now here is just one more for you to do. They are really very much the same.

Example 4

If $e^{x+y} = x^2y^2$, find an expression for $\dfrac{dy}{dx}$

$$e^{x+y} - x^2y^2 = 0. \quad \text{Let } z = e^{x+y} - x^2y^2 = 0$$

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

So continue with the good work and finish it off, finally getting that

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots$$

Then move to Frame 23

23

$$\boxed{\frac{dy}{dx} = \frac{2xy^2 - e^{x+y}}{e^{x+y} - 2x^2y}}$$

Because $z = e^{x+y} - x^2y^2 = 0$

$$\frac{\partial z}{\partial x} = e^{x+y} - 2xy^2; \quad \frac{\partial z}{\partial y} = e^{x+y} - 2x^2y$$

$$\therefore \quad \frac{dz}{dx} = (e^{x+y} - 2xy^2) + (e^{x+y} - 2x^2y)\frac{dy}{dx}$$

$$\text{But } z = 0 \quad \therefore \quad \frac{dz}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(e^{x+y} - 2xy^2)}{(e^{x+y} - 2x^2y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2xy^2 - e^{x+y}}{(e^{x+y} - 2x^2y)}$$

That is how they are all done. But now there is one more process that you must know how to tackle.

So on to Frame 24

Change of variables

24

If z is a function of x and y, i.e. $z = f(x, y)$, and x and y are themselves functions of two other variables u and v, then z is also a function of u and v. We may therefore need to find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$. How do we go about it?

$$z = f(x, y) \qquad \therefore \ \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

Divide both sides by δu:

$$\frac{\delta z}{\delta u} = \frac{\partial z}{\partial x}\cdot\frac{\delta x}{\delta u} + \frac{\partial z}{\partial y}\cdot\frac{\delta y}{\delta u}$$

If v is kept constant for the time being, then $\dfrac{\delta x}{\delta u}$ when $\delta u \to 0$ becomes $\dfrac{\partial x}{\partial u}$ and $\dfrac{\delta y}{\delta u}$ becomes $\dfrac{\partial y}{\partial u}$.

$$\left.\begin{aligned} \therefore \quad & \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \text{and} \quad & \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}\right\} \quad \text{Note these}$$

Next frame

25

Here is an example of this work.
If $z = x^2 + y^2$, where $x = r\cos\theta$ and $y = r\sin 2\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\text{and} \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\text{Now,} \quad \frac{\partial z}{\partial x} = 2x \qquad \left[\frac{\partial z}{\partial y} = 2y\right]$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \left[\frac{\partial y}{\partial r} = \sin 2\theta\right]$$

$$\therefore \quad \frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$\text{and} \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \ \text{ and } \ \frac{\partial y}{\partial \theta} = 2r\cos 2\theta$$

$$\therefore \quad \frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$\frac{\partial z}{\partial \theta} = 4yr\cos 2\theta - 2xr\sin\theta$$

And in these two results, the symbols x and y can be replaced by $r\cos\theta$ and $r\sin 2\theta$ respectively.

26

One more example:

If $z = e^{xy}$ where $x = \ln(u+v)$ and $y = \sin(u-v)$, find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$\text{We have } \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = y.e^{xy}.\frac{1}{u+v} + x.e^{xy}.\cos(u-v)$$

$$= e^{xy}\left\{\frac{y}{u+v} + x.\cos(u-v)\right\}$$

$$\text{and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = y.e^{xy}.\frac{1}{u+v} + x.e^{xy}.\{-\cos(u-v)\}$$

$$= e^{xy}\left\{\frac{y}{u+v} - x.\cos(u-v)\right\}$$

Now move on to Frame 27

27

Here is one for you to do on your own. All that it entails is to find the various partial derivatives and to substitute them in the established results:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

So you do this one:

If $z = \sin(x+y)$, where $x = u^2 + v^2$ and $y = 2uv$, find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$

The method is the same as before.

When you have completed the work, check with the result in Frame 28

28

$z = \sin(x+y)$; $x = u^2 + v^2$; $y = 2uv$

$$\frac{\partial z}{\partial x} = \cos(x+y); \qquad \frac{\partial z}{\partial y} = \cos(x+y)$$

$$\frac{\partial x}{\partial u} = 2u \qquad \frac{\partial y}{\partial u} = 2v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$= \cos(x+y).2u + \cos(x+y).2v$$

$$= 2(u+v)\cos(x+y)$$

$$\text{Also } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$\frac{\partial x}{\partial v} = 2v; \qquad \frac{\partial y}{\partial v} = 2u$$

$$\frac{\partial z}{\partial v} = \cos(x+y).2v + \cos(x+y).2u$$

$$= 2(u+v)\cos(x+y)$$

29

You have now reached the end of this Programme and know quite a bit about partial differentiation. We have established some important results during the work, so let us list them once more.

1 *Small increments*

$$z = f(x,y) \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y \qquad \text{(a)}$$

2 *Rates of change*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \qquad \text{(b)}$$

3 *Implicit functions*

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \qquad \text{(c)}$$

4 *Change of variables*

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\} \qquad \text{(d)}$$

All that now remains is the **Can You?** checklist and the **Test exercise**, so move to Frames 30 and 31 and work through them carefully at your own speed.

☑ Can You?

Checklist 11

30

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Derive the first- and second-order partial derivatives of a function of two real variables? 1 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply partial differentiation to rate-of-change problems? 8 to 23

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply partial differentiation to change-of-variable problems? 24 to 29

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 11

31 Take your time and work carefully. The questions are just like those you have been doing quite successfully.

1 Use partial differentiation to determine expressions for $\frac{dy}{dx}$ in the following cases:

(a) $x^3 + y^3 - 2x^2y - 0$ (b) $e^x \cos y - e^y \sin x$

(c) $\sin^2 x - 5 \sin x \cos y + \tan y = 0$

2 The base radius of a cone, r, is decreasing at the rate of 0·1 cm/s while the perpendicular height, h, is increasing at the rate of 0·2 cm/s. Find the rate at which the volume, V, is changing when $r = 2$ cm and $h = 3$ cm.

3 If $z = 2xy - 3x^2y$ and x is increasing at 2 cm/s, determine at what rate y must be changing in order that z shall be neither increasing nor decreasing at the instant when $x = 3$ cm and $y = 1$ cm.

4 If $z = x^4 + 2x^2y + y^3$ and $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in their simplest forms.

Further problems 11

32

1 If $F = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$, show that
$\frac{\partial F}{\partial u} = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial v} = -y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y}$.

2 Given that $z = x^3 + y^3$ and $x^2 + y^2 = 1$, determine an expression for $\frac{dz}{dx}$ in terms of x and y.

3 If $z = f(x, y) = 0$, show that $\frac{dy}{dx} = -\frac{\partial z}{\partial x} / \frac{\partial z}{\partial y}$. The curves $2y^2 + 3x - 8 = 0$ and $x^3 + 2xy^3 + 3y - 1 = 0$ intersect at the point (2, −1). Find the tangent of the angle between the tangents to the curves at this point.

4 If $u = (x^2 - y^2)f(t)$ where $t = xy$ and f denotes an arbitrary function, prove that $\frac{\partial^2 u}{\partial x.\partial y} = (x^2 - y^2)\{t.f''(t) + 3f'(t)\}$. [*Note*: $f''(t)$ is the second derivative of $f(t)$ w.r.t. t.]

5 If $V = xy/(x^2 + y^2)^2$ and $x = r\cos\theta$, $y = r\sin\theta$, show that
$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$

6 If $u = f(x, y)$ where $x = r^2 - s^2$ and $y = 2rs$, prove that
$r\dfrac{\partial u}{\partial r} - s\dfrac{\partial u}{\partial s} = 2(r^2 + s^2)\dfrac{\partial u}{\partial x}$.

7 If $f = F(x, y)$ and $x = re^{\theta}$ and $y = re^{-\theta}$, prove that
$$2x\frac{\partial f}{\partial x} = r\frac{\partial f}{\partial r} + \frac{\partial f}{\partial \theta} \text{ and } 2y\frac{\partial f}{\partial y} = r\frac{\partial f}{\partial r} - \frac{\partial f}{\partial \theta}.$$

8 If $z = x\ln(x^2 + y^2) - 2y\tan^{-1}\left(\dfrac{y}{x}\right)$ verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z + 2x$.

9 By means of partial differentiation, determine $\dfrac{dy}{dx}$ in each of the following cases:

(a) $xy + 2y - x = 4$
(b) $x^3y^2 - 2x^2y + 3xy^2 - 8xy = 5$
(c) $\dfrac{4y}{x} + \dfrac{2x}{y} = 3$

10 If $z = 3xy - y^3 + (y^2 - 2x)^{3/2}$, verify that:
$$\text{(a)}\ \frac{\partial^2 z}{\partial x.\partial y} = \frac{\partial^2 z}{\partial y.\partial x} \text{ and (b) } \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x.\partial y}\right)^2$$

11 If $f = \dfrac{1}{\sqrt{1 - 2xy + y^2}}$, show that $y\dfrac{\partial f}{\partial y} = (x - y)\dfrac{\partial f}{\partial x}$.

12 If $z = x.f\left(\dfrac{y}{x}\right) + F\left(\dfrac{y}{x}\right)$, prove that:
$$\text{(a)}\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - F\left(\frac{y}{x}\right) \qquad \text{(b)}\ x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x.\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

13 If $z = e^{k(r-x)}$, where k is a constant, and $r^2 = x^2 + y^2$, prove:
$$\text{(a)}\ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2zk\frac{\partial z}{\partial x} = 0 \qquad \text{(b)}\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 2k\frac{\partial z}{\partial x} = \frac{kz}{r}$$

14 If $z = f(x - 2y) + F(3x + y)$, where f and F are arbitrary functions, and if
$\dfrac{\partial^2 z}{\partial x^2} + a\dfrac{\partial^2 z}{\partial x.\partial y} + b\dfrac{\partial^2 z}{\partial y^2} = 0$, find the values of a and b.

15 If $z = xy/(x^2 + y^2)^2$, verify that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

16 If $\sin^2 x - 5\sin x\cos y + \tan y = 0$, find $\dfrac{dy}{dx}$ by using partial differentiation.

17 Find $\dfrac{dy}{dx}$ by partial differentiation, when $x\tan y = y\sin x$.

▶

18 If $V = \tan^{-1}\left\{\frac{2xy}{x^2 - y^2}\right\}$, prove that:

(a) $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 0$ (b) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

19 Prove that, if $z = 2xy + x.f\left(\frac{y}{x}\right)$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + 2xy$.

20 (a) Find $\frac{dy}{dx}$ given that $x^2y + \sin xy = 0$.

(b) Find $\frac{dy}{dx}$ given that $x \sin xy = 1$.

Curves and curve fitting

Frames 1 to 67

Learning outcomes

When you have completed this Programme you will be able to:

- Draw sketch graphs of standard curves
- Determine the equations of asymptotes parallel to the x- and y-axes
- Sketch the graphs of curves with asymptotes, stationary points and other features
- Fit graphs to data using the 'straight-line' forms
- Fit graphs to data using the method of least squares

Introduction

1

The purpose of this Programme is eventually to devise a reliable method for establishing the relationship between two variables, corresponding values of which have been obtained as a result of tests or experimentation. These results in practice are highly likely to include some errors, however small, owing to the imperfect materials used, the limitations of the measuring devices and the shortcomings of the operator conducting the test and recording the results.

There are methods by which we can minimize any further errors or guesswork in processing the results and, indeed, eradicate some of the errors already inherent in the recorded results, but before we consider this important section of the work, some revision of the shape of standard curves and the systematic sketching of curves from their equations would be an advantage.

Standard curves

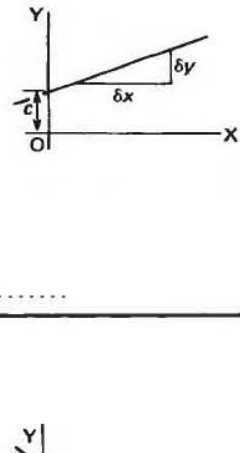
2

(a) Straight line

The equation is a first-degree relationship and can always be expressed in the form $y = mx + c$, where

m denotes the gradient, i.e. $\dfrac{\delta y}{\delta x}$

c denotes the intercept on the y-axis.

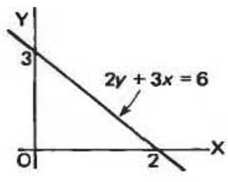


Any first-degree equation gives a straight-line graph.
To find where the line crosses the x-axis, put $y = 0$.
To find where the line crosses the y-axis, put $x = 0$.
Therefore, the line $2y + 3x = 6$ crosses the axes at ....

3

$\boxed{(2, 0) \text{ and } (0, 3)}$

Because when $x = 0$, $2y = 6$ $\therefore$ $y = 3$
and when $y = 0$, $3x = 6$ $\therefore$ $x = 2$

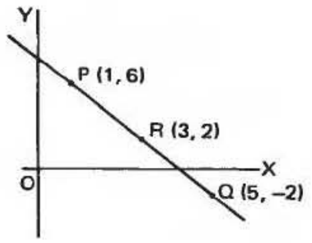


We can establish the equation of a given straight line by substituting in $y = mx + c$ the x- and y-coordinates of any three points on the line. Of course, two points are sufficient to determine values of m and c, but the third point is taken as a check.

So, if (1, 6), (3, 2) and (5, −2) lie on a straight line, its equation is ....

4

$$\boxed{y = -2x + 8}$$



$$\begin{array}{ll} (1, 6) & 6 = 1m + c \\ (5, -2) & -2 = 5m + c \\ & \therefore\ -8 = 4m \quad \therefore\ m = -2 \\ & y = -2x + c \end{array}$$

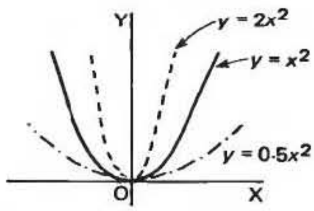
When $x = 1$, $y = 6$ $\therefore$ $6 = -2 + c$ $\therefore$ $c = 8$ $\therefore$ $y = -2x + 8$

Check: When $x = 3$, $y = -6 + 8 = 2$ which agrees with the third point.

(b) Second-degree curves

The basic second-degree curve is $y = x^2$, a parabola symmetrical about the y-axis and existing only for $y \geq 0$.

$y = ax^2$ gives a thinner parabola if $a > 1$ and a flatter parabola if $a < 1$.

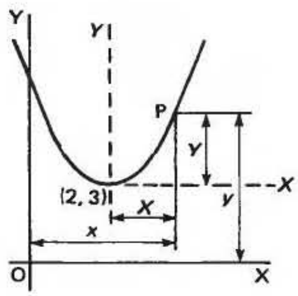


The general second-degree curve is $y = ax^2 + bx + c$, where the three coefficients, a, b and c, determine the position of the vertex and the 'width' of the parabola.

Change of vertex: If the parabola $y = x^2$ is moved parallel to itself to a vertex position at (2, 3), its equation relative to the new axes is $Y = X^2$.

Considering a sample point P, we see that:

$Y = y - 3$ and $X = x - 2$



So, in terms of the original variables, x and y, the equation of the new parabola is . . . .

5

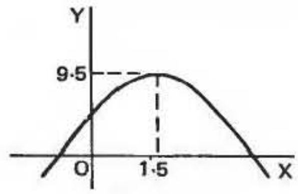
$$\boxed{y = x^2 - 4x + 7}$$

Because $Y = X^2$ becomes $y - 3 = (x - 2)^2$ i.e. $y - 3 = x^2 - 4x + 4$ which simplifies to $y = x^2 - 4x + 7$.

▶

Note: If the coefficient of x^2 is negative, the parabola is inverted e.g. $y = -2x^2 + 6x + 5$.

The vertex is at (1·5, 9·5).



The parabola cuts the y-axis at $y =$ ………… and the x-axis at $x =$ ………… and $x =$ …………

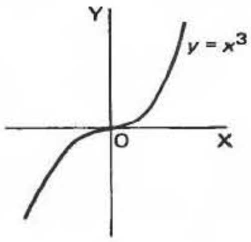
6

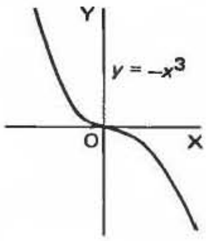
$\boxed{y = 5;\ x = -0{\cdot}68 \text{ and } x = 3{\cdot}68}$

(c) Third-degree curves

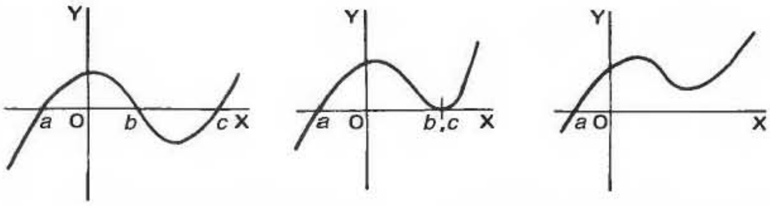
The basic third-degree curve is $y = x^3$ which passes through the origin. For x positive, y is positive and for x negative, y is negative.

Writing $y = -x^3$ turns the curve upside down:





In general, a third-degree curve has a more accentuated double bend and cuts the x-axis in three points which may have (a) three real and different values, (b) two values the same and one different, or (c) one real value and two complex values.



Now let us collect our ideas so far by working through a short exercise.
Move on to the next frame

7

As an exercise, sketch the graphs of the following, indicating relevant information. Do not plot the graphs in detail.

1 $y = 2x - 5$

2 $y = \dfrac{x}{3} + 7$

3 $y = -2x + 4$

4 $2y + 5x - 6 = 0$

5 $y = x^2 + 4$

6 $y = (x - 3)^2$

7 $y = (x + 2)^2 - 4$

8 $y = x - x^2$

9 $y = x^3 - 4$

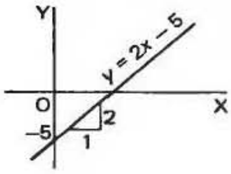
10 $y = 2 - (x + 3)^3$

When you have completed the whole set, check your results with those in the next frame

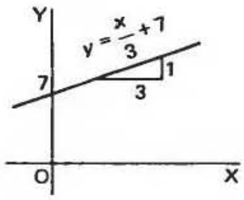
8

Here are the results:

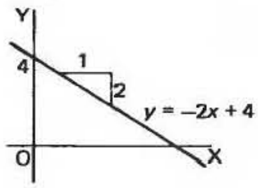
1



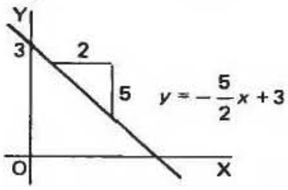
2



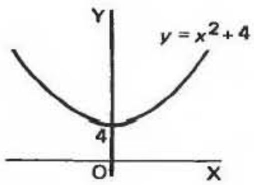
3



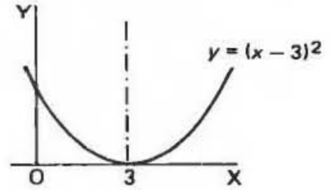
4



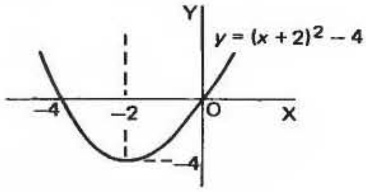
5



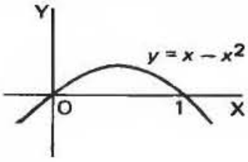
6



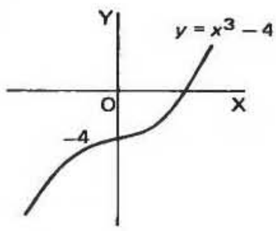
7



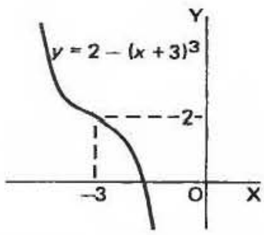
8



9



10

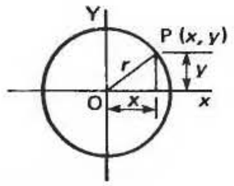


Now we will revise a further set of curves. Next frame

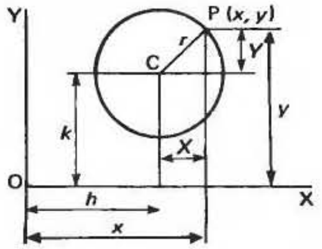
9

(d) Circle

The simplest case of the circle is with centre at the origin and radius r:



The equation is then $x^2 + y^2 = r^2$



Moving the centre to a new point (h, k) gives $X^2 + Y^2 = r^2$

where $Y = y - k$

and $X = x - h$

$\therefore \; (x - h)^2 + (y - k)^2 = r^2$

The general equation of a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

where the centre is the point $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Note that, for a second-degree equation to represent a circle:

(a) the coefficients of x^2 and y^2 are identical

(b) there is no product term in xy.

So the equation $x^2 + y^2 + 2x - 6y - 15 = 0$ represents a circle with centre .... and radius ....

10

$\boxed{\text{centre } (-1, 3);\ \text{radius } 5}$

Because

$$\left.\begin{aligned} 2g &= 2 \quad \therefore g = 1 \\ 2f &= -6 \quad \therefore f = -3 \end{aligned}\right\} \quad \therefore \text{ centre } (-g, -f) = (-1, 3)$$

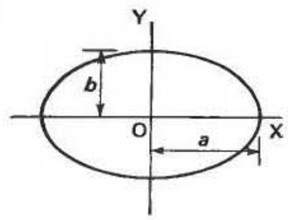
also $c = -15 \quad \therefore$ radius $= \sqrt{g^2 + f^2 - c} = \sqrt{1 + 9 + 15} = \sqrt{25} = 5$.

(e) Ellipse

The equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where $a =$ semi-major axis $\quad \therefore y = 0, \; x = \pm a$

and $b =$ semi-minor axis $\quad \therefore x = 0, \; y = \pm b$



Of course, when a^2 and b^2 are equal (say r^2) we obtain ....

11

$\boxed{\text{the equation of a circle, i.e. } x^2 + y^2 = r^2}$

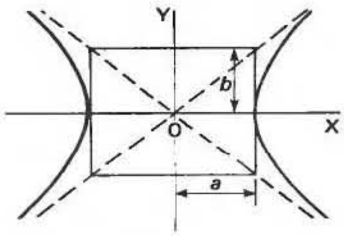
(f) Hyperbola

The equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

When $y = 0$, $x = \pm a$.

When $x = 0$, $y^2 = -b^2$ $\therefore$ the curve does not cross the y-axis.



Note that the opposite arms of the hyperbola gradually approach two straight lines (asymptotes).

Rectangular hyperbola

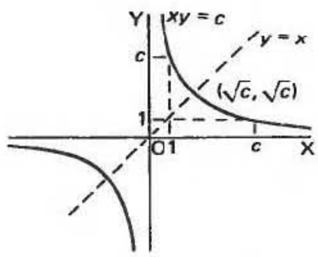
If the asymptotes are at right-angles to each other, the curve is then a *rectangular hyperbola*. A more usual form of the rectangular hyperbola is to rotate the figure through 45° and to make the asymptotes the axes of x and y. The equation of the curve relative to the new axes then becomes

$$xy = c \qquad \text{i.e. } y = \frac{c}{x}$$

Three points are easily located.

If $xy = c$:

(a) when $x = 1$, $y = c$
(b) when $y = 1$, $x = c$
(c) the line $y = x$ cuts $xy = c$ at the point $(\pm\sqrt{c}, \pm\sqrt{c})$.



These three points are a great help in sketching a rectangular hyperbola. Rectangular hyperbolas frequently occur in practical considerations.

Now for another short exercise, so move on to the next frame

12

As an exercise, sketch the graphs of the following, showing relevant facts:

1 $x^2 + y^2 = 12{\cdot}25$

2 $x^2 + 4y^2 = 100$

3 $x^2 + y^2 - 4x + 6y - 3 = 0$

4 $2x^2 - 3x + 4y + 2y^2 = 0$

5 $\dfrac{x^2}{36} - \dfrac{y^2}{49} = 1$

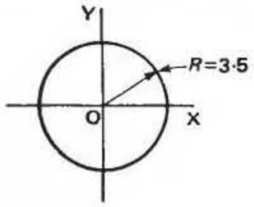
6 $xy = 5$

When you have sketched all six, compare your results with those in the next frame

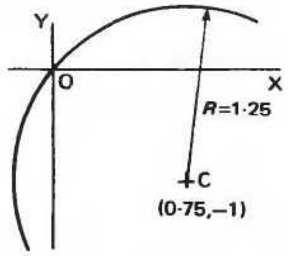
13

The sketch graphs are as follows:

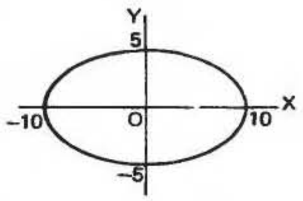
1



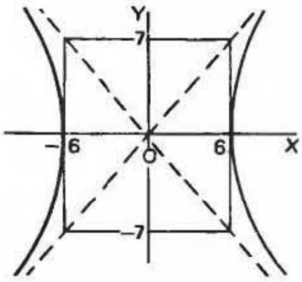
4



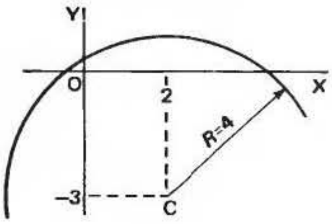
2



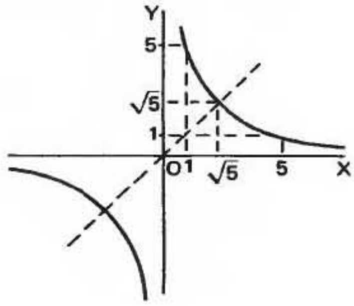
5



3



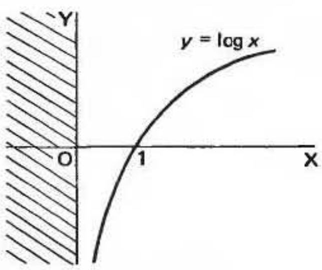
6



On to Frame 14 for a third set of curves frequently occurring

14

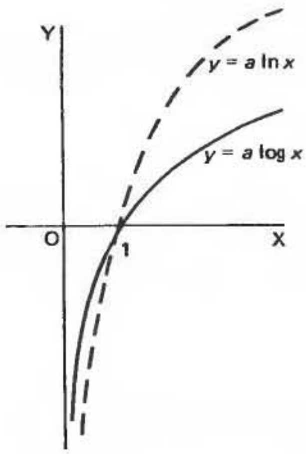
(g) Logarithmic curves



If $y = \log x$, then when $x = 1$, $y = \log 1 = 0$ i.e. the curve crosses the x-axis at $x = 1$.

Also, $\log x$ does not exist for $x < 0$.
$y = \log x$ flattens out as $x \to \infty$, but continues to increase at an ever-decreasing rate.

▶



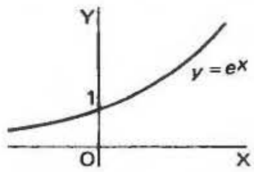
The graph of $y = \ln x$ also has the same shape and crosses the x-axis at $x = 1$, but the function values are different.

The graphs of $y = a \log x$ and $y = a \ln x$ are similar, but with all ordinates multiplied by the constant factor a.

Continued in the next frame

15

(h) Exponential curves



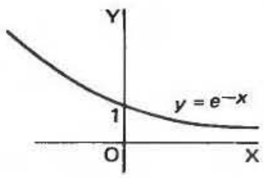
$y = e^x$ crosses the y-axis at $x = 0$

i.e. $y = e^0 = 1$

As $x \to \infty$, $y \to \infty$

as $x \to -\infty$, $y \to 0$

Sometimes known as the *growth curve.*

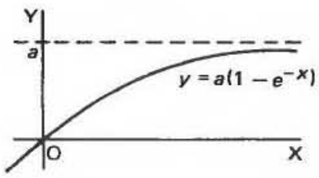


$y = e^{-x}$ also crosses the y-axis at $y = 1$.

As $x \to \infty$, $y \to 0$

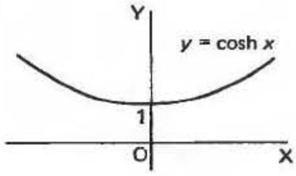
as $x \to -\infty$, $y \to \infty$

Sometimes known as the *decay curve.*



In electrical work, we also frequently have curves of the form $y = a(1 - e^{-x})$. This is an inverted exponential curve, passing through the origin and tending to $y = a$ as asymptote as $x \to \infty$ (since $e^{-x} \to 0$ as $x \to \infty$).

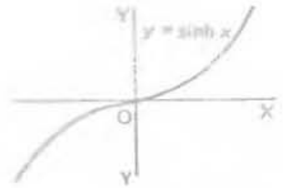
(i) Hyperbolic curves



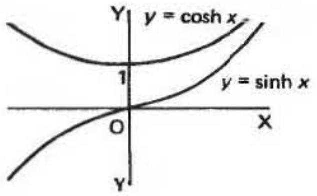
Combination of the curves for $y = e^x$ and $y = e^{-x}$ gives the hyperbolic curves of

$$y = \cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and}$$

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$



We have already dealt with these functions in detail in Programme 3, so refer back if you need to revise them further.

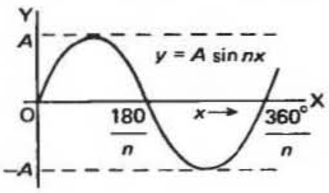


If we draw the two graphs on the same axes, we see that $y = \sinh x$ is always outside $y = \cosh x$, i.e. for any particular value of x, $\cosh x > \sinh x$.

We have one more type of curve to list, so move on to the next frame

16

(j) Trigonometrical curves

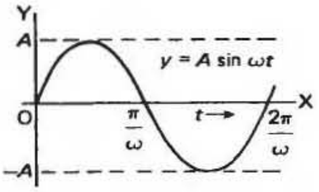


The most common occurring in practice is the sine curve.

(a) $y = A \sin nx$

$$\text{Period} = \frac{360^\circ}{n}$$

$$\text{Amplitude} = A$$



(b) $y = A \sin \omega t$

$$\text{Period} = \frac{2\pi}{\omega}$$

$$\text{Amplitude} = A$$

Now, as an exercise, we can sketch a further selection of curves.

Sketch the following pairs of curves on the same axes. Label each graph clearly and show relevant information.

1 $y = \cosh x$ and $y = 2\cosh x$

2 $y = \sinh x$ and $y = \sinh 2x$

3 $y = e^x$ and $y = e^{3x}$

4 $y = e^{-x}$ and $y = 2e^{-x}$

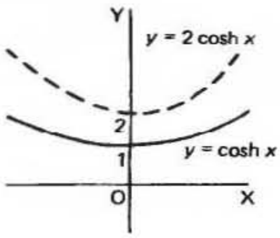
5 $y = 5\sin x$ and $y = 3\sin 2x$

6 $y = 4\sin \omega t$ and $y = 2\sin 3\omega t$

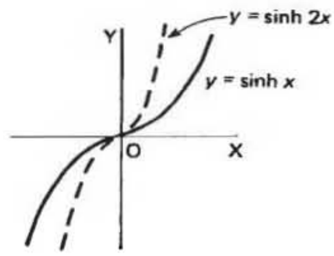
17

Here they are:

1

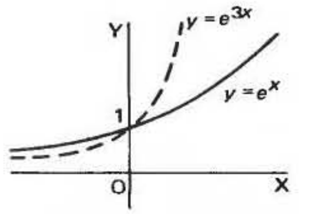


2

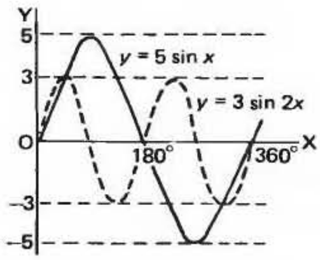


▶

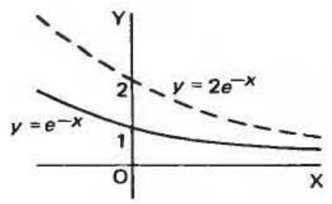
3



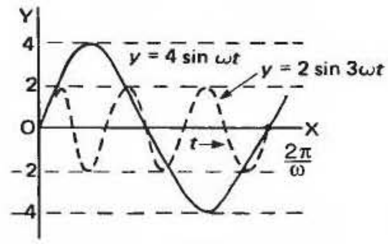
5



4



6



Asymptotes

18

We have already made references to asymptotes in the previous work and it is always helpful to know where asymptotes occur when sketching curves of functions.

Definition: An *asymptote* to a curve is a straight line to which the curve approaches as the distance from the origin increases. It can also be thought of as a tangent to the curve at infinity, i.e. the curve touches the asymptote at two coincident points at infinity.

Condition for infinite roots: If we consider the equation

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n = 0$$

and substitute $x = \frac{1}{y}$, the equation becomes

$$a_0\frac{1}{y^n} + a_1\frac{1}{y^{n-1}} + a_2\frac{1}{y^{n-2}} + \dots + a_{n-1}\frac{1}{y} + a_n = 0$$

If we now multiply through by y^n and reverse the order of the terms, we have:

$$a_ny^n + a_{n-1}y^{n-1} + \dots + a_2y^2 + a_1y + a_0 = 0$$

If $a_0 = 0$ and $a_1 = 0$, the equation reduces to:

$$a_ny^n + a_{n-1}y^{n-1} + \dots + a_2y^2 = 0$$

$$\therefore\ y^2 = 0 \text{ or } a_ny^{n-2} + a_{n-1}y^{n-3} + \dots + a_2 = 0$$

Therefore with the stated condition, $y^2 = 0$ gives two zero roots for y, i.e. two infinite roots for x, since $y = \frac{1}{x}$ and hence $x = \frac{1}{y}$.

▶

Therefore, the original equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

will have two infinite roots if $a_0 = 0$ and $a_1 = 0$.

Determination of an asymptote

From the result we have just established, to find an asymptote to $y = f(x)$:

(a) substitute $y = mx + c$ in the given equation and simplify
(b) equate to zero ....

19

the coefficients of the two highest powers of x and so determine the values of m and c

Let us work through an example to see how it develops.

20

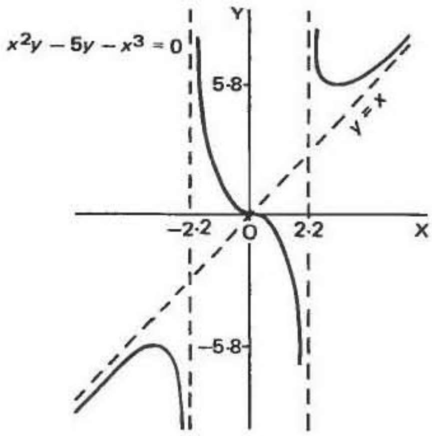
To find the asymptote to the curve $x^2y - 5y - x^3 = 0$.

Substitute $y = mx + c$ in the equation:

$$x^2(mx + c) - 5(mx + c) - x^3 = 0$$
$$mx^3 + cx^2 - 5mx - 5c - x^3 = 0$$
$$(m - 1)x^3 + cx^2 - 5mx - 5c = 0$$

Equating to zero the coefficients of the two highest powers of x:

$$\left.\begin{array}{ll} m - 1 = 0 & \therefore\ m = 1 \\ c = 0 & \qquad c = 0 \end{array}\right\} \quad \therefore \text{ Asymptote is } y = x$$



In fact, the graph of $x^2y - 5y - x^3 = 0$ is as shown on the left and we can see that the curve approaches $y = x$ as $x \to \infty$ and as $x \to -\infty$.

From the graph, however, it appears that there are two further asymptotes which are the lines .... and ....

21

$x = -2{\cdot}2$ and $x = 2{\cdot}2$

These are two lines parallel to the y-axis.

Asymptotes parallel to the x- and y-axes

These can be found by a simple rule.

For the curve $y = f(x)$, the asymptotes parallel to the x-axis can be found by equating the coefficient of the highest power of x to zero. Similarly, the asymptotes parallel to the y-axis can be found by equating the coefficient of the highest power of y to zero.

Example 1

Find the asymptotes, if any, of $y = \dfrac{x-2}{2x+3}$.

First get everything on one line by multiplying by the denominator, $(2x + 3)$:

$y(2x+3) = x - 2 \quad \therefore \; 2xy + 3y - x + 2 = 0$

(a) Asymptote parallel to x-axis: Equate the coefficient of the highest power of x to zero.

$(2y-1)x + 3y + 2 = 0$

$\therefore \; 2y - 1 = 0 \quad \therefore \; 2y = 1 \quad \therefore \; y = 0{\cdot}5 \quad \therefore \; y = 0{\cdot}5$ is an asymptote.

(b) Asymptote parallel to y-axis: Equate the coefficient of the highest power of y to zero. $\therefore$ .... is also an asymptote.

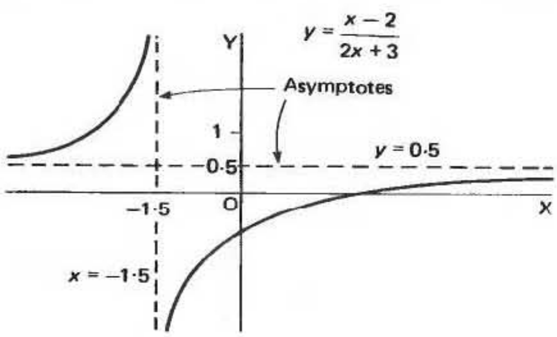
22

$\boxed{x = -1{\cdot}5}$

Because rearranging the equation to obtain the highest power of y:

$(2x+3)y - x + 2 = 0$

$\therefore \; 2x + 3 = 0 \quad \therefore \; x = -1{\cdot}5 \quad \therefore \; x = -1{\cdot}5$ is an asymptote.



In fact, the graph is positioned as shown above. The only asymptotes are $y = 0{\cdot}5$ (parallel to the x-axis) and $x = -1{\cdot}5$ (parallel to the y-axis).

Let us do another.

▶

Example 2

Find the asymptotes of the curve $x^2(x^2+2) = y^3(x+5)$.

(a) Parallel to the x-axis: $x^4 + 2x^2 = (x+5)y^3$.

Equate the coefficient of the highest power of x to zero.

Highest power of x is x^4. Its coefficient is 1, which gives $1 = 0$. This not the equation of a line. Therefore, there is no asymptote parallel to the x-axis.

(b) Parallel to the y-axis: This gives ....

23

$$\boxed{x = -5 \text{ is an asymptote}}$$

Because for an asymptote parallel to the y-axis, we equate the highest power of y to zero. $\therefore\ x+5=0 \quad \therefore\ x=-5$ Therefore, $x=-5$ is an asymptote parallel to the y-axis.

Now, to find a general asymptote, i.e. not necessarily parallel to either axis, we carry out the method described earlier, which was to ....

24

> substitute $y = mx + c$ in the equation and equate the coefficients of the two highest powers of x to zero

If we do that with the equation of this example, we get ....

Work it right through to the end

25

$$\boxed{y = x - \frac{5}{3} \text{ is also an asymptote}}$$

Because substituting $y = mx + c$ in $x^2(x^2+2) = y^3(x+5)$ we have

$$\begin{aligned}
x^4 + 2x^2 &= (m^3x^3 + 3m^2x^2c + 3mxc^2 + c^3)(x+5) \\
&= m^3x^4 + 3m^2x^3c + 3mx^2c^2 + c^3x \\
&\quad + 5m^3x^3 + 15m^2x^2c + 15mxc^2 + 5c^3
\end{aligned}$$

$$\therefore\ (m^3-1)x^4 + (5m^3+3m^2c)x^3 + (15m^2c+3mc^2-2)x^2 + (15mc^2+c^3)x + 5c^3 = 0$$

Equating to zero the coefficients of the two highest powers of x:

$$m^3 - 1 = 0 \qquad \therefore\ m^3 = 1 \qquad \therefore\ m = 1$$

$$5m^3 + 3m^2c = 0 \qquad \therefore\ 5 + 3c = 0 \qquad \therefore\ c = \frac{-5}{3}$$

$$\therefore\ y = x - \frac{5}{3} \text{ is an asymptote}$$

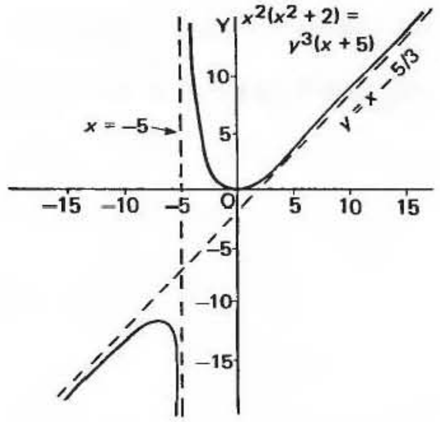
▶

There are, then, two asymptotes:

$$x = -5$$

$$\text{and} \quad y = x - \frac{5}{3}$$

In fact, the graph is shown on the right.



Here are two similar problems for you to do as an exercise on your own. Just apply the rules. There are no tricks.

Determine the asymptotes, if any, for the following curves:

1 $x^4 - 2x^3y + 10x^2 - 7y^2 = 0$
2 $x^3 - xy^2 + 4y^2 - 5 = 0$

26

1	$y = \frac{x}{2}$ is the only asymptote
2	$y = x + 2;\quad y = -x - 2;\quad x = 4$

1 There are no asymptotes parallel to the axes.

Substituting $y = mx + c$ and collecting up terms gives:

$$(1 - 2m)x^4 - 2cx^3 + (10 - 7m^2)x^2 - 14mcx - 7c^2 = 0$$

$$\left.\begin{aligned} &\text{from which } 1 - 2m = 0 \quad \therefore\ m = \frac{1}{2} \\ &c = 0 \end{aligned}\right\} \quad \therefore\ y = \frac{x}{2} \text{ is the only asymptote.}$$

2 $x = 4$ is an asymptote parallel to the y-axis.

There is no asymptote parallel to the x-axis.

Putting $y = mx + c$ and simplifying produces:

$$(1 - m^2)x^3 + (4m^2 - 2mc)x^2 + (8mc - c^2)x + 4c^2 - 5 = 0$$

so that $\quad 1 - m^2 = 0 \quad \therefore\ m^2 = 1 \quad \therefore\ m = \pm 1$

and $\quad 4m^2 - 2mc = 0 \quad \therefore\ 2m = c.$ When $m = 1$, $c = 2$ and

when $m = -1$, $c = -2 \quad \therefore\ y = x + 2$ and $y = -x - 2$ are asymptotes.

Now let us apply these methods in a wider context starting in the next frame

Systematic curve sketching, given the equation of the curve

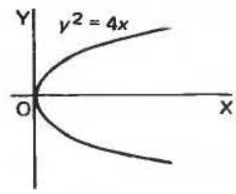
27

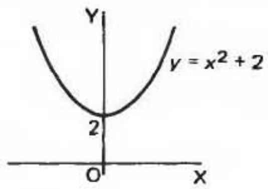
If, for $y = f(x)$, the function $f(x)$ is known, the graph of the function can be plotted by calculating x- and y-coordinates of a number of selected points. This, however, can be a tedious occupation and considerable information about the shape and positioning of the curve can be obtained by a systematic analysis of the given equation. There is a list of steps we can take.

1 Symmetry

Inspect the equation for symmetry:

(a) If only even powers of y occur, the curve is symmetrical about the x-axis.
(b) If only even powers of x occur, the curve is symmetrical about the y-axis.





(c) If only even powers of y and also only even powers of x occur, then ....

28

the curve is symmetrical about both axes

For example:

$25x^2 + 16y^2 = 400$ is symmetrical about both axes.

$y^2 + 3y - 2 = (x^2 + 7)^2$ is symmetrical about the y-axis, but not about the x-axis since ....

29

both odd and even powers of y occur

2 Intersection with the axes

Points at which the curve crosses the x- and y-axes.

Crosses the x-axis: Put $y = 0$ and solve for x.

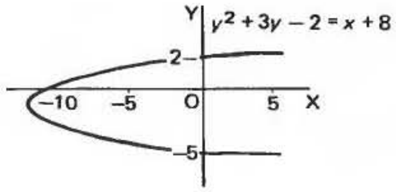
Crosses the y-axis: Put $x = 0$ and solve for y.

So, the curve $y^2 + 3y - 2 = x + 8$ crosses the x- and y-axes at ....

30

x-axis at $x = -10$
y-axis at $y = 2$ and $y = -5$

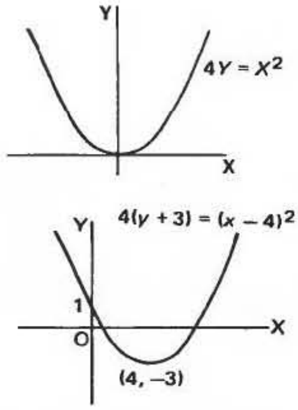
In fact, the curve is:



3 Change of origin

Look for a possible change of origin to simplify the equation. For example, for the curve $4(y+3) = (x-4)^2$, if we change the origin by putting $Y = y+3$ and $X = x-4$, the equation becomes $4Y = X^2$ which is a parabola symmetrical about the axis of Y.

So the curve relative to the original x- and y-axes is positioned thus:



31

4 Asymptotes

We have already dealt with asymptotes in some detail. We investigate (a) asymptotes parallel to the axes, and (b) those of a more general nature.

(a) *Parallel to the axes*

(i) Express the equation 'on one line', i.e. remove algebraic fractions.

(ii) Equate to zero the coefficient of the highest power of y to find the asymptote parallel to the y-axis.

(iii) Equate to zero the coefficient of the highest power of x to find the asymptote parallel to the x-axis.

As an example, find the asymptotes parallel to the axes for the curve

$$y = \frac{(x-1)(x+6)}{(x+3)(x-4)}.$$

32

$$\boxed{x = -3; \quad x = 4; \quad y = 1}$$

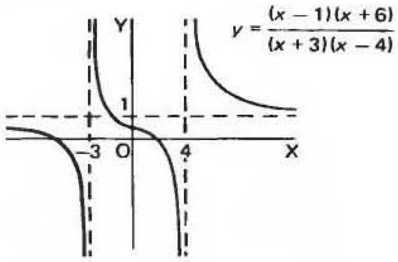
Because $y(x+3)(x-4) = (x-1)(x+6)$

Asymptotes parallel to the y-axis: $(x+3)(x-4) = 0 \quad \therefore \; x = -3$ and $x = 4$.

Rearranging the equation gives $(y-1)x^2 - (y+5)x - 12y + 6 = 0$

Asymptote parallel to the x-axis: $y - 1 = 0 \quad \therefore \; y = 1$

The graph of the function is as shown to the right:



(b) *General asymptotes*

Substitute $y = mx + c$ and equate the coefficient of the two highest powers of x to zero to determine m and c.

Thus, for the curve $x^4 - 2x^3y + 5x^2 - 4y^2 = 0$, the asymptote is ....

33

$$\boxed{y = \frac{x}{2}}$$

Because substituting $y = mx + c$ and simplifying the left-hand side, gives:

$$(1 - 2m)x^4 - 2cx^3 + (5 - 4m^2)x^2 - 8mcx - 4c^2 = 0$$

$$\left.\begin{aligned} \therefore \; 1 - 2m = 0 \quad &\therefore \; m = \tfrac{1}{2} \\ 2c = 0 \quad &\therefore \; c = 0 \end{aligned}\right\} \quad \therefore \; y = \frac{x}{2} \text{ is an asymptote.}$$

5 Large and small values of x and y

If x or y is *small*, higher powers of x or y become negligible and hence only lower powers of x or y appearing in the equation provide an approximate simpler form.

Similarly, if x or y is *large*, the higher powers have predominance and lower powers can be neglected, i.e. when x is large

$$y^2 = 2x^2 - 7x + 4$$

approximates to $y^2 = 2x^2$, i.e. $y = \pm x\sqrt{2}$.

▶

6 Stationary points

Maximum and minimum values; points of inflexion. We have dealt at length in a previous Programme with this whole topic. We will just summarize the results at this stage:

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is negative (curve concave downwards) then the point is a *maximum*

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is positive (curve concave upwards) then the point is a *minimum*

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ with a change of sign through the stationary point then the point is a *point of inflexion.*

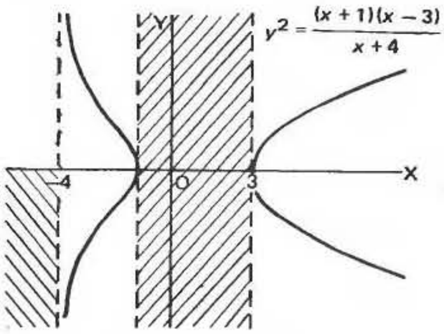
7 Limitations

Restrictions on the possible range of values that x or y may have. For example, consider:

$$y^2 = \frac{(x+1)(x-3)}{x+4}$$

For $x < -4$,	y^2 is negative	$\therefore$	no real values of y.
For $-4 < x < -1$,	y^2 is positive	$\therefore$	real values of y exist.
For $-1 < x < 3$,	y^2 is negative	$\therefore$	no real values of y.
For $3 < x$,	y^2 is positive	$\therefore$	real values of y exist.

The curve finally looks like this:



34

In practice, not all of these considerations are applicable in any one particular case. Let us work in detail through one such example.

Sketch the curve whose equation is $y = \dfrac{(x+2)(x-3)}{x+1}$.

(a) *Symmetry*: First write the equation 'on the line'.

$$y(x+1) = (x+2)(x-3) = x^2 - x - 6$$

Both odd and even powers of x occur, $\therefore$ no symmetry about y-axis.

Only odd powers of y occur, $\therefore$ no symmetry about x-axis.

(b) *Crossing the axes*: This is done simply by putting $y = 0$ and $x = 0$, so, in this example, the curve crosses the axes at ....

35

x-axis at $x = -2$ and at $x = 3$
y-axis at $y = -6$

(c) *Stationary points*: The first essential is to find the values of x at which $\dfrac{dy}{dx} = 0$. Obtain an expression for $\dfrac{dy}{dx}$ and solve $\dfrac{dy}{dx} = 0$ for values of x.

This gives $\dfrac{dy}{dx} = 0$ at $x =$ ....

36

no real values of x

Because if $y = \dfrac{x^2 - x - 6}{x+1}$, $\dfrac{dy}{dx} = \dfrac{(x+1)(2x-1) - (x^2 - x - 6)}{(x+1)^2}$

$$= \frac{x^2 + 2x + 5}{(x+1)^2}$$

For stationary points, $\dfrac{dy}{dx} = 0 \quad \therefore \; x^2 + 2x + 5 = 0 \quad \therefore \; x = \dfrac{-2 \pm \sqrt{-16}}{2}$

i.e. x is complex. Therefore, there are no stationary points on the graph.

(d) *When x is very small*: $y \approx -\dfrac{x+6}{x+1}$ i.e. $y \approx -6$.

When x is very large: $y \approx \dfrac{x^2}{x}$ i.e. $x \; \therefore \; y \approx x$.

(e) *Asymptotes*:

(i) First find any asymptotes parallel to the axes. These are ....

37

$$\boxed{\text{parallel to the } y\text{-axis: } x = -1}$$

Because $y(x+1) - x^2 + x + 6 = 0 \quad \therefore\ x + 1 = 0 \quad \therefore\ x = -1$.

(ii) Now investigate the general asymptote, if any.
This gives ....

38

$$\boxed{y = x - 2}$$

This is obtained, as usual, by putting $y = mx + c$ in the equation:

$$(mx + c)(x + 1) = (x + 2)(x - 3)$$
$$\therefore\ mx^2 + mx + cx + c = x^2 - x - 6$$
$$(m - 1)x^2 + (m + c + 1)x + c + 6 = 0$$

Equating the coefficients of the two highest powers of x to zero:

$$m - 1 = 0 \quad \therefore\ m = 1$$

and $m + c + 1 = 0 \quad \therefore\ c + 2 = 0 \quad \therefore\ c = -2$

$$\therefore\ y = x - 2 \text{ is an asymptote.}$$

So, collecting our findings together, we have

(a) No symmetry about the x- or y-axis.

(b) Curve crosses the x-axis at $x = -2$ and at $x = 3$.

(c) Curve crosses the y-axis at $y = -6$.

(d) There are no stationary points on the curve.

(e) Near $x = 0$, the curve approximates to $y = -6$.

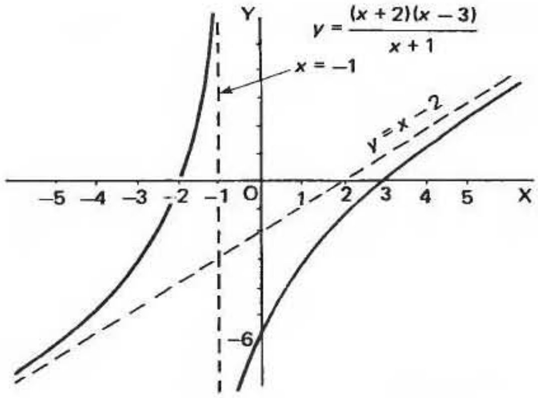
(f) For numerically large values of x, the curve approximates to $y = x$, i.e. when x is large and positive, y is large and positive and when x is large and negative, y is large and negative.

(g) The only asymptotes are $x = -1$ (parallel to the y-axis) and $y = x - 2$.

With these facts before us, we can now sketch the curve.
Do that and then check your result with that shown in the next frame

39

Here is the graph as it should appear. You can, of course, always plot an odd point or two in critical positions if extra help is needed.



Now let us move on to something rather different

Curve fitting

40

Readings recorded from a test or experiment normally include errors of various kinds and therefore the points plotted from these data are scattered about the positions they should ideally occupy. Unless very few readings are taken, it can be assumed that the inherent errors will be of a random nature, resulting in some of the values being slightly too high and some slightly too low. Having plotted the points, we then draw as the graph the middle line of this narrow band of points. It may well be that the line drawn does not pass through any of the actual plotted points, but from now on it is this line which is used to determine the relationship between the two variables.

1 Straight-line law

For example, values of V and h are recorded in a test:

h	6·0	10	14	18	21	25
V	5·5	7·0	9·5	12·5	13·5	16·5

If the law relating V and h is $V = ah + b$, where a and b are constants,

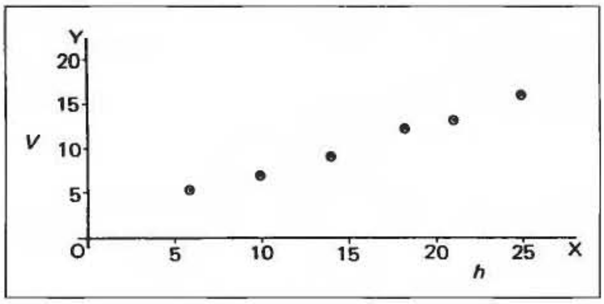
(a) plot the graph of V against h

(b) determine the values of a and b.

(a) Plotting the points is quite straightforward. Do it carefully on squared paper.

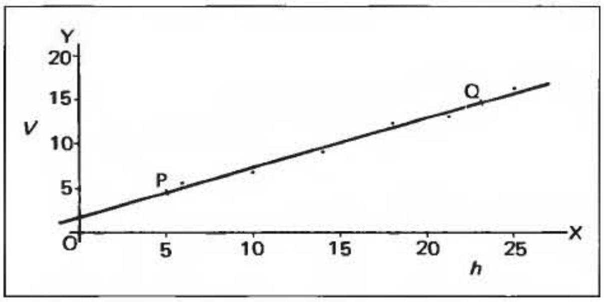
You get . . . .

41



We now estimate by eye the straight position for a straight-line graph drawn down the middle of this band of points. Draw the line on your graph.

42



Henceforth, we shall use this line as representing the equation and ignore the actual points we plotted.

(b) The law is first degree, i.e. $V = ah + b$

Compare this with $y = mx + c$.

If we therefore find the values of m and c in the normal way, these values will be those of a and b.

We now select two convenient points *on the line* and read off their x- and y-coordinates. For instance, P (5, 4·5) and Q (23, 15). Substituting these values in $y = mx + c$ gives two equations from which we can find the values of m and c, which are ....

43

$$\boxed{m = 0{\cdot}583; \quad c = 1{\cdot}58}$$

Because we have $\left.\begin{aligned} 4{\cdot}5 &= 5m + c \\ 15{\cdot}0 &= 23m + c \end{aligned}\right\}$ $\therefore\ 10{\cdot}5 = 18m \quad \therefore\ m = \dfrac{10{\cdot}5}{18} = 0{\cdot}583$

and then $4{\cdot}5 = 5(0{\cdot}583) + c \quad \therefore\ c = 4{\cdot}5 - 2{\cdot}917 \quad \therefore\ c = 1{\cdot}583$

The equation of the line is $y = 0{\cdot}583x + 1{\cdot}58$

and the law relating V and h is $V = 0{\cdot}583h + 1{\cdot}58$

Provided we can express the law in straight-line form, they are all tackled in the same manner.

2 Graphs of the form $y = ax^n$, where a and n are constants

To convert this into 'straight-line form', we take logarithms of both sides. The equation then becomes ....

44

$$\boxed{\log y = n\log x + \log a}$$

If we compare this result with $Y = mX + c$ we see we have to plot .... along the Y-axis and .... along the X-axis to obtain a straight-line graph.

45

$\log y$ along the Y-axis
$\log x$ along the X-axis

$$\left.\begin{aligned}\log y &= n\log x + \log a\\ Y &= mX + c\end{aligned}\right\}$$

If we then find m and c from the straight line as before, then $m = n$
and $c = \log a \quad \therefore\ a = 10^c$.

Let us work through an example.

Values of x and y are related by the equation $y = ax^n$:

x	2	5	12	25	32	40
y	5·62	13·8	52·5	112	160	200

Determine the values of the constants a and n.

$$y = ax^n \quad \therefore\ \log y = n\log x + \log a$$
$$Y = mX + c$$

We must first compile a table showing the corresponding values of $\log x$ and $\log y$.

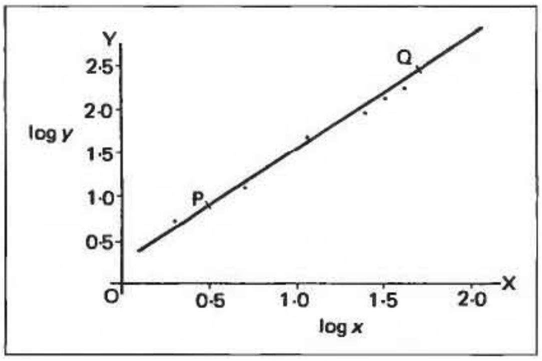
Do that and check with the results shown before moving on

46

$\log x$	0·3010	0·6990	1·079	1·398	1·505	1·602
$\log y$	0·750	1·14	1·72	2·05	2·20	2·30

Now plot these points on graph paper, $\log x$ along the x-axis
$\log y$ along the y-axis
and draw the estimated best straight-line graph.

47



If we select from the graph two points, P (0·50, 0·94) and Q (1·70, 2·45) we can calculate the values of m and c which are $m =$ .... and $c =$ ....

48

$$\boxed{m = 1{\cdot}258; \quad c = 0{\cdot}311}$$

Because $Y = mX + c \quad \therefore \quad 2{\cdot}45 = 1{\cdot}70m + c$

$$0{\cdot}94 = 0{\cdot}50m + c$$

$$\therefore \quad 1{\cdot}51 = 1{\cdot}20m \qquad \therefore \quad m = 1{\cdot}258; \text{ then } c = 0{\cdot}311.$$

Therefore $Y = 1{\cdot}258X + 0{\cdot}311$

$$\log y = n \log x + \log a$$

$$\therefore \quad n = 1{\cdot}258 \text{ and } \log a = 0{\cdot}311 \quad \therefore \quad a = 2{\cdot}05$$

Therefore, the equation is $y = 2{\cdot}05x^{1{\cdot}26}$.

3 Graphs of the form $y = ae^{nx}$

Exponential relationships occur frequently in technical situations. As before, our first step is to convert the equation to 'straight-line form' by taking logs of both sides. We could use common logarithms as we did previously, but the work involved is less if we use natural logarithms.

So, taking natural logarithms of both sides, we express the equation in the form ....

49

$$\boxed{\ln y = nx + \ln a}$$

If we compare this with the straight-line equation, we have:

$$\ln y = nx + \ln a$$
$$Y = mX + c$$

which shows that if we plot values of $\ln y$ along the Y-axis and values of just x along the X-axis, the value of m will give the value of n, and the value of c will be $\ln a$, hence $a = e^c$.

▶

Let us do an example.

The following values of W and T are related by the law $W = ae^{nT}$ where a and n are constants:

T	3·0	10	15	30	50	90
W	3·857	1·974	1·733	0·4966	0·1738	0·0091

We need values of $\ln W$, so compile a table showing values of T and $\ln W$.

50

T	3·0	10	15	30	50	90
$\ln W$	1·35	0·68	0·55	−0·70	−1·75	−4·70

$W = ae^{nT}$ $\quad\quad \therefore \ \ln W = nT + \ln a$

$$Y = mX + c$$

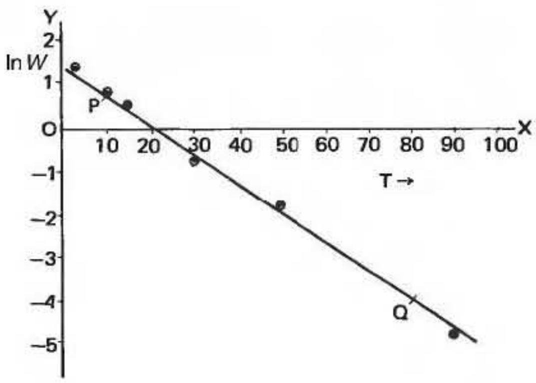
Therefore, we plot $\ln W$ along the Y-axis
and T along the X-axis
to obtain a straight-line graph, from which $m = n$
and $c = \ln a \quad \therefore \ a = e^c$.

So, plot the points; draw the best straight-line graph; and from it determine the values of n and a. The required law is therefore

$W = \ldots\ldots\ldots\ldots$

51

$$\boxed{W = 4{\cdot}13e^{-0{\cdot}067T}}$$



$P = (10, 0{\cdot}75)$;
$Q = (80, -3{\cdot}93)$

$$-3{\cdot}93 = 80m + c$$
$$0{\cdot}75 = 10m + c$$
$$\therefore \ -4{\cdot}68 = 70m$$
$$\therefore \ m = -0{\cdot}0668$$
$$\therefore \ c = 0{\cdot}75 + 0{\cdot}668$$
$$= 1{\cdot}418$$
$$m = n = -0{\cdot}0668$$
$$c = \ln a = 1{\cdot}418$$
$$\therefore \ a = 4{\cdot}13$$

$\therefore$ The required law is

$W = 4{\cdot}13e^{-0{\cdot}067T}$

If we can express the equation or law in straight-line form, the same method can be applied.

How about these? How could you arrange for each of the following to give a straight-line graph?

1 $y = ax^2 + b$ **3** $y = \frac{a}{x} + b$

2 $y = ax + b$ **4** $y = ax^2 + bx$

Check your suggestions with the next frame

52

Here they are:

Equation	*Straight-line form*	*Plot*
$y = ax^2 + b$	$y = ax^2 + b$	y against x^2
$y = ax + b$	$y = ax + b$	y against x
$y = \frac{a}{x} + b$	$y = a\frac{1}{x} + b$	y against $\frac{1}{x}$
$y = ax^2 + bx$	$\frac{y}{x} = ax + b$	$\frac{y}{x}$ against x

And now just one more. If we convert $y = \frac{a}{x+b}$ to straight-line form, it becomes …………

53

$$\boxed{y = -\frac{1}{b}xy + \frac{a}{b}}$$

Because

$$\text{if } y = \frac{a}{x+b}, \text{ then } xy + by = a \quad \therefore \ by = -xy + a$$

$$\therefore \ y = -\frac{1}{b}xy + \frac{a}{b}$$

That is, if we plot values of y against values of the product xy, we shall obtain a straight-line graph from which $m = -\frac{1}{b}$ and $c = \frac{a}{b}$.

From these, a and b can be easily found.

Finally, to what is perhaps the most important part of this Programme.
Let us start with a new frame

Method of least squares

54

All the methods which involve drawing the best straight line by eye are approximate only and depend upon the judgement of the operator. Quite considerable variation can result from various individuals' efforts to draw the 'best straight line'.

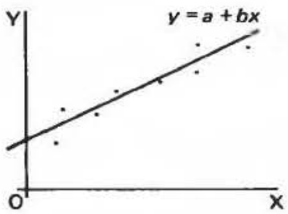
The *method of least squares* determines the best straight line entirely by calculation, using the set of recorded results. The form of the equation has to be chosen and this is where the previous revision will be useful.

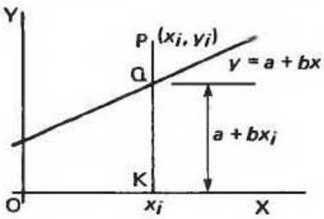
Let us start with the case of a linear relationship

55

Fitting a straight-line graph

We have to fit a straight line $y = a + bx$ to a set of plotted points (x_1, y_1), $(x_2, y_2) \dots (x_n, y_n)$ so that the sum of the squares of the distances to this straight line from the given set of points is a minimum. The distance of any point from the line is measured along an ordinate, i.e. in the y-direction.





If we take a sample point P (x_i, y_i):

QK is the value of $y = a + bx$ at $x = x_i$, i.e. $a + bx_i$.

PQ is the difference between PK and QK, i.e. $y_i - a - bx_i$.

$$\therefore \; PQ^2 = (y_i - a - bx_i)^2$$

Therefore, the sum S of the squares of these differences for all n such points is given by $S = \sum_{i=1}^{n}(y_i - a - bx_i)^2$.

We have to determine the values a and b so that S shall be a minimum. The right-hand side contains two unknowns a and b. Therefore, for the sum of the squares to be a minimum:

$$\frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial b} = 0$$

$$\frac{\partial S}{\partial a} = -2\sum_{i=1}^{n}(y_i - a - bx_i) = 0; \quad \frac{\partial S}{\partial b} = -2\sum_{i=1}^{n}x_i(y_i - a - bx_i) = 0$$

The first gives $\sum_{i=1}^{n} y_i - na - b\sum_{i=1}^{n} x_i = 0$ $\left[\text{Note: } \sum_{i=1}^{n} a = na.\right]$

$$\text{i.e. } an + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \qquad \text{(a)}$$

▶

The second gives $\sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 = 0$

i.e. $a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} {x_i}^2 = \sum_{i=1}^{n} x_i y_i$ (b)

Equations (a) and (b) are called the *normal equations* of the problem and we will now write them without the suffixes, remembering that the x- and y-values are those of the recorded values.

$$\left.\begin{aligned} an + b\Sigma x &= \Sigma y \\ a\Sigma x + b\Sigma x^2 &= \Sigma xy \end{aligned}\right\} \text{ for all the } n \text{ given pairs of values.}$$

From these *normal equations*, the specific values of a and b can be determined.

We will now work through some examples

Example 1 56

Apply the method of least squares to fit a straight-line relationship for the following points:

x	−2·4	−0·8	0·3	1·9	3·2
y	−5·0	−1·5	2·5	6·4	11·0

For this set, $n = 5$ and the normal equations are

$$\left.\begin{aligned} an + b\sum x &= \sum y \\ a\sum x + b\sum x^2 &= \sum xy \end{aligned}\right\} \text{ where } y = a + bx.$$

Therefore, we need to sum the values of x, y, x^2 and xy. This is best done in table form:

	x	y	x^2	xy
	−2·4	−5·0	5·76	12·0
	−0·8	−1·5	0·64	1·2
	0·3	2·5	0·09	0·75
	1·9	6·4	3·61	12·16
	3·2	11·0	10·24	35·2
$\sum$	2·2	13·4	20·34	61·31

The normal equations now become

....

and ....

57

$$\boxed{\begin{aligned} 5a + 2{\cdot}2b &= 13{\cdot}4 \\ 2{\cdot}2a + 20{\cdot}34b &= 61{\cdot}31 \end{aligned}}$$

Dividing through each equation by the coefficient of a gives:

$$\begin{aligned} a + 0{\cdot}440b &= \ \ 2{\cdot}68 \\ \underline{a + 9{\cdot}245b} &\underline{= 27{\cdot}87} \\ \therefore\ 8{\cdot}805b &= 25{\cdot}19 \quad \therefore\ b = 2{\cdot}861 \\ \therefore\ a = 2{\cdot}68 &- 1{\cdot}2588 \quad \therefore\ a = 1{\cdot}421 \end{aligned}$$

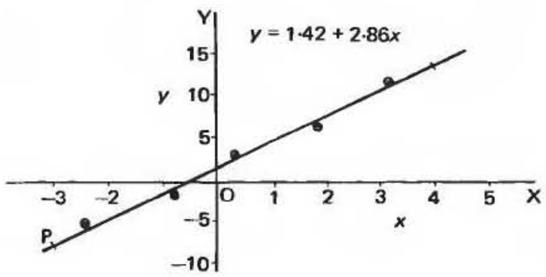
Therefore, the best straight line for the given values is

$y = 1{\cdot}42 + 2{\cdot}86x$

To see how well the method works, plot the set of values for x and y and also the straight line whose equation we have just found on the same axes.

58

The result should look like this:



Any relationship that can be expressed in straight-line form can be dealt with in the same way.

Example 2

It is required to fit the best rectangular hyperbola $xy = c$ to the set of values given below:

x	0·5	1·0	2·0	3·0	4·0	5·0
y	62	28	17	9·0	7·0	5·0

▶

In this case, $n = 6$. Also $y = c.\dfrac{1}{x}$

Ordinate difference between a point and the curve $= y_i - \dfrac{c}{x_i}$

The sum of the squares $S = \sum_{i=1}^{n}\left(y_i - \dfrac{c}{x_i}\right)^2$ and for S to be a minimum,

$$\frac{\partial S}{\partial c} = 0 \quad \frac{\partial S}{\partial c} = -2\sum_{i=1}^{n}\frac{1}{x_i}\left(y_i - \frac{c}{x_i}\right) = 0$$

$$\therefore \quad \sum_{i=1}^{n}\frac{y_i}{x_i} - c\sum_{i=1}^{n}\frac{1}{x_i^2} = 0$$

So this time we need values of x, y, $\dfrac{1}{x}$, $\dfrac{y}{x}$ and $\dfrac{1}{x^2}$:

x	y	$\dfrac{1}{x}$	$\dfrac{y}{x}$	$\dfrac{1}{x^2}$
0·5	62	2·0	124	4·0
1·0	28	1·0	28	1·0
2·0	17	0·5	8·5	0·25
3·0	9	0·333	3	0·111
4·0	7	0·25	1·75	0·0625
5·0	5	0·2	1	0·04
		$\sum$	166·25	5·4635

From this we can find c and the equation of the required hyperbola which is therefore ....

59

$$\boxed{xy = 30{\cdot}4}$$

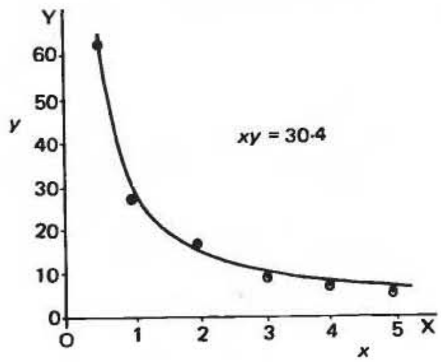
Because

$$\sum_{i=1}^{n}\frac{y}{x} - c\sum_{i=1}^{n}\frac{1}{x^2} = 0$$

$\therefore\ 166{\cdot}25 = 5{\cdot}4635c$

$\therefore\ c = 30{\cdot}4$

Here is the graph showing the plotted values and also the rectangular hyperbola we have just obtained:



60

Example 3

Values of x and y are related by the law $y = a \ln x$. Determine the value of a to provide the best fit for the following set of values:

x	0·3	0·9	1·7	2·5	4·0	10·0
y	−7·54	−0·672	3·63	4·41	8·22	12·2

$y = a \ln x$ will give a straight line if we plot y against $\ln x$. Therefore, let $u = \ln x$.

Ordinate difference $= y_i - a \ln x_i = y_i - au_i \quad \therefore \; S = \sum_{i=1}^{n} (y_i - au_i)^2$

For minimum S, $\dfrac{\partial S}{\partial a} = 0. \qquad \therefore \; -2\sum_{i=1}^{n} u_i(y_i - au_i) = 0$

$$\therefore \; \sum_{i=1}^{n} u_i y_i - a\sum_{i=1}^{n} u_i^2 = 0$$

Therefore we need values of x, $\ln x$ (i.e. u), y, uy and u^2:

x	u $(= \ln x)$	y	uy	u^2
0·3	−1·204	−7·54	9·078	1·450
0·9	−0·105	−0·672	0·0708	0·0111
1·7	0·531	3·63	1·9262	0·2816
2·5	0·916	4·41	4·0408	0·8396
4·0	1·3863	8·22	11·3953	1·9218
10·0	2·3026	12·20	28·092	5·302

Now total up the appropriate columns and finish the problem off. The equation is finally ....

61

$$\boxed{y = 5{\cdot}57 \ln x}$$

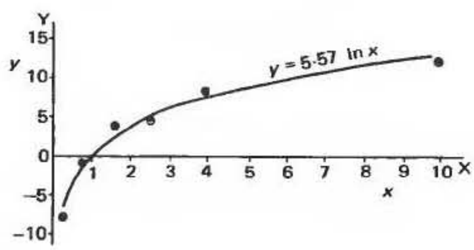
Because

$\sum uy = 54{\cdot}603$ and $\sum u^2 = 9{\cdot}806$

and since $\sum uy - a\sum u^2 = 0$,

then $a = \dfrac{\sum uy}{\sum u^2} = 5{\cdot}568$

$\therefore \; y = 5{\cdot}57 \ln x$



62

Example 4

A test on the length of tool life at different cutting speeds gave the following results:

Speed v (m/min)	120	130	170	200
Life T (min)	62	25	7·2	2·8

If the law relating v and T is $v = aT^k$ determine the constants a and k to give the best fit.

$$v = aT^k \qquad \therefore \ \log v = k \log T + \log a$$
$$Y = mX + c$$

$X = \log T$	1·792	1·398	0·8573	0·4472
$Y = \log v$	2·079	2·114	2·230	2·301

Arguing as before, $S = \sum_{i=1}^{n}(Y_i - mX_i - c)^2$

$$\frac{\partial S}{\partial m} = -2\sum_{i=1}^{n} X_i(Y_i - mX_i - c) = 0 \qquad \therefore \ \sum_{i=1}^{n} X_iY_i - m\sum_{i=1}^{n} X_i^2 - c\sum_{i=1}^{n} X_i = 0$$

$$\frac{\partial S}{\partial c} = -2\sum_{i=1}^{n}(Y_i - mX_i - c) = 0 \qquad \therefore \ \sum_{i=1}^{n} Y_i - m\sum_{i=1}^{n} X_i - nc = 0$$

So we now need columns for X, Y, XY and X^2.

Compile the appropriate table and finish off the problem, so finding the required law ....

63

$$\boxed{v = 237T^{-0{\cdot}173}}$$

Here is the working:

X	Y	XY	X^2
1·792	2·079	3·726	3·211
1·398	2·114	2·955	1·954
0·8573	2·230	1·912	0·735
0·4472	2·301	1·029	0·200
$\sum$ 4·4945	8·724	9·622	6·100

▶

The normal equations give $\quad 9{\cdot}622 - m6{\cdot}100 - c4{\cdot}495 = 0$

and $\quad 8{\cdot}724 - m4{\cdot}495 - c4 = 0.$

Dividing through each equation by the coefficient of m, we have:

$m + 0{\cdot}7369c = 1{\cdot}5774$

$\underline{m + 0{\cdot}8899c = 1{\cdot}9408}$

$\therefore\ 0{\cdot}1530c = 0{\cdot}3634 \qquad \therefore\ c = 2{\cdot}375$

Then $m = 1{\cdot}5774 - 0{\cdot}7369(2{\cdot}375) = 1{\cdot}5774 - 1{\cdot}7503 = -0{\cdot}1729$

$\therefore\ Y = -0{\cdot}173X + 2{\cdot}375$

$m = n \qquad \therefore\ n = -0{\cdot}173$

$c = \log a \qquad \therefore\ \log a = 2{\cdot}3752 \quad \therefore\ a = 237{\cdot}25$

$\therefore$ The law is $v = 237T^{-0{\cdot}173}$

The same principles can be applied to fit curves of higher degree to sets of values but, in general, fitting a second-degree curve involves three normal equations in three unknowns and curves of higher degree involve an increasing number of simultaneous equations, so that the working tends to become unwieldy without some form of computing facility.

All that now remains is to check down the **Revision summary** that follows and then to work through the **Can You?** checklist and **Test exercise** as usual.

64

Revision summary

1 *Standard curves* Refer back to Frames 2 to 17.

2 *Asymptotes* Rewrite the given equation, if necessary, on one line.

(a) *Asymptotes parallel to the axes*

Parallel to the x-axis: equate to zero the coefficient of the highest power of x.

Parallel to the y-axis: equate to zero the coefficient of the highest power of y.

(b) *General asymptotes*

Substitute $y = mx + c$ in the equation of the curve and equate to zero the coefficients of the two highest powers of x.

3 *Systematic curve sketching.*

(a) *Symmetry*

Even powers only of x: curve symmetrical about y-axis.

Even powers only of y: curve symmetrical about x-axis.

Even powers only of both x and y: curves symmetrical about both axes.

(b) *Intersection with the axes* Put $x = 0$ and $y = 0$.

(c) *Change of origin* to simplify analysis.

(d) *Asymptotes*

(e) *Large and small values* of x or y.

(f) *Stationary points*

(g) *Limitations* on possible range of values of x or y.

4 *Curve fitting.*

(a) Express the law or equation in straight-line form.

(b) Plot values and draw the best straight line as the middle line of the band of points.

(c) Determine m and c from the x- and y-coordinates of two points *on the line*. Check with the coordinates of a third point.

(d) Line of 'best fit' can be calculated by the *method of least squares*. Refer back to Frames 54 to 63.

Can You?

Checklist 12

65

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Draw sketch graphs of standard curves? 1 to 17

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the equations of asymptotes parallel to the x- and y-axes? 18 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Sketch the graphs of curves with asymptotes, stationary points and other features? 27 to 39

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Fit graphs to data using the 'straight-line' forms? 40 to 53

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Fit graphs to data using the method of least squares? 54 to 63

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 12

66

1 Without detailed plotting of points, sketch the graphs of the following showing relevant information on the graphs:

(a) $y = (x-3)^2 + 5$

(b) $y = 4x - x^2$

(c) $4y^2 + 24y - 14 - 16x + 4x^2 = 0$

(d) $5xy = 40$

(e) $y = 6 - e^{-2x}$

▶

2 Determine the asymptotes of the following curves:

(a) $x^2y - 9y + x^3 = 0$

(b) $y^2 = \dfrac{x(x-2)(x+4)}{x-4}$

3 Analyse and sketch the graph of the function $y = \dfrac{(x-3)(x+5)}{x+2}$

4 Express the following in 'straight-line' form and state the variables to be plotted on the x- and y-axes to give a straight line:

(a) $y = Ax + Bx^2$ (c) $y = \dfrac{A}{B+x}$

(b) $y = x + Ae^{kx}$ (d) $x^2(y^2 - 1) = k$

5 The force, P newtons, required to keep an object moving at a speed, V metres per second, was recorded.

P	126	178	263	398	525	724
V	1·86	2·34	2·75	3·63	4·17	4·79

If the law connecting P and V is of the form $V = aP^k$, where a and k are constants, apply the method of least squares to obtain the values of a and k that give the best fit to the given set of values.

Further problems 12

67

1 For each of the following curves, determine the asymptotes parallel to the x- and y-axes:

(a) $xy^2 + x^2 - 1 = 0$ (d) $y = \dfrac{x(x+4)}{(x+3)(x+2)}$

(b) $x^2y^2 = 4(x^2 + y^2)$ (e) $x^2y^2 - x^2 = y^2 + 1$

(c) $y = \dfrac{x^2 - 3x + 5}{x-3}$ (f) $y^2 = \dfrac{x}{x-2}$

2 Determine all the asymptotes of each of the following curves:

(a) $x^3 - xy^2 + 4x - 16 = 0$

(b) $xy^3 - x^2y + 3x^3 - 4y^3 - 1 = 0$

(c) $y^3 + 2y^2 - x^2y + y - x + 4 = 0$

3 Analyse and sketch the graphs of the following functions:

(a) $y = x + \dfrac{1}{x}$

(b) $y = \dfrac{1}{x^2 + 1}$

(c) $y^2 = \dfrac{x}{x - 2}$

(d) $y = \dfrac{(x-1)(x+4)}{(x-2)(x-3)}$

(e) $y(x+2) = (x+3)(x-4)$

(f) $x^2(y^2 - 25) = y$

(g) $xy^2 - x^2y + x + y = 2$

4 Variables x and y are thought to be related by the law $y = a + bx^2$. Determine the values of a and b that best fit the set of values given.

x	5·0	7·5	12	15	25
y	13·1	28·1	70·2	109	301

5 By plotting a suitable graph, show that P and W are related by a law of the form $P = a\sqrt{W} + b$, where a and b are constants, and determine the values of a and b.

W	7·0	10	15	24	40	60
P	9·76	11·0	12·8	15·4	18·9	22·4

6 If $R = a + \dfrac{b}{d^2}$, find the best values for a and b from the set of corresponding values given below:

d	0·1	0·2	0·3	0·5	0·8	1·5
R	5·78	2·26	1·60	1·27	1·53	1·10

7 Two quantities, x and y, are related by the law $y = \dfrac{a}{1 - bx^2}$, where a and b are constants. Using the values given below, draw a suitable graph and hence determine the best values of a and b.

x	4	6	8	10	11	12
y	4·89	5·49	6·62	9·00	11·4	16·1

8 The pressure p and volume v of a mass of gas in a container are related by the law $pv^n = c$, where n and c are constants. From the values given below, plot a suitable graph and hence determine the values of n and c.

v	4·60	7·20	10·1	15·3	20·4	30·0
p	14·2	7·59	4·74	2·66	1·78	1·04

9 The current, I milliamperes, in a circuit is measured for various values of applied voltage V volts. If the law connecting I and V is $I = aV^n$, where a and n are constants, draw a suitable graph and determine the values of a and n that best fit the set of recorded values.

V	8	12	15	20	28	36
I	41·1	55·6	65·8	81·6	105	127

10 Values of x and y are thought to be related by a law of the form $y = ax + b\ln x$, where a and b are constants. By drawing a suitable graph, test whether this is so and determine the values of a and b.

x	10·4	32·0	62·8	95·7	136	186
y	8·14	12·8	16·3	19·2	22·1	25·3

11 The following pairs of values of x and y are thought to satisfy the law $y = ax^2 + \dfrac{b}{x}$. Draw a suitable graph to confirm that this is so and determine the values of the constants a and b.

x	1	3	4	5	6	7
y	5·18	15·9	27·0	41·5	59·3	80·4

12 In a test on breakdown voltages, V kilovolts, for insulation of different thicknesses, t millimetres, the following results were obtained:

t	2·0	3·0	5·0	10	14	18
V	153	200	282	449	563	666

If the law connecting V and t is $V = at^n$, draw a suitable graph and determine the values of the constants a and n.

13 The torque, T newton metres, required to rotate shafts of different diameters, D millimetres, on a machine is shown below. If the law is $T = aD^n$, where a and n are constants, draw a suitable graph and hence determine the values of a and n.

D	7·0	10	18	25	40
T	0·974	1·71	4·33	7·28	15·3

Series 1

Frames

1 to 56

Learning outcomes

When you have completed this Programme you will be able to:

- Manipulate arithmetic and geometric series
- Manipulate series of powers of the natural numbers
- Determine the limiting values of arithmetic and geometric series
- Determine the limiting values of simple indeterminate forms
- Apply various convergence tests to infinite series
- Distinguish between absolute and conditional convergence

Sequences

1

A sequence is a set of quantities, $u_1, u_2, u_3, \ldots$, stated in a definite order and each term formed according to a fixed pattern, i.e. $u_r = f(r)$.

e.g. $1, 3, 5, 7, \ldots$ is a sequence (the next term would be 9).

$2, 6, 18, 54, \ldots$ is a sequence (the next term would be 162).

$1^2, -2^2, 3^2, -4^2, \ldots$ is a sequence (the next term would be 5^2).

Also $1, -5, 37, 6, \ldots$ is a sequence, but its pattern is more involved and the next term cannot readily be anticipated.

A *finite* sequence contains only a finite number of terms.

An *infinite* sequence is unending.

So which of the following constitutes a finite sequence:

(a) all the natural numbers, i.e. $1, 2, 3, \ldots$ etc.
(b) the page numbers of a book
(c) the telephone numbers in a telephone directory.

2

The page numbers of a book
The telephone numbers in a directory

Clearly, the page numbers are in fixed order and terminate at the last page. The telephone numbers form a complicated sequence, ordered by the alphabetical letters of the surnames of the subscribers. The natural numbers form an infinite sequence, since they never come to an end.

Series

3

A *series* is formed by the sum of the terms of a sequence.

e.g. $1, 3, 5, 7, \ldots$ is a sequence

but $1 + 3 + 5 + 7 + \ldots$ is a series.

We shall indicate the terms of a series as follows:

u_1 will represent the first term, u_2 the second term, u_3 the third term, etc., so that u_r will represent the rth term and u_{r+1} the $(r+1)$th term, etc.

Also, the sum of the first 5 terms will be indicated by S_5.

So the sum of the first n terms will be stated as ....

4

$\boxed{S_n}$

You will already be familiar with two special kinds of series which have many applications. These are (a) *arithmetic series* and (b) *geometric series*. Just by way of revision, however, we will first review the important results relating to these two series.

Arithmetic series (or arithmetic progression), denoted by AP

5

An example of an AP is the series:

$2 + 5 + 8 + 11 + 14 + \dots$

You will note that each term can be written from the previous term by simply adding on a constant value 3. This regular increment is called the *common difference* and is found by selecting any term and subtracting from it the previous term

e.g. $11 - 8 = 3$; $5 - 2 = 3$; etc.

Move on to the next frame

6

The *general arithmetic series* can therefore be written:

$a + (a + d) + (a + 2d) + (a + 3d) + \dots$ (a)

where a = first term and d = common difference.

You will remember that:

(a) the nth term $= a + (n - 1)d$ (b)

(b) the sum of the first n terms is given by

$S_n = \dfrac{n}{2}(2a + \overline{n-1}.d)$ (c)

Make a note of these three items in your record book.

By way of warming up, find the sum of the first 20 terms of the series:

$10 + 6 + 2 - 2 - 6 \dots$ etc.

Then move to Frame 7

7

$$\boxed{S_{20} = -560}$$

Because for the series $10 + 6 + 2 - 2 - 6 \ldots$ etc.

$a = 10$ and $d = 2 - 6 = -4$

$$S_n = \frac{n}{2}(2a + \overline{n-1}.d)$$

$$\therefore\ S_{20} = \frac{20}{2}(20 + 19[-4])$$

$$= 10(20 - 76) = 10(-56) = -560$$

Here is another example:

If the 7th term of an AP is 22 and the 12th term is 37, find the series.

$$\left.\begin{array}{ll} \text{We know 7th term} = 22 & \therefore\ a + 6d = 22 \\ \text{and 12th term} = 37 & \therefore\ a + 11d = 37 \end{array}\right\} \begin{array}{ll} 5d = 15 & \therefore\ d = 3 \\ & \therefore\ a = 4 \end{array}$$

So the series is $4 + 7 + 10 + 13 + 16 + \ldots$ etc.

Here is one for you to do:

The 6th term of an AP is -5 and the 10th term is -21. Find the sum of the first 30 terms.

8

$$\boxed{S_{30} = -1290}$$

Because

$$\left.\begin{array}{ll} \text{6th term} = -5 & \therefore\ a + 5d = -5 \\ \text{10th term} = -21 & \therefore\ a + 9d = -21 \end{array}\right\} \begin{array}{ll} 4d = -16 & \therefore\ d = -4 \\ & a = 15 \end{array}$$

$$\therefore\ a = 15,\ d = -4,\ n = 30,\ S_n = \frac{n}{2}(2a + \overline{n-1}.d)$$

$$\therefore\ S_{30} = \frac{30}{2}(30 + 29[-4]) = 15(30 - 116) = 15(-86) = -1290$$

Arithmetic mean

We are sometimes required to find the arithmetic mean of two numbers, P and Q. This means that we have to insert a number A between P and Q, so that $P + A + Q$ forms an AP.

$$A - P = d \text{ and } Q - A = d$$

$$\therefore\ A - P = Q - A \qquad 2A = P + Q \qquad \therefore\ A = \frac{P + Q}{2}$$

The arithmetic mean of two numbers, then, is simply their average.

Therefore, the arithmetic mean of 23 and 58 is ....

9

The arithmetic mean of 23 and 58 is $\boxed{40{\cdot}5}$

If we are required to insert 3 arithmetic means between two given numbers, P and Q, it means that we have to supply three numbers, A, B, C between P and Q, so that $P + A + B + C + Q$ forms an AP.

For example: Insert 3 arithmetic means between 8 and 18.

Let the means be denoted by A, B, C.

Then $8 + A + B + C + 18$ forms an AP.

First term, $a = 8$; fifth term $= a + 4d = 18$

$$\left.\begin{aligned} \therefore\ a &= 8 \\ a + 4d &= 18 \end{aligned}\right\} 4d = 10 \qquad \therefore\ d = 2{\cdot}5$$

$$\left.\begin{aligned} A &= 8 + 2{\cdot}5 = 10{\cdot}5 \\ B &= 8 + 5 = 13 \\ C &= 8 + 7{\cdot}5 = 15{\cdot}5 \end{aligned}\right\} \text{Required arithmetic means are } 10{\cdot}5,\ 13,\ 15{\cdot}5$$

Now you find five arithmetic means between 12 and 21·6.

Then move to Frame 10

10

$\boxed{13{\cdot}6,\ 15{\cdot}2,\ 16{\cdot}8,\ 18{\cdot}4,\ 20}$

Here is the working:

Let the 5 arithmetic means be A, B, C, D, E.

Then $12 + A + B + C + D + E + 21{\cdot}6$ forms an AP.

$\therefore\ a = 12; \quad a + 6d = 21{\cdot}6$

$\therefore\ 6d = 9{\cdot}6 \qquad \therefore\ d = 1{\cdot}6$

$$\begin{aligned} \text{Then } A &= 12 + 1{\cdot}6 = 13{\cdot}6 \qquad & A &= 13{\cdot}6 \\ B &= 12 + 3{\cdot}2 = 15{\cdot}2 & B &= 15{\cdot}2 \\ C &= 12 + 4{\cdot}8 = 16{\cdot}8 & C &= 16{\cdot}8 \\ D &= 12 + 6{\cdot}4 = 18{\cdot}4 & D &= 18{\cdot}4 \\ E &= 12 + 8{\cdot}0 = 20{\cdot}0 & E &= 20 \end{aligned}$$

So that is it! Once you have done one, the others are just like it.

Now we will see how much you remember about *geometric series*.

So, on to Frame 11

Geometric series (geometric progression), denoted by GP

11

An example of a GP is the series:

$1 + 3 + 9 + 27 + 81 + \ldots$ etc.

Here you can see that any term can be written from the previous term by multiplying it by a constant factor 3. This constant factor is called the *common ratio* and is found by selecting any term and dividing it by the previous one:

e.g. $27 \div 9 = 3$; $9 \div 3 = 3$; etc.

A GP therefore has the form:

$a + ar + ar^2 + ar^3 + \ldots$ etc.

where a = first term, r = common ratio.
So in the geometric series $5 - 10 + 20 - 40 + \ldots$ etc. the common ratio, r, is ....

12

$$\boxed{r = \frac{20}{-10} = -2}$$

The general geometric series is therefore:

$a + ar + ar^2 + ar^3 + \ldots$ etc. (d)

and you will remember that:

(1) the nth term $= ar^{n-1}$ (e)

(2) the sum of the first n terms is given by:

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{(f)}$$

Make a note of these items in your record book.

So, now you can do this one:

For the series $8 + 4 + 2 + 1 + \frac{1}{2} + \ldots$etc., find the sum of the first 8 terms.

Then on to Frame 13

13

$$\boxed{S_8 = 15\frac{15}{16}}$$

Because for the series 8, 4, 2, 1, ... etc.

$$a = 8;\ r = \frac{2}{4} = \frac{1}{2};\quad S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore\ S_8 = \frac{8\left(1 - \left[\frac{1}{2}\right]^8\right)}{1 - \frac{1}{2}}$$

$$= \frac{8\left(1 - \frac{1}{256}\right)}{1 - \frac{1}{2}} = 16\cdot\frac{255}{256} = \frac{255}{16} = 15\frac{15}{16}$$

Now here is another example.

If the 5th term of a GP is 162 and the 8th term is 4374, find the series.

We have 5th term = 162 $\therefore\ ar^4 = 162$

8th term = 4374 $\therefore\ ar^7 = 4374$

$$\frac{ar^7}{ar^4} = \frac{4374}{162}\quad \therefore\ r^3 = 27\quad \therefore\ r = 3$$

$$\therefore\ a = \ldots\ldots\ldots\ldots$$

14

$$\boxed{a = 2}$$

Because

$ar^4 = 162;\ ar^7 = 4374$ and $r = 3$

$$\therefore\ a.3^4 = 162\quad \therefore\ a = \frac{162}{81}\quad \therefore\ a = 2$$

$\therefore$ The series is: $2 + 6 + 18 + 54 + \ldots$ etc.

Of course, now that we know the values of a and r, we could calculate the value of any term or the sum of a given number of terms.

For this same series, find

(a) the 10th term

(b) the sum of the first 10 terms.

When you have finished, move to Frame 15

15

$$\boxed{39\,366, \quad 59\,048}$$

Because $a = 2$; $r = 3$

(a) 10th term $= ar^9 = 2 \times 3^9 = 2(19\,683) = 39\,366$

(b) $S_{10} = \dfrac{a(1-r^{10})}{1-r} = \dfrac{2(1-3^{10})}{1-3}$

$$= \frac{2(1-59\,049)}{-2} = 59\,048$$

Geometric mean (GM)

The geometric mean of two given numbers P and Q is a number A such that $P + A + Q$ form a GP.

$$\frac{A}{P} = r \text{ and } \frac{Q}{A} = r$$

$$\therefore \frac{A}{P} = \frac{Q}{A} \quad \therefore A^2 = PQ \quad A = \sqrt{PQ}$$

So the geomectric mean of 2 numbers is the square root of their product.

Therefore, the geometric mean of 4 and 25 is ....

16

$$\boxed{A = \sqrt{4 \times 25} = \sqrt{100} = 10}$$

To insert 3 GMs between two given numbers, P and Q, means to insert 3 numbers, A, B, C, such that $P + A + B + C + Q$ form a GP.

For example, insert 4 geometric means between 5 and 1215.

Let the means be A, B, C, D. Then $5 + A + B + C + D + 1215$ form a GP.

i.e. $a = 5$ and $ar^5 = 1215$

$$\therefore r^5 = \frac{1215}{5} = 243 \quad \therefore r = 3$$

$$\therefore A = 5 \times 3 = 15$$

$$B = 5 \times 9 = 45$$

$$C = 5 \times 27 = 135$$

$$D = 5 \times 81 = 405$$

The required geometric means are: 15, 45, 135, 405

Now here is one for you to do: Insert two geometric means between 5 and 8·64.

Then on to Frame 17

17

Required geometric means are 6·0, 7·2

Because

Let the means be A and B.

Then $5 + A + B + 8{\cdot}64$ form a GP.

$\therefore\ a = 5;\quad \therefore\ ar^3 = 8{\cdot}64;\quad \therefore\ r^3 = 1{\cdot}728;\quad \therefore\ r = 1{\cdot}2$

$$\left.\begin{array}{l} A = 5 \times 1{\cdot}2 = 6{\cdot}0 \\ B = 5 \times 1{\cdot}44 = 7{\cdot}2 \end{array}\right\} \text{ Required means are } 6{\cdot}0 \text{ and } 7{\cdot}2$$

Arithmetic and geometric series are, of course, special kinds of series. There are other special series that are worth knowing. These consist of the series of the powers of the natural numbers. So let us look at these in the next frame.

Series of powers of the natural numbers

18

1 The series $1 + 2 + 3 + 4 + 5 + \ldots + n$ etc. $= \sum_{r=1}^{n} r$.

This series, you will see, is an example of an AP, where $a = 1$ and $d = 1$. The sum of the first n terms is given by:

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + 4 + 5 = \ldots + n$$

$$= \frac{n}{2}(2a + \overline{n-1}d) = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

So, the sum of the first 100 natural numbers is ....

Then on to Frame 19

19

$$\boxed{\sum_{r=1}^{100} r = 5050}$$

Because

$$\sum_{r=1}^{n} r = \frac{100(101)}{2} = 50(101) = 5050$$

▶

2 That was easy enough. Now let us look at this one: To establish the result for the sum of n terms of the series $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2$, we make use of the identity:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

We write this as:

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Replacing n by $n-1$, we get

$$n^3 - (n-1)^3 = 3(n-1)^2 + 3(n-1) + 1$$

and again $(n-1)^3 - (n-2)^3 = 3(n-2)^2 + 3(n-2) + 1$

and $(n-2)^3 - (n-3)^3 = 3(n-3)^2 + 3(n-3) + 1$

Continuing like this, we should eventually arrive at:

$$3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1$$
$$2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

If we now add all these results together, we find on the left-hand side that all the terms disappear except the first and the last.

$$(n+1)^3 - 1^3 = 3\left\{n^2 + (n-1)^2 + (n-2)^2 + \ldots + 2^2 + 1^2\right\}$$
$$+ 3\left\{n + (n-1) + (n-2) + \ldots + 2 + 1\right\} + n(1)$$
$$= 3.\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n$$

$$\therefore\ n^3 + 3n^2 + 3n + \cancel{1} - \cancel{1} = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n = 3\sum_{r=1}^{n} r^2 + 3\frac{n(n+1)}{2} + n$$

$$\therefore\ n^3 + 3n^2 + 2n = 3\sum_{r=1}^{n} r^2 + \frac{3}{2}(n^2 + n)$$

$$\therefore\ 2n^3 + 6n^2 + 4n = 6\sum_{r=1}^{n} r^2 + 3n^2 + 3n$$

$$6\sum_{r=1}^{n} r^2 = 2n^3 + 3n^2 + n$$

$$\therefore\ \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

So, the sum of the first 12 terms of the series $1^2 + 2^2 + 3^2 + \ldots$ is ....

20

$$\boxed{650}$$

Because $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$

so $\sum_{r=1}^{12} r^2 = \dfrac{12(13)(25)}{6} = 26(25) = 650$

3 The sum of the cubes of the natural numbers is found in much the same way. This time, we use the identity:

$$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$$

We rewrite it as before:

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

If we now do the same trick as before and replace n by $(n-1)$ over and over again, and finally total up the results we get the result:

$$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Note in passing that $\sum_{r=1}^{n} r^3 = \left\{\sum_{r=1}^{n} r\right\}^2$

21

Let us collect together these last three results. Here they are:

1 $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ (g)

2 $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$ (h)

3 $\sum_{r=1}^{n} r^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$ (i)

These are handy results, so copy them into your record book.

Now move on to Frame 22 and we can see an example of the use of these results

22

Find the sum of the series $\sum_{n=1}^{5} n(3+2n)$

$$S_5 = \sum_{n=1}^{5} n(3+2n) = \sum_{n=1}^{5}(3n+2n^2)$$

$$= \sum_{n=1}^{5} 3n + \sum_{n=1}^{5} 2n^2 = 3\sum_{n=1}^{5} n + 2\sum_{n=1}^{5} n^2$$

$$= 3.\frac{5.6}{2} + 2.\frac{5.6.11}{6} = 45 + 110 = 155$$

It is just a question of using the established results. Here is one for you to do in the same manner.

Find the sum of the series $\sum_{n=1}^{4}(2n+n^3)$

Working in Frame 23

23

$\boxed{120}$

Because $S_4 = \sum_{n=1}^{4}(2n + n^3)$

$$= 2\sum_{n=1}^{4} n + \sum_{n=1}^{4} n^3$$

$$= \frac{2.4.5}{2} + \left\{\frac{4.5}{2}\right\}^2$$

$$= 20 + 100 = 120$$

Remember:

$$\text{Sum of the first } n \text{ natural numbers} = \frac{n(n+1)}{2}$$

$$\text{Sum of squares of the first } n \text{ natural numbers} = \frac{n(n+1)(2n+1)}{6}$$

$$\text{Sum of cubes of the first } n \text{ natural numbers} = \left\{\frac{n(n+1)}{2}\right\}^2$$

Infinite series

24

So far, we have been concerned with a finite number of terms of a given series. When we are dealing with the sum of an infinite number of terms of a series, we must be careful about the steps we take.

For example, consider the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

This we recognize as a GP in which $a = 1$ and $r = \frac{1}{2}$. The sum of the first n terms is therefore given by:

$$S_n = \frac{1\left(1 - \left[\frac{1}{2}\right]^n\right)}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$$

Now if n is very large, 2^n will be very large and therefore $\frac{1}{2^n}$ will be very small.

In fact, as $n \to \infty$, $\frac{1}{2^n} \to 0$. The sum of all the terms in this infinite series is therefore given by S_∞ = the limiting value of S_n as $n \to \infty$.

i.e. $S_\infty = \underset{n\to\infty}{Lim}\, S_n = 2(1 - 0) = 2$

This result means that we can make the sum of the series as near to the value 2 as we please by taking a sufficiently large number of terms.

Next frame

25

This is not always possible with an infinite series, for in the case of an AP things are very different.

Consider the infinite series $1 + 3 + 5 + 7 + \ldots$

This is an AP in which $a = 1$ and $d = 2$.

$$\text{Then } S_n = \frac{n}{2}(2a + \overline{n-1}.d) = \frac{n}{2}(2 + \overline{n-1}.2)$$

$$= \frac{n}{2}(2 + 2n - 2)$$

$$S_n = n^2$$

Of course, in this case, if n is large then the value of S_n is very large. In fact, if $n \to \infty$, then $S_n \to \infty$, which is not a definite numerical value and therefore of little use to us.

This always happens with an AP: if we try to find the 'sum to infinity', we always obtain $+\infty$ or $-\infty$ as the result, depending on the actual series.

Move on now to Frame 26

26

In the previous two frames, we made two important points:

(a) We cannot evaluate the sum of an infinite number of terms of an AP because the result is always infinite.

(b) We can sometimes evaluate the sum of an infinite number of terms of a GP since, for such a series, $S_n = \dfrac{a(1-r^n)}{1-r}$ and *provided* $|r| < 1$, then as $n \to \infty$, $r^n \to 0$. In that case $S_\infty = \dfrac{a(1-0)}{1-r} = \dfrac{a}{1-r}$, i.e. $S_\infty = \dfrac{a}{1-r}$.

So, find the 'sum to infinity' of the series

$20 + 4 + 0{\cdot}8 + 0{\cdot}16 + 0{\cdot}032 + \ldots\ldots\ldots\ldots$

27

$$\boxed{S_\infty = 25}$$

Because, for

$$20 + 4 + 0{\cdot}8 + 0{\cdot}16 + 0{\cdot}032 + \ldots$$

$$a = 20; \quad r = \frac{0{\cdot}8}{4} = 0{\cdot}2 = \frac{1}{5}$$

$$\therefore\ S_\infty = \frac{a}{1-r} = \frac{20}{1-\frac{1}{5}} = \frac{5}{4}.(20) = 25$$

Limiting values

28

In this Programme, we have already seen that we have sometimes to determine the limiting value of S_n as $n \to \infty$. Before we leave this topic, let us look a little further into the process of finding limiting values. A few examples will suffice.

So move on to Frame 29

29

Example 1

To find the limiting value of $\dfrac{5n+3}{2n-7}$ as $n \to \infty$.

We cannot just substitute $n = \infty$ in the expression and simplify the result, since ∞ is not an ordinary number and does not obey the normal rules. So we do it this way:

$$\frac{5n+3}{2n-7} = \frac{5+3/n}{2-7/n} \quad \text{(dividing top and bottom by } n\text{)}$$

$$\underset{n\to\infty}{Lim}\left\{\frac{5n+3}{2n-7}\right\} = \underset{n\to\infty}{Lim}\frac{5+3/n}{2-7/n}$$

Now when $n \to \infty$, $3/n \to 0$ and $7/n \to 0$

$$\therefore \underset{n\to\infty}{Lim}\frac{5n+3}{2n-7} = \underset{n\to\infty}{Lim}\frac{5+3/n}{2-7/n} = \frac{5+0}{2-0} = \frac{5}{2}$$

We can always deal with fractions of the form $\dfrac{c}{n}$, $\dfrac{c}{n^2}0$, $\dfrac{c}{n^3}$, etc., because when $n \to \infty$, each of these tends to zero, which is a precise value.

Let us try another example.

On to the next frame then

30

Example 2

To find the limiting value of $\dfrac{2n^2+4n-3}{5n^2-6n+1}$ as $n \to \infty$.

First of all, we divide top and bottom by the highest power of n which is involved, in this case n^2.

$$\frac{2n^2+4n-3}{5n^2-6n+1} = \frac{2+4/n-3/n^2}{5-6/n+1/n^2}$$

$$\therefore \underset{n\to\infty}{Lim}\frac{2n^2+4n-3}{5n^2-6n+1} = \underset{n\to\infty}{Lim}\frac{2+4/n-3/n^2}{5-6/n+1/n^2}$$

$$= \frac{2+0-0}{5-0+0} = \frac{2}{5}$$

Example 3

To find $\underset{n\to\infty}{Lim}\dfrac{n^3-2}{2n^3+3n-4}$

In this case, the first thing is to ....

Move on to Frame 31

31

Divide top and bottom by n^3

Right. So we get:

$$\frac{n^3-2}{2n^3+3n-4}=\frac{1-2/n^3}{2+3/n^2-4/n^3}$$

$$\therefore \lim_{n\to\infty}\frac{n^3-2}{2n^3+3n-4}=\dots\dots\dots\dots$$

Finish it off. Then move on to Frame 32

32

$$\boxed{\frac{1}{2}}$$

Next frame

Convergent and divergent series

33

A series in which the sum (S_n) of n terms of the series tends to a definite value, as $n \to \infty$, is called a *convergent* series. If S_n does not tend to a definite value as $n \to \infty$, the series is said to be *divergent*.

For example, consider the GP: $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots$

We know that for a GP, $S_n=\frac{a(1-r^n)}{1-r}$, so in this case since $a=1$ and $r=\frac{1}{3}$, we have:

$$S_n=\frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}=\frac{1-\frac{1}{3^n}}{\frac{2}{3}}=\frac{3}{2}\left(1-\frac{1}{3^n}\right)$$

$$\therefore \text{ As } n\to\infty,\ \frac{1}{3^n}\to 0 \quad \therefore \lim_{n\to\infty} S_n=\frac{3}{2}$$

The sum of n terms of this series tends to the definite value of $\frac{3}{2}$ as $n \to \infty$. It is therefore a .... series.
(convergent/divergent)

34

convergent

If S_n tends to a definite value as $n \to \infty$, the series is *convergent*.

If S_n does not tend to a definite value as $n \to \infty$, the series is *divergent*.

Here is another series. Let us investigate this one.

$1 + 3 + 9 + 27 + 81 + \dots$

This is also a GP with $a = 1$ and $r = 3$.

$$\therefore \; S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-3^n)}{1-3} = \frac{1-3^n}{-2}$$
$$= \frac{3^n - 1}{2}$$

Of course, when $n \to \infty$, $3^n \to \infty$ also.

$\therefore \; \lim_{n\to\infty} S_n = \infty$ (which is not a definite numerical value)

So in this case, the series is ....

35

divergent

We can make use of infinite series only when they are convergent and it is necessary, therefore, to have some means of testing whether or not a given series is, in fact, convergent.

Of course, we could determine the limiting value of S_n as $n \to \infty$, as we did in the examples a moment ago, and this would tell us directly whether the series in question tended to a definite value (i.e. was convergent) or not.

That is the fundamental test, but unfortunately, it is not always easy to find a formula for S_n and we have therefore to find a test for convergence which uses the terms themselves.

Remember the notation for series in general. We shall denote the terms by $u_1 + u_2 + u_3 + u_4 + \dots$

So now move on to Frame 36

Test for convergence

36

Test 1. A series cannot be convergent unless its terms ultimately tend to zero, i.e. unless $\lim_{n\to\infty} u_n = 0$

If $\lim_{n\to\infty} u_n \neq 0$, the series is divergent.

This is almost just common sense, for if the sum is to approach some definite value as the value of n increases, the numerical value of the individual terms must diminish.